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Incidence modules for symplectic spaces in characteristic two

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ABSTRACT

We study the permutation action of a finite symplectic group of characteristic 2 on the set of subspaces of its standard module which are either totally isotropic or else complementary to totally isotropic subspaces with respect to the alternating form. A general formula is obtained for the 2-rank of the incidence matrix for the inclusion of one-dimensional subspaces in the distinguished subspaces of a fixed dimension.

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1. Introduction

Let V be a finite-dimensional vector space over \mathbf{F}_q , $q = 2^t$. We assume in addition that V has a nonsingular alternating form $b(\cdot, \cdot)$. Then the dimension of V must be an even number $2m$. We fix a basis $e_1, e_2, \dots, e_m, f_m, \dots, f_1$ of V , with corresponding coordinates $x_1, x_2, \dots, x_m, y_m, \dots, y_1$ so that $b(e_i, f_j) = \delta_{ij}$, $b(e_i, e_j) = 0$, and $b(f_i, f_j) = 0$, for all i and j .

For any subspace W of V , let $W^\perp = \{v \in V \mid b(v, w) = 0, \forall w \in W\}$ be its complement. A subspace W is called *totally isotropic* if $b(\mathbf{x}, \mathbf{x}') = 0$ for any two vectors, $\mathbf{x}, \mathbf{x}' \in W$. We will be interested in totally isotropic subspaces of V ; these necessarily have dimensions $r \leq m$. Also, we will consider the

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complements (with respect to the alternating form) of totally isotropic subspaces, which of course will have dimensions $r \geq m$.

For $1 \leq r \leq 2m - 1$, let $\mathcal{I}_r = \mathcal{I}_r(t)$ denote the set of totally isotropic subspaces, or complements of such, of dimension r . Let $B_{r,1} = B_{r,1}(t)$ denote the $(0, 1)$ -incidence matrix of the natural inclusion relation between \mathcal{I}_1 and \mathcal{I}_r . Its rows are indexed by \mathcal{I}_r and its columns by \mathcal{I}_1 , with the entry corresponding to $Y \in \mathcal{I}_1$ and $X \in \mathcal{I}_r$ equal to 1 if and only if Y is contained in X . We now state a theorem giving the 2-ranks of the matrices $B_{r,1}$, that is, their ranks when the entries are considered as elements of \mathbf{F}_2 . In order to simplify this and several other statements we will employ the notational convention that $\delta(P) = 1$ if a statement P holds, and $\delta(P) = 0$ otherwise.

Theorem 1.1. *Let $m \geq 2$ and $1 \leq r \leq 2m - 1$. Let A be the $(2m - r) \times (2m - r)$ -matrix whose (i, j) -entry is*

$$a_{i,j} = \binom{2m}{2j - i} - \binom{2m}{2j + i + 2r - 4m - 2 - 2(m - r)\delta(r \leq m)}.$$

Then

$$\text{rank}_2(B_{r,1}(t)) = 1 + \text{Trace}(A^t).$$

The significance of the entries $a_{i,j}$ is that they are the dimensions of certain representations of the symplectic group $\text{Sp}(V)$ which are restrictions of representations of the algebraic group $\text{Sp}(2m, \bar{\mathbf{F}}_q)$, where $\bar{\mathbf{F}}_q$ is an algebraic closure of \mathbf{F}_q .

Example 1.2. When $m = r = 2$, the matrix A is

$$\begin{pmatrix} 4 & 4 \\ 1 & 5 \end{pmatrix},$$

whose eigenvalues are $\frac{9 \pm \sqrt{17}}{2} = \left(\frac{1 \pm \sqrt{17}}{2}\right)^2$. Thus,

$$\text{rank}_2(B_{2,1}(t)) = 1 + \left(\frac{1 + \sqrt{17}}{2}\right)^{2t} + \left(\frac{1 - \sqrt{17}}{2}\right)^{2t}.$$

This formula was previously proved in [8] by using very detailed information about the extensions of simple modules for $\text{Sp}(4, q)$; such cohomological information is unavailable when V has higher dimension. Moreover, in [8] the numbers $\frac{1 \pm \sqrt{17}}{2}$ arise in a rather mysterious fashion from the combinatorics involved. Theorem 1.1, whose proof does not depend on [8], tells us that squares of these numbers are the eigenvalues of a matrix whose entries are the dimensions of modules for $\text{Sp}(4, q)$. This explanation is more natural.

