A TIME DOMAIN BOUNDARY INTEGRAL EQUATION METHOD FOR LAYERED MEDIUM SCATTERING

November 13th, 2015

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On each $\Omega_\ell$, $\ell = 0, 1, \ldots, N$, the solution $u_\ell$ satisfies the wave equation:
\[ c_\ell^{-2} \ddot{u}_\ell = \kappa_\ell \Delta u_\ell, \quad u_\ell(0) = \dot{u}_\ell(0) = 0. \]
Denote $m_\ell = c_\ell \sqrt{\kappa_\ell}$.

On each $\Gamma_\ell$, $\ell = 1, 2, \ldots, N$, the solution satisfies a pair of jump cond:
\[
\begin{align*}
\gamma^-_\ell u_\ell - \gamma^+_\ell u_{p_\ell} &= \beta^D_\ell \\
\kappa_\ell \partial^-_\ell u_\ell - \kappa_{p_\ell} \partial^+_\ell u_{p_\ell} &= \beta^N_\ell
\end{align*}
\]
Here $p_\ell$ is the parent domain (next slide).

For short, $\left\lfloor \gamma_\ell \right\rfloor u = \beta_\ell$. 
On each $\Omega_\ell$, $\ell = 0, 1, \ldots, N$, the solution $u_\ell$ satisfies the wave equation:
$$c_\ell^{-2} \ddot{u}_\ell = \kappa_\ell \Delta u_\ell, \ u_\ell(0) = \dot{u}_\ell(0) = 0.$$ Denote $m_\ell = c_\ell \sqrt{\kappa_\ell}$.

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$$\begin{align*}
\gamma^-_\ell u_\ell - \gamma^+_\ell u_{p_\ell} &= \beta^D_\ell \\
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\end{align*}$$
Here $p_\ell$ is the parent domain (next slide).

For short, $[\gamma_\ell]u = \beta_\ell$. 

Parent: Enclosed by

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| b₀ | 2 | 1 | 4 | 3 |   |

Diagram:

- Ω₀
- Ω₁
- Ω₂
- Ω₃
- Ω₄
- Ω₅
- Γ₁
- Γ₂
- Γ₃
- Γ₄
- Γ₅
Let $\kappa_\ell = \kappa_{p \ell} = 1$. If $u_\ell$ is $H^1_\Delta(\mathbb{R}^d \setminus \Gamma_\ell)$-valued distributional solution to the wave equation $\dddot{u}_\ell = m_\ell^2 \Delta u_\ell$, then $u_\ell$ has the following representation:

$$u_\ell = S_{\ell}^{m\ell} * (\kappa_\ell \partial^- \ell u_\ell - \kappa_{p \ell} \partial^+ \ell u_{p \ell}) - D_{\ell}^{m\ell} * (\gamma^- \ell u_\ell - \gamma^+ \ell u_{p \ell})$$

$$= \begin{bmatrix} -D_{\ell}^{m\ell} & \kappa^{-1}_\ell S_{\ell}^{m\ell} \end{bmatrix} * \begin{bmatrix} \gamma^- \ell u_\ell - \gamma^+ \ell u_{p \ell} \\ \kappa_\ell \partial^- \ell u_\ell - \kappa_{p \ell} \partial^+ \ell u_{p \ell} \end{bmatrix}$$

$$:= G^{\ell} * \begin{bmatrix} \gamma \ell \end{bmatrix} u$$

Fundamental solution in $\mathbb{R}^2$ is $T_m(x, t) = \frac{1}{2\pi} \left( t^2 - \frac{|x|^2}{m^2} \right)^{-1/2} H(t - \frac{|x|}{m})$.

The single layer potential is

$$(S_{\ell}^{m\ell} * \phi)(x, t) := \int_0^t \int_{\Gamma_\ell} T_{m\ell}(x - y, \tau) \phi(y, t - \tau) \, d\Gamma(y) \, d\tau, x \notin \Gamma_\ell,$$

and the double layer potential is

$$(D_{\ell}^{m\ell} * \lambda)(x, t) := \int_0^t \int_{\Gamma_\ell} \partial_\nu(y) T_{m\ell}(x - y, \tau) \lambda(y, t - \tau) \, d\Gamma(y) \, d\tau, x \notin \Gamma_\ell.$$

\[ u_1 = \mathcal{G}_1^1 * \gamma_1^- u_1 \]
\[ u_1 = \mathcal{G}_1 \ast \gamma^+_{1} u_1 \]

\[ u_1 = -\mathcal{G}_2 \ast \gamma^+_{2} u_1 \]
MORE COMPLICATED

\[ u_1 = g_1^1 \ast \gamma_1^- u_1 \]

\[ u_1 = -g_2^1 \ast \gamma_2^+ u_1 \]

\[ u_1 = -\sum_{k=2}^{3} g_k^1 \ast \gamma_k^+ u_1 \]
\[ u_1 = \mathcal{G}_1^1 \ast \gamma_1^- u_1 \]

\[ u_1 = -\mathcal{G}_2^1 \ast \gamma_2^+ u_1 \]

\[ u_1 = -\sum_{k=2}^{3} \mathcal{G}_k^1 \ast \gamma_k^+ u_1 \]

\[ u_1 = \mathcal{G}_1^1 \ast \gamma_1^- u_1 - \sum_{k=2}^{3} \mathcal{G}_k^1 \ast \gamma_k^+ u_1 \]
1. Write down the potential representation for $\Omega_\ell$. Take interior trace and $\kappa_\ell$ times the interior normal derivative on both sides.

2. Do the same thing for the other side of $\Gamma_\ell$: $\Omega_{p_\ell}$.

3. Replace exterior data by transmission condition $\gamma^+_{l_\ell} u_{p_\ell} = \varphi_\ell - \beta_\ell$.
   Treat $\varphi_\ell = \gamma^-_{l_\ell} u_\ell, \ell = 1, 2, \ldots, N$ as unknowns.

4. Subtract (2) from (1).

\[
\begin{align*}
\gamma^+_{l_\ell} u_{p_\ell} &= \left( \frac{\delta_0 \otimes \text{Id}}{2} - A^p_{l_\ell} \right) \ast \gamma^+_{l_\ell} u_{p_\ell} + R^p_{l_\ell, p_\ell} \ast \gamma^-_{l_\ell} u_{p_\ell} - \sum_{k \in b_\ell, k \neq \ell} R^p_{l_\ell, k} \ast \gamma^+_k u_{p_\ell} \\
\end{align*}
\]
1. Write down the potential representation for $\Omega_\ell$. Take interior trace and $\kappa_\ell$ times the interior normal derivative on both sides.

2. Do the same thing for the other side of $\Gamma_\ell$: $\Omega_{p_\ell}$.

3. Replace exterior data by transmission condition $\gamma^+_{p_\ell} u_{p_\ell} = \varphi_\ell - \beta_\ell$. Treat $\varphi_\ell = \gamma^-_\ell u_\ell$, $\ell = 1, 2, \ldots, N$ as unknowns.

4. Subtract (2) from (1).

\[
\gamma^-_\ell u_\ell = \gamma^-_\ell g^\ell_\ell * \gamma^-_\ell u_\ell - \sum_{k \in d_\ell} \gamma^-_\ell g^\ell_k * \gamma^+_k u_\ell
\] (1)

\[
\gamma^+_{p_\ell} u_{p_\ell} = \left( \delta_0 \otimes \text{Id} \right) - A^p_\ell \right) * \gamma^+_{p_\ell} u_{p_\ell} + R^p_{\ell, p_\ell} * \gamma^-_{p_\ell} u_{p_\ell} - \sum_{k \in b_\ell, k \neq \ell} R^p_{\ell, k} * \gamma^+_k u_{p_\ell}
\] (2)
1. Write down the potential representation for $\Omega_\ell$. Take interior trace and $\kappa_\ell$ times the interior normal derivative on both sides.
2. Do the same thing for the other side of $\Gamma_\ell$: $\Omega_{p\ell}$.
3. Replace exterior data by transmission condition $\gamma_\ell^+ u_{p\ell} = \varphi_\ell - \beta_\ell$. Treat $\varphi_\ell = \gamma_\ell^- u_\ell, \ell = 1, 2, \ldots, N$ as unknowns.
4. Subtract (2) from (1).