We denote by $k[\mathcal{P}]$ the space of functions from $\mathcal{P} = \mathcal{I}_1$ to $k = \mathbf{F}_q$. The action of $\text{Sp}(V)$ on \mathcal{P} makes $k[\mathcal{P}]$ into a permutation module. Since $\text{Sp}(V)$ acts transitively on \mathcal{I}_r , the 2-rank of $B_{r,1}$ is equal to the dimension of the $k\text{Sp}(V)$ -submodule of $k[\mathcal{P}]$ generated by the characteristic function of one element of \mathcal{I}_r . Let us denote this submodule by \mathcal{C}_r . Since the sum of all the characteristic functions of totally isotropic r -subspaces contained in a given totally isotropic $(r + 1)$ -subspace is equal to the characteristic function of the latter, with a similar relation for complements of totally isotropic subspaces, we have

$$\mathcal{C}_{2m-1} \subset \cdots \subset \mathcal{C}_2 \subset \mathcal{C}_1 = k[\mathcal{P}]. \tag{1.1}$$

Functions in $k[\mathcal{P}]$ can be written as polynomials in the coordinates. For example, the characteristic function of the isotropic subspace W_0 defined by $x_1 = 0, x_2 = 0, \dots, x_m = 0$ is

$$\chi_{W_0} = \prod_{i=1}^m (1 - x_i^{q-1}). \tag{1.2}$$

Theorem 1.1 is proved by relating this polynomial description to the actions of the groups $GL(V)$ and $Sp(V)$ on $k[\mathcal{P}]$. We deduce it as a numerical corollary of Theorem 5.2 which gives a basis of \mathcal{C}_r of the right size.

When $q = p^f$ is odd, the p -rank of the corresponding incidence matrices has been computed in all dimensions in [4]. It was noted in [4] that for $m = 2$ and $r = 2$, there is a single formula in p and t which gives the rank for both even and odd values of q , even though there is no uniform proof. This circumstance is somewhat coincidental, however, since the examples in Section 7 show that for $m = r = 3$, the ranks are not given by the same function of p and t for the even characteristic and odd characteristic cases. The even-odd distinction reflects some fundamental differences in the $kSp(V)$ -submodule structures of $k[\mathcal{P}]$ between the even and odd cases, which are apparent in the action of $Sp(V)$ on the graded pieces of the truncated polynomial algebra. In a sense which will become clear, these modules are the basic building blocks of $k[\mathcal{P}]$. When p is odd, the action of $Sp(V)$ on each homogeneous piece of the truncated polynomial algebra is semisimple, but for $p = 2$, the submodule structure is much richer. In characteristic 2 the truncated polynomial algebra is an exterior algebra. The scalar extensions of the exterior powers to the algebraic closure \bar{k} are examples of *tilting modules* for the algebraic group $Sp(V \otimes \bar{k})$, as described in [5]; these have filtrations by *Weyl modules* and their duals. (See [7] for definitions.) These filtrations are very important in our work; indeed the entries of the matrix A in Theorem 1.1 are the dimensions of submodules which appear as terms in them. In order to keep prerequisites to a minimum, our treatment of filtrations of exterior powers in Section 4 is self-contained, requiring no general facts about Weyl modules, but sufficient to define the filtrations and compute the dimensions of the subquotients.