\[
\gamma_\ell^- u_\ell = \left( \frac{\delta_0 \otimes \text{Id}}{2} + \mathcal{A}_\ell \right) * \gamma_\ell^- u_\ell - \sum_{k \in d_\ell} \mathcal{R}_{\ell,k} * \gamma_k^+ u_\ell \tag{1}
\]

\[
\gamma_\ell^+ u_{p\ell} = \left( \frac{\delta_0 \otimes \text{Id}}{2} - \mathcal{A}_{p\ell} \right) * \gamma_\ell^+ u_{p\ell} + \mathcal{R}_{\ell,p\ell} * \gamma_{p\ell}^- u_{p\ell} - \sum_{k \in b_\ell, k \neq \ell} \mathcal{R}_{\ell,k} * \gamma_k^+ u_{p\ell} \tag{2}
\]
1. Write down the potential representation for $\Omega_\ell$. Take interior trace and $\kappa_\ell$ times the interior normal derivative on both sides.

2. Do the same thing for the other side of $\Gamma_\ell$: $\Omega_{p\ell}$.

3. Replace exterior data by transmission condition $\gamma^+_{\ell} u_{p\ell} = \varphi_{\ell} - \beta_{\ell}$. Treat $\varphi_{\ell} = \gamma^+_{\ell} u_{\ell}$, $\ell = 1, 2, \ldots, N$ as unknowns.

4. Subtract (2) from (1).

$$\gamma^-_{\ell} u_{\ell} = \left( \frac{\delta_0 \otimes \text{Id}}{2} + A_{\ell} \right) \ast \gamma^-_{\ell} u_{\ell} - \sum_{k \in d_{\ell}} R_{\ell,k} \ast \gamma^+_{k} u_{\ell}$$ (1)

$$\gamma^+_{\ell} u_{p\ell} = \left( \frac{\delta_0 \otimes \text{Id}}{2} - A_{p\ell} \right) \ast \gamma^+_{\ell} u_{p\ell} + R_{\ell,p\ell} \ast \gamma^-_{p\ell} u_{p\ell} - \sum_{k \in b_{\ell}, k \neq \ell} R_{\ell,k} \ast \gamma^+_{k} u_{p\ell}$$ (2)
1. Write down the potential representation for $\Omega_\ell$. Take interior trace and $\kappa_\ell$ times the interior normal derivative on both sides.

2. Do the same thing for the other side of $\Gamma_\ell$: $\Omega_{p\ell}$.

3. Replace exterior data by transmission condition $\gamma_\ell^+ u_{p\ell} = \varphi_\ell - \beta_\ell$. Treat $\varphi_\ell = \gamma_\ell^- u_\ell$, $\ell = 1, 2, \ldots, N$ as unknowns.

4. Subtract (2) from (1).

\[
\gamma_\ell^- u_\ell = \left( \frac{\delta_0 \otimes \text{Id}}{2} + A_\ell \right) * \gamma_\ell^- u_\ell - \sum_{k \in d_\ell} R_{\ell,k} * \gamma_k^+ u_{p\ell} \tag{1}
\]

\[
\gamma_\ell^+ u_{p\ell} = \left( \frac{\delta_0 \otimes \text{Id}}{2} - A_{p\ell} \right) * \gamma_\ell^+ u_{p\ell} + R_{\ell,p\ell} * \gamma_{p\ell}^- u_{p\ell} - \sum_{k \in b_\ell, k \neq \ell} R_{\ell,k} * \gamma_k^+ u_{p\ell} \tag{2}
\]
Solve $\varphi_{\ell}, \ell = 1, 2, \ldots, N$ from the following TDBIEs.

- $\Omega_{\ell}$ encloses and is enclosed by other domains.
- $\Omega_{\ell}$ adjacent to exterior domain.
- $\Omega_{\ell}$ doesn’t enclose other domains.

\[
\left(\mathcal{A}_\ell^l + \mathcal{A}_\ell^{p_{\ell}}\right) \ast \varphi_{\ell} - \sum_{k\in d_{\ell}} \mathcal{R}_{\ell,k} \ast \varphi_k + \sum_{k\in b_{\ell}, k\neq \ell} \mathcal{R}_{\ell,k}^{p_{\ell}} \ast \varphi_k - \mathcal{R}_{\ell,p_{\ell}}^{p_{\ell}} \ast \varphi_{p_{\ell}}
\]

\[
= \left(\frac{\delta_0 \otimes \text{Id}}{2} + \mathcal{A}_\ell^{p_{\ell}}\right) \ast \beta_{\ell} + \sum_{k\in b_{\ell}, k\neq \ell} \mathcal{R}_{\ell,k}^{p_{\ell}} \ast \beta_k - \sum_{k\in d_{\ell}} \mathcal{R}_{\ell,k} \ast \beta_k
\]

and obtain solution by substituting back to the potential representation

\[
u_{\ell} = \left(\mathcal{G}_{\ell}^l \ast \varphi_{\ell} - \sum_{k\in d_{\ell}} \mathcal{G}_k^l \ast (\varphi_k - \beta_k)\right)
\]

\[
u_0 = - \sum_{k\in d_0} \mathcal{G}_k^0 \ast (\varphi_k - \beta_k)
\]
Solve $\varphi_{\ell}$, $\ell = 1, 2, \ldots, N$ from the following TDBIEs.

- $\Omega_{\ell}$ encloses and is enclosed by other domains.
- $\Omega_{\ell}$ adjacent to exterior domain.
- $\Omega_{\ell}$ doesn’t enclose other domains.

\[
(A_{\ell}^e + A_{\ell}^p) * \varphi_{\ell} - \sum_{k \in d_{\ell}} R_{\ell,k} * \varphi_{k} + \sum_{k \in b_{\ell}, k \neq \ell} R_{\ell,k}^p * \varphi_{k} = \left( \frac{\delta_0 \otimes \text{Id}}{2} + A_{\ell}^p \right) * \beta_{\ell} + \sum_{k \in b_{\ell}, k \neq \ell} R_{\ell,k}^p * \beta_{k} - \sum_{k \in d_{\ell}} R_{\ell,k} * \beta_{k}
\]

and obtain solution by substituting back to the potential representation

\[
u_{\ell} = \left( G_{\ell}^e * \varphi_{\ell} - \sum_{k \in d_{\ell}} G_{k}^e * (\varphi_{k} - \beta_{k}) \right)
\]

\[
u_0 = - \sum_{k \in d_0} G_{k}^0 * (\varphi_{k} - \beta_{k})
\]
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- $\Omega_\ell$ encloses and is enclosed by other domains.
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- $\Omega_\ell$ doesn’t enclose other domains.

\[
(\mathcal{A}_\ell^l + \mathcal{A}_\ell^p) * \varphi_\ell + \sum_{k \in b_\ell, k \neq \ell} \mathcal{R}_{\ell,k}^p * \varphi_k - \mathcal{R}_{\ell,p\ell}^p * \varphi_{p\ell} = \left(\frac{\delta_0 \otimes \text{Id}}{2} + \mathcal{A}_\ell^p\right) * \beta_\ell + \sum_{k \in b_\ell, k \neq \ell} \mathcal{R}_{\ell,k}^p * \beta_k - \sum_{k \in d_\ell} \mathcal{R}_{\ell,k}^l * \beta_k
\]

and obtain solution by substituting back to the potential representation

\[
u_\ell = \left(\mathcal{G}_\ell^l * \varphi_\ell\right)
\]

\[
u_0 = - \sum_{k \in d_0} \mathcal{G}_k^0 * (\varphi_k - \beta_k)
\]
Solve $\varphi_\ell, \ell = 1, 2, \ldots, N$ from the following TDBIEs.

- $\Omega_\ell$ encloses and is enclosed by other domains.
- $\Omega_\ell$ adjacent to exterior domain.
- $\Omega_\ell$ doesn’t enclose other domains.

\[
(\mathcal{A}_\ell^l + \mathcal{A}_{\ell}^{pe}) \ast \varphi_\ell + \sum_{k \in b_\ell, k \neq \ell} \mathcal{R}_{\ell,k}^{pe} \ast \varphi_k - \mathcal{R}_{\ell,p_\ell}^{pe} \ast \varphi_{p_\ell}
\]

\[
= \left( \frac{\delta_0 \otimes \text{Id}}{2} + \mathcal{A}_{\ell}^{pe} \right) \ast \beta_\ell + \sum_{k \in b_\ell, k \neq \ell} \mathcal{R}_{\ell,k}^{pe} \ast \beta_k - \sum_{k \in d_\ell} \mathcal{R}_{\ell,k}^l \ast \beta_k
\]

and obtain solution by substituting back to the potential representation

\[
u_\ell = (\mathcal{G}_\ell^l \ast \varphi_\ell)
\]

\[
u_0 = -\sum_{k \in d_0} \mathcal{G}_k^0 \ast (\varphi_k - \beta_k)
\]
The exact TDBIE system in matrix form for the example is