The exterior powers are related to $k[\mathcal{P}]$ through its $kGL(V)$ -module structure. The $kGL(V)$ -submodule lattice of $k[\mathcal{P}]$ has a simple description [1] in terms of a partially ordered set \mathcal{H} of certain t -tuples of natural numbers. Each t -tuple in $\mathcal{H} \cup \{(0, 0, \dots, 0)\}$ corresponds to a $GL(V)$ composition factor of $k[\mathcal{P}]$ and each composition factor is isomorphic to the t -fold twisted tensor product of exterior powers. These items are discussed in detail in Section 2.

In Section 3 we shall define a set of polynomials which maps bijectively to a basis of $k[\mathcal{P}]$. It is from this special basis, whose elements will be called *symplectic basis functions* (SBFs), that we will find subsets which give bases of the incidence submodules \mathcal{C}_r for all r . Each SBF is a t -fold product

$$f = f_0 f_1^2 \cdots f_{t-1}^{2^{t-1}}, \tag{1.3}$$

where each “digit” f_j is the function from V to k induced by a homogeneous square-free polynomial. Homogeneous square-free polynomials correspond bijectively to elements of the exterior power of the same degree, and the factorization (1.3) is compatible with the twisted tensor product factorization mentioned above. Thus each of these functions has a well-defined \mathcal{H} -type. We also introduce the notions of the *class* and *level* of each “digit” to further subdivide our set of special functions. Since the class and level are defined for each digit they apply also to elements of exterior powers. In Section 4, we show that the filtration of the exterior powers by levels consists of $kSp(V)$ -submodules, and compute the dimensions of the subquotients.

The proof of Theorem 5.2 is then given in a series of lemmas in Section 5. A modified version of the proof is given in Section 6 to handle the $q = 2$ case. Theorem 5.2 tells us that a basis of \mathcal{C}_r consists of all those SBFs of certain \mathcal{H} -types and whose digits satisfy certain conditions on their levels. Thus, using the dimension computations of Section 4, we obtain Theorem 1.1 as a corollary of Theorem 5.2.

2. $k[\mathcal{P}]$ as a permutation module for $GL(V)$

In this section, V is a $2m$ -dimensional vector space over $k = \mathbf{F}_q$, $q = 2^t$. For the time being, we do not equip V with an alternating form. So we simply consider $k[\mathcal{P}]$ as a $kGL(V)$ -module. Let $k[X_1, X_2, \dots, X_{2m}]$ denote the polynomial ring, in $2m$ indeterminates. Since every function on V is given by a polynomial in the $2m$ coordinates x_i , the map $X_i \mapsto x_i$ defines a surjective k -algebra homomorphism $k[X_1, X_2, \dots, X_{2m}] \rightarrow k[V]$, with kernel generated by the elements $X_i^q - X_i$. Furthermore, this map is simply the coordinate description of the following canonical map. The polynomial ring is isomorphic to the symmetric algebra $S(V^*)$ of the dual space of V ; so we have a natural evaluation map $S(V^*) \rightarrow k[V]$. This canonical description makes it clear that the map is equivariant with respect to the natural actions of $GL(V)$ on these spaces. A basis for $k[V]$ is obtained by taking monomials in $2m$ coordinates x_i such that the degree in each variable is at most $q - 1$. We will call these the *basis monomials* of $k[V]$. Noting that the functions on $V \setminus \{0\}$ which descend to \mathcal{P} are precisely those which are invariant under scalar multiplication by k^* , we obtain from the monomial basis of $k[V]$ a basis of $k[\mathcal{P}]$,

$$\left\{ \prod_{i=1}^{2m} x_i^{b_i} \mid 0 \leq b_i \leq q - 1, \sum_i b_i \equiv 0 \pmod{q - 1}, (b_1, \dots, b_{2m}) \neq (q - 1, \dots, q - 1) \right\}.$$

We refer the elements of the above set as *the basis monomials* of $k[\mathcal{P}]$.