\[
\begin{bmatrix}
A_1^1 + A_1^0 & R_{1,2}^0 & -R_{1,3}^1 & -R_{1,4}^1 & -R_{2,5}^2 \\
R_{2,1}^0 & A_2^2 + A_2^0 & R_{3,4}^1 & A_4^4 + A_4^1 & A_5^5 + A_5^2 \\
-R_{3,1}^1 & A_3^3 + A_3^1 & R_{4,3}^1 & A_4^4 & A_5^5 + A_5^2 \\
-R_{4,1}^1 & -R_{5,2}^2 & A_5^5 + A_5^2 & A_5^5 + A_5^2 & A_5^5 + A_5^2
\end{bmatrix}
\begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4 \\
\varphi_5
\end{bmatrix}
\]

The system can also be expressed as

\[
\begin{bmatrix}
A_1^0 & R_{1,2}^0 & -R_{1,3}^1 & -R_{1,4}^1 & -R_{2,5}^2 \\
A_2^2 & A_2^0 & A_3^1 & R_{3,4}^1 & A_4^4 \\
-A_3^3 & R_{4,3}^1 & A_4^4 & A_5^5 & A_5^5 \\
-A_4^1 & -R_{5,2}^2 & -R_{5,2}^2 & A_5^5 & A_5^5 \\
-A_5^5 & A_5^5 & A_5^5 & A_5^5 & A_5^5
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5
\end{bmatrix}
\]

\[
\begin{bmatrix}
u_0^0 \\
u_0^1 \\
u_0^2 \\
u_0^3 \\
u_0^4 \\
u_0^5
\end{bmatrix} = \begin{bmatrix}
-G_1^0 & -G_2^0 \\
G_1^1 & -G_3^1 \\
-G_2^1 & -G_4^1 \\
G_3^3 & -G_5^2 \\
G_4^4 & G_5^5 \\
G_5^5 & G_5^5
\end{bmatrix}
\begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4 \\
\varphi_5
\end{bmatrix}
\]

\[
\begin{bmatrix}
G_1^0 & G_2^0 & G_3^1 & G_4^1 & G_5^2 \\
G_3^0 & G_4^0 & G_5^2 & G_5^2 & G_5^2
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5
\end{bmatrix}
\]
Discretization: Choose $Y_h(\Gamma_\ell) \subset H^{1/2}(\Gamma_\ell)$, $X_h(\Gamma_\ell) \subset H^{-1/2}(\Gamma_\ell)$ for $\ell = 1, 2, \ldots, N$. Find $\varphi^h = ((\varphi_1^D, \varphi_1^N), (\varphi_2^D, \varphi_2^N), \ldots, (\varphi_N^D, \varphi_N^N)) \in \prod_{\ell=1}^N (Y_h(\Gamma_\ell) \times X_h(\Gamma_\ell))$ such that

$$\langle A_{\text{lhs}} \ast \varphi^h, \psi \rangle = \langle A_{\text{rhs}} \ast \beta, \psi \rangle \quad \forall \psi \in \prod_{\ell=1}^N (Y_h(\Gamma_\ell) \times X_h(\Gamma_\ell))$$

where the angled bracket is defined as

$$\langle \varphi, \psi \rangle = \sum_{\ell=1}^N \left( \langle \psi^N_\ell, \varphi^D_\ell \rangle_{H^{-1/2}(\Gamma_\ell) \times H^{1/2}(\Gamma_\ell)} + \langle \varphi^N_\ell, \psi^D_\ell \rangle_{H^{-1/2}(\Gamma_\ell) \times H^{1/2}(\Gamma_\ell)} \right)$$

$$u^h_\ell = \begin{cases} G^\ell_\ell \ast \varphi^h_\ell, & d_\ell = \emptyset, \\
- \sum_{k \in d_\ell} G^\ell_k \ast (\varphi_k - \beta_k), & \ell = 0, \\
G^\ell_\ell \ast \varphi^h_\ell - \sum_{k \in d_\ell} G^\ell_k \ast (\varphi_k - \beta_k), & \text{otherwise}.
\end{cases}$$
Proposition (Stability)