2.1. Types and \mathcal{H} -types

We recall the definitions of two t -tuples associated with each basis monomial. Let

$$f = \prod_{i=1}^{2m} x_i^{b_i} = \prod_{j=0}^{t-1} \prod_{i=1}^{2m} (x_i^{a_{ij}})^{2^j}, \tag{2.1}$$

be a basis monomial of $k[\mathcal{P}]$, where $b_i = \sum_{j=0}^{t-1} a_{ij} 2^j$ and $0 \leq a_{ij} \leq 1$. Let $\lambda_j = \sum_{i=1}^{2m} a_{ij}$. The t -tuple $\lambda = (\lambda_0, \dots, \lambda_{t-1})$ is called the *type* of f . The set of all types of monomials is denoted by Λ .

In [1], there is another t -tuple associated with each basis monomial of $k[\mathcal{P}]$, which we will call its \mathcal{H} -type. This tuple will lie in the set $\mathcal{H} \cup \{(0, 0, \dots, 0)\}$, where

$$\mathcal{H} = \{ \mathbf{s} = (s_0, s_1, \dots, s_{t-1}) \mid \forall j, 1 \leq s_j \leq 2m - 1, 0 \leq 2s_{j+1} - s_j \leq 2m \}.$$

The \mathcal{H} -type \mathbf{s} of f is uniquely determined by the type via the equations

$$\lambda_j = 2s_{j+1} - s_j, \quad 0 \leq j \leq t - 1,$$

where the subscripts are taken modulo t . Moreover, these equations determine a bijection between the set Λ of types of basis monomials of $k[\mathcal{P}]$ and the set $\mathcal{H} \cup \{(0, 0, \dots, 0)\}$. We will consider \mathcal{H} as a partially ordered set under the natural order induced by the product order on t -tuples of natural numbers.

2.2. Composition factors

The types, or equivalently the \mathcal{H} -types parametrize the composition factors of $k[\mathcal{P}]$ in the following sense. The $kGL(V)$ -module $k[\mathcal{P}]$ is multiplicity-free. We can associate to each \mathcal{H} -type $\mathbf{s} \in \mathcal{H} \cup \{(0, 0, \dots, 0)\}$ a composition factor, which we shall denote by $L(\mathbf{s})$, such that these simple modules are all nonisomorphic, with $L((0, 0, \dots, 0)) \cong k$. The simple modules $L(\mathbf{s})$, $\mathbf{s} \in \mathcal{H}$, occur as

subquotients of $k[\mathcal{P}]$ in the following way. For each $\mathbf{s} \in \mathcal{H}$, we let $Y(\mathbf{s})$ be the subspace spanned by monomials of \mathcal{H} -types in $\mathcal{H}_{\mathbf{s}} = \{\mathbf{s}' \in \mathcal{H} \mid \mathbf{s}' \leq \mathbf{s}\}$, and $Y(\mathbf{s})$ has a unique simple quotient, isomorphic to $L(\mathbf{s})$.

The isomorphism type of the simple module $L(\mathbf{s})$ is most easily described in terms of the corresponding type $(\lambda_0, \dots, \lambda_{t-1}) \in \Lambda$. Let S^λ be the degree λ component in the truncated polynomial ring $k[X_1, X_2, \dots, X_{2m}]/(X_i^2; 1 \leq i \leq 2m)$. Here λ ranges from 0 to $2m$. The truncated polynomial ring is isomorphic to the exterior algebra $\bigwedge(V^*)$; so the dimension of S^λ is $\binom{2m}{\lambda}$. The simple module $L(\mathbf{s})$ is isomorphic to the twisted tensor product

$$S^{\lambda_0} \otimes (S^{\lambda_1})^{(2)} \otimes \dots \otimes (S^{\lambda_{t-1}})^{(2^{t-1})}. \tag{2.2}$$