Let \( \beta^D_\ell \in \mathcal{W}_+^2(H^{1/2}(\Gamma_\ell)) \), \( \beta^N_\ell \in \mathcal{W}_+^1(H^{-1/2}(\Gamma_\ell)) \), then the solution lies in the space \( u^h_\ell \in C_+(H^1(\mathbb{R}^d \setminus \Gamma)) \cap C_1^1(L^2(\mathbb{R}^d \setminus \Gamma)) \) and there exists a constant \( C \) independent of \( h \) and \( t \) such that

\[
\sum_{\ell=0}^{N} \|u^h_\ell(t)\|_{1,\mathbb{R}^d \setminus \Gamma} \leq C \sum_{\ell=1}^{N} (H_2(\beta^D_\ell, t|H^{1/2}(\Gamma_\ell)) + H_1(\beta^N_\ell, t|H^{-1/2}(\Gamma_\ell)))
\]

for all \( t \geq 0 \).

\[
\mathcal{W}_+^k(X) := \{ f \in C^{k-1}(\mathbb{R}; X), f \equiv 0 \text{ in } (-\infty, 0), f^{(k)} \in L^1(\mathbb{R}; X) \}
\]

\[
H_k(f, t|X) := \sum_{\ell=0}^{k} \int_{0}^{t} \|f^{(\ell)}(\tau)\|_X \, d\tau
\]

\[
C_+^k(X) := \{ f \in C^k(\mathbb{R}; X) : f \equiv 0 \text{ in } (-\infty, 0) \}
\]
**Proposition (Error Estimate)**

If \( \varphi_D^\ell \in \mathcal{W}^2_+(H^{1/2}(\Gamma_\ell)) \), \( \varphi_N^\ell \in \mathcal{W}^1_+(H^{-1/2}(\Gamma_\ell)) \), then the solution lies in the space \( \varepsilon^h_\ell := u_\ell - u_h^\ell \in C_+(H^1(\mathbb{R}^d \setminus \Gamma)) \cap C^1_+(L^2(\mathbb{R}^d \setminus \Gamma)) \) and there exists a constant \( C \) independent of \( h \) and \( t \) such that

\[
\sum_{\ell=0}^{N} \| \varepsilon^h_\ell(t) \|_{1,\mathbb{R}^d \setminus \Gamma} \leq C \sum_{\ell=1}^{N} \left( H_2(\varphi_D^\ell - \Pi_{Y_h}^\ell \varphi_D^\ell, t|H^{1/2}(\Gamma_\ell)) + H_1(\varphi_N^\ell - \Pi_{X_h}^\ell \varphi_N^\ell, t|H^{-1/2}(\Gamma_\ell)) \right)
\]

for all \( t \geq 0 \).

\( \Pi_{Y_h}^\ell \) and \( \Pi_{X_h}^\ell \) are the orthogonal projectors to the spaces \( Y_h(\Gamma_\ell) \) and \( X_h(\Gamma_\ell) \) respectively.
Sketch of the proof: \( u^h_\ell \) satisfies the system when \( \varphi^D_\ell = 0, \varphi^N_\ell = 0 \) while the error \( \epsilon^h_\ell \) satisfies the system when \( \beta^D_\ell = 0, \beta^N_\ell = 0 \).

For \( \ell = 0, 1, \ldots, N \),

\[
c_\ell^{-2} \ddot{w}_\ell = \kappa_\ell \Delta w_\ell, \text{ in } L^2(\mathbb{R}^d \setminus \Gamma),
\]

For \( \ell = 1, 2, \ldots, N \),

\[
\begin{align*}
\gamma^+_\ell w_\ell - \gamma^-_\ell w_{p_\ell} & \in X^\circ_h(\Gamma_\ell) \\
\left[ \gamma_\ell \right] w_{p_\ell} + \left[ \gamma_\ell \right] w_\ell & = \beta^D_\ell \text{ in } H^{1/2}(\Gamma_\ell) \\
\left[ \gamma_\ell \right] w_\ell + \varphi^D_\ell & \in Y_h(\Gamma_\ell) \\
\left[ \gamma_{\ell'} \right] w_{p_\ell} & = 0, \ell' \neq \ell, p_\ell
\end{align*}
\]

\[
\begin{align*}
\kappa_\ell \partial^+ \ell w_\ell - \kappa_{p_\ell} \partial^- \ell w_{p_\ell} & \in Y^\circ_h(\Gamma_\ell) \\
\kappa_\ell \left[ \partial_\ell \right] w_\ell + \kappa_{p_\ell} \left[ \partial_\ell \right] w_{p_\ell} & = \beta^N_\ell \text{ in } H^{-1/2}(\Gamma_\ell) \\
\left[ \partial_\ell \right] w_\ell + \varphi^N_\ell & \in X_h(\Gamma_\ell) \\
\left[ \partial_{\ell'} \right] w_{p_\ell} & = 0, \ell' \neq \ell, p_\ell
\end{align*}
\]

where \( X^\circ_h(\Gamma_\ell), Y^\circ_h(\Gamma_\ell) \) are annihilator sets of \( X_h(\Gamma_\ell) \) and \( Y_h(\Gamma_\ell) \) respectively.
SPATIAL SEMIDISCRETE ANALYSIS (SEMIGROUP)

The system can be written as a first order system.

\[
W_h := \begin{cases} 
(w_0, \ldots, w_N) \in H^1(\mathbb{R}^d \setminus \Gamma)^{N+1} \\
\end{cases} 
\begin{aligned}
\gamma_+^\ell u_\ell - \gamma_-^\ell u_{p_\ell} &\in X_0^o(\Gamma_\ell) \\
[\gamma^\ell] u_{p_\ell} &= -[\gamma^\ell] u_\ell \in Y_0(\Gamma_\ell) \\
[\gamma^\ell'] u_{p_\ell} &= 0 \text{ on } \Gamma_{\ell'}, \ell' \neq \ell, p_\ell \\
&\quad \text{for } \ell = 1, 2, \ldots, N.
\end{aligned}
\]

\[
V_h := \begin{cases} 
(v_0, \ldots, v_N) \in H(\text{div}, \mathbb{R}^d \setminus \Gamma)^{N+1} \\
\end{cases} 
\begin{aligned}
\gamma_{\nu\ell}^\nu v_\ell - \gamma_-^{\nu\ell} v_{p_\ell} &\in Y_0^o(\Gamma_\ell) \\
[\gamma_{\nu\ell}] v_{p_\ell} &= -[\gamma_{\nu\ell}] v_\ell \in X_0(\Gamma_\ell) \\
[\gamma_{\nu\ell'}] v_{p_\ell} &= 0 \text{ on } \Gamma_{\ell'}, \ell' \neq \ell, p_\ell \\
&\quad \text{for } \ell = 1, 2, \ldots, N.
\end{aligned}
\]

We prove the following operator \( A \) is maximal dissipative.

\[
W_h \times V_h \rightarrow H := \left( L^2(\mathbb{R}^d \setminus \Gamma) \times L^2(\mathbb{R}^d \setminus \Gamma) \right)^{N+1}
\]

\[
W = (w_\ell, v_\ell)_{\ell=0}^n \mapsto AW = (c_\ell^2 \nabla \cdot v_\ell, \kappa_\ell \nabla w_\ell)_{\ell=0}^n
\]

The original system can be written as \( \dot{W} = AW + F \). According to semigroup theory,

\[
\|W(t)\|_H \leq \int_0^t \|F(\tau)\|_H \, d\tau.
\]
We only illustrate the CQ method by the example of computing the single layer potential. CQ needs a priori knowledge of the convolution kernel in Laplace domain: \( T_{m\ell}(x, s) = \mathcal{L}\{T_m\} = \frac{i}{4} H_0^{(1)}\left(\frac{i}{m} s|x|\right) \).

\[
(S_{\ell}^m \ast \phi)(x, t) = \int_0^t \int_{\Gamma} T_m(x - y, \tau)\phi(y, t - \tau) \, d\Gamma(y) \, d\tau 
\approx \sum_{0 \leq j\kappa \leq t} \int_{\Gamma} \omega_j(x - y, \kappa)\phi(y, t - j\kappa) \, d\Gamma(y)
\]

where \( \omega_j \) are the Taylor coefficients of

\[
T_m(x, \frac{1}{\kappa} \delta(\zeta)) = \sum_{j=0}^{\infty} \omega_j(x, \kappa)\zeta^j,
\]

\( \delta(\zeta) = \frac{3}{2} - 2\zeta + \frac{1}{2}\zeta^2 \) is the generating polynomial of BDF-2, \( \kappa \) its time step.
TEMPORAL SEMIDISCRETE ERROR

Proposition (Error Estimate)