2.3. Submodule structure

The reason for considering \mathcal{H} -types is that they allow a simple description of the submodule structure of the $kGL(V)$ -module $k[\mathcal{P}]$. The space $k[\mathcal{P}]$ has a $kGL(V)$ -decomposition

$$k[\mathcal{P}] = k \oplus Y_{\mathcal{P}}, \tag{2.3}$$

where $Y_{\mathcal{P}}$ is the kernel of the map $k[\mathcal{P}] \rightarrow k, f \mapsto |\mathcal{P}|^{-1} \sum_{Q \in \mathcal{P}} f(Q)$. The $kGL(V)$ -module $Y_{\mathcal{P}}$ is an indecomposable module whose composition factors are parametrized by \mathcal{H} . Then [1, Theorem A] states that given any $kGL(V)$ -submodule of $Y_{\mathcal{P}}$, the set of \mathcal{H} -types of its composition factors is an ideal in the partially ordered set \mathcal{H} and that this correspondence is an order isomorphism from the submodule lattice of $Y_{\mathcal{P}}$ to the lattice of ideals in \mathcal{H} .

The submodules of $Y_{\mathcal{P}}$ can also be described in terms of basis monomials [1, Theorem B]. Any submodule of $Y_{\mathcal{P}}$ has a basis consisting of the basis monomials which it contains. Moreover, the \mathcal{H} -types of these basis monomials are precisely the \mathcal{H} -types of the composition factors of the submodule. Furthermore, in any composition series, the images of the monomials of a fixed \mathcal{H} -type form a basis of the composition factor of that \mathcal{H} -type.

3. Symplectic basis functions

We now define a set of square-free homogeneous polynomials of degree λ which we will use to construct symplectic basis functions (SBFs). These square-free homogeneous polynomials are in the indeterminates $X_1, X_2, \dots, X_m, Y_m, \dots, Y_1$, and will have the form given in the following definition.

Definition 3.1. Suppose the set of integers $\{1, \dots, m\}$ is partitioned into five disjoint sets, R, R', S, T , and U . We require $|R| = |R'| = \rho$, and take the cardinalities of the other sets to be σ, τ , and ν , respectively, such that $2\rho + 2\sigma + \tau = \lambda$. We further impose a pairing between $R = \{r_1, \dots, r_\rho\}$ and $R' = \{r'_1, \dots, r'_\rho\}$, so that r_i is paired with $r'_i, 1 \leq i \leq \rho$. We write $W_i = X_i Y_i, 1 \leq i \leq m$, and write Z_i to denote either X_i or else Y_i . Then we shall say that a square-free homogeneous polynomial F belongs to the class $(\rho, \sigma, \tau, \nu)$ if it can be written as

$$F = \prod_{i=1}^{\rho} (W_{r_i} + W_{r'_i}) \prod_{i \in S} W_i \prod_{i \in T} Z_i.$$

We denote by P_ℓ^λ the set of polynomials of those classes $(\rho, \sigma, \tau, \nu)$ with $2\rho + 2\sigma + \tau = \lambda$ and $\sigma \leq \ell$. Then σ will be called the *level* of F , and ℓ will be called the *level* of P_ℓ^λ .

We now construct a basis for $k[\mathcal{P}]$. We begin with a basis for the space of square-free homogeneous polynomials of degree λ in $X_1, Y_1, \dots, X_m, Y_m$. Note that the linear span of $P_{\ell-1}^\lambda$ is a subspace of the span of P_ℓ^λ .

Definition 3.2. Let λ be a fixed integer, $0 \leq \lambda \leq 2m$. Choose B_0^λ to be a linearly independent subset of P_0^λ of maximum size. For $1 \leq \ell \leq \lambda/2$, suppose that $B_{\ell-1}^\lambda$ has been chosen. Then we choose $P_\ell'^\lambda \subset P_\ell^\lambda$ such that $B_\ell^\lambda = B_{\ell-1}^\lambda \cup P_\ell'^\lambda$ forms a basis for the span of P_ℓ^λ . Finally, let

$$\mathcal{B} = \bigcup_{0 \leq \lambda \leq 2m} B_{\lfloor \lambda/2 \rfloor}^\lambda.$$

Note that $B_{\lfloor \lambda/2 \rfloor}^\lambda$ is a basis for the space of all square-free homogeneous polynomials of degree λ . Thus \mathcal{B} forms a basis of the space of square-free polynomials in $X_1, Y_1, \dots, X_m, Y_m$. Polynomials of the form $\mathcal{F} = F_0 F_1^2 \dots F_{t-1}^{2^{t-1}}$, where $F_j \in \mathcal{B}$, $0 \leq j \leq t-1$, form a basis for polynomials in these indeterminates, where the exponent of each indeterminate is at most $q-1$.