If $\beta_D^\ell \in \mathcal{W}^8_+(H^{1/2}(\Gamma_\ell))$, $\varphi^N_\ell \in \mathcal{W}^7_+(H^{-1/2}(\Gamma_\ell))$, then there exists a constant $C$ independent of $h$ and $t$ such that the temporal semidiscrete error $\epsilon_{h,\kappa}^\ell := u_\ell^h - u_{h,\kappa}^\ell$ satisfies

$$
\sum_{\ell=0}^N \|\epsilon_{h,\kappa}^\ell(t)\|_{1,\mathbb{R}^d \setminus \Gamma} \leq C \kappa^2 t^2 \sum_{\ell=1}^N (H_8(\beta_D^\ell, t | H^{1/2}(\Gamma_\ell))) + H_7(\beta^N_\ell, t | H^{-1/2}(\Gamma_\ell)))
$$

for all $t \geq 0$. 
Let $w_{\ell} := u_{\ell}^h - u_{\ell}^{h,\kappa}$. Its Laplace transform is $\tilde{w}_{\ell}(s) = \mathcal{L}\{w_{\ell}\}$. Its zeta transform is $\hat{w}_{\ell}(\zeta) = \mathcal{Z}\{w_{\ell}\}$.

In Laplace domain, $c_{\ell}^{-2}s^2\tilde{w}_{\ell} = \kappa_{\ell}\Delta\tilde{w}_{\ell} - c_{\ell}^{-2}(\ddot{u}_{\ell}^h - s^2\ddot{u}_{\ell}^h)$.

CQ essentially replaces $s$ with $\frac{1}{\kappa}\delta(\zeta)$ and looks for solution in zeta domain,

$$c_{\ell}^{-2}\frac{1}{\kappa^2}\delta(\zeta)^2\hat{w}_{\ell} = \kappa_{\ell}\Delta\hat{w}_{\ell} - c_{\ell}^{-2}(\ddot{u}_{\ell}^h - \frac{1}{\kappa^2}\delta(\zeta)^2\ddot{u}_{\ell}^h).$$

Written in time domain,

$$\left(\frac{9}{4} - m^2\kappa^2\Delta\right)w_{\ell}[n] - 6w_{\ell}[n-1] + \frac{11}{2}w_{\ell}[n-2] - 2w_{\ell}[n-3] + \frac{1}{4}w_{\ell}[n-4] = \kappa^2 g_{\ell}[n].$$

We prove Laplacian with transmission conditions is non-negative self-adjoint. Using a result from functional calculus e.g. Hasse 2006,

$$\|w_{\ell}[n]\| = \sum_{j=0}^{n} \|\eta_j(-\kappa^2\Delta)\|\kappa^2\|g_{\ell}[n-j]\| \leq \kappa^2 \sum_{j=0}^{n} \sup_{x \geq 0} |\eta_j(x)|\|g_{\ell}[n-j]\|.$$

$^1w_{\ell}[n] = w_{\ell}(n\tau)$. $g_{\ell}[n]$ is the local truncation error of BDF-2 applied to second order derivative.
NUMERICAL EXPERIMENT 1

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Let $u_0 = 0$, $u_i(x, t) = \sin(\pi z_i) H(z_i)$, $z_i = c_i(t - l_i) - (d_i \cdot x)$, $i = 1, 2, 3$. Solve the BDF2-CQ-BIE system with $\beta_i = \gamma_i u_i$. Run the experiment up to $T = 4$. Divide each $\Gamma_\ell$ into $M$ elements. $X_h$ piecewise constant function space. $Y_h$ continuous piecewise linear function space.

$$E_\ell^D = \| \varphi^{Dir}_\ell - \varphi^{Dir,h}_\ell \|_{L^2(\Gamma_\ell)},$$
$$E_\ell^N = \| \varphi^{Neu}_\ell - \varphi^{Neu,h}_\ell \|_{L^2(\Gamma_\ell)},$$
$$E_p = \| u_2(x_{obs}, T) - u_2^h(x_{obs}, T) \|,$$
$x_{obs} = (-0.5, 0.5)$.

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CONCLUSION AND FUTURE WORK

Done:

✔ General way of handling layered medium scattering
✔ Spatial semidiscrete analysis

To do:

✗ Scatterers adjacent to each other
✗ Temporal semidiscrete analysis

Thanks for your attention. Questions?