The evaluation map $\phi : k[X_1, \dots, X_m, Y_m, \dots, Y_1] \rightarrow k[V]$, sending X_i to x_i and Y_i to y_i , $1 \leq i \leq m$, restricts to a bijection from the space of square-free homogeneous polynomials in $X_1, \dots, X_m, Y_m, \dots, Y_1$ to those in $x_1, \dots, x_m, y_m, \dots, y_1$. If F is a square-free polynomial belonging to class $(\rho, \sigma, \tau, \nu)$, then we shall refer to the function $\phi(F)$ as belonging to that class also.

Definition 3.3. Let $(s_0, \dots, s_{t-1}) \in \mathcal{H}$ and let $f = f_0 f_1^2 \dots f_{t-1}^{2^{t-1}} \in k[\mathcal{P}]$ such that the degree of f_j is $\lambda_j = 2s_{j+1} - s_j$ (subscripts modulo t), and $f_j \in \phi(B_{\lfloor \lambda_j/2 \rfloor}^{\lambda_j})$, for each j , $0 \leq j \leq t-1$. Then we shall say that f is a *symplectic basis function* (SBF), of the specified \mathcal{H} -type.

3.1. Linear substitutions in factorizable functions

We will frequently be dealing with functions, such as basis monomials or SBFs, which can be factorized as a product

$$f = f_0 f_1^2 \dots f_{t-1}^{2^{t-1}} \in k[\mathcal{P}] \tag{3.1}$$

of functions $f_j \in k[V]$ which are images under ϕ of homogeneous polynomials. We wish to think of f_j as the *jth digit of f*. However the representation (3.1) is not unique in general; $f = x_1^2$ can be factorized by taking $f_0 = x_1^2$ or by taking $f_1 = x_1$. Thus when we refer to the digit of such a function, it must be with a particular factorization in mind. If we impose the additional requirement that the f_j in (3.1) be square-free, then the digits are determined up to scalars and such a function f has a well-defined \mathcal{H} -type, as in the case of basis monomials and SBFs. The more general notion of digit is needed to discuss transformations of functions. Let $b = b_0 b_1^2 \dots b_{t-1}^{2^{t-1}}$ be a function which has a well-defined \mathcal{H} -type. An element $g \in GL(V)$ acts on $k[V]$ as a linear substitution, and we have

$$gb = (gb_0)(gb_1)^2 \dots (gb_{t-1})^{2^{t-1}}. \tag{3.2}$$

The digits gb_j in (3.2) are the homogeneous polynomials obtained from the b_j through linear substitution, and may contain monomials with square factors. Like any function, gb can be rewritten as a linear combination of factorizable functions with square-free digits, such as basis monomials. We want to take a closer look at this rewriting process, keeping track of the relationship between the \mathcal{H} -type of b and the \mathcal{H} -types of the terms in the possible rewritten forms of gb . The rewriting as a combination of basis monomials is done as follows. The product (3.2) is first distributed into a sum of products of monomials, each monomial having the factorized form (3.1), possibly with square factors in the digits. Then if the first digit has a factor x_i^2 , the factor x_i^2 is replaced by 1 and the second digit is multiplied by x_i . We will call this a *carry* from the first digit to the second. Note that this process causes the degree of the first digit to decrease by 2 while that of the second increases by one. We perform all possible carries from the first digit to the second, leaving the first digit square-free. We then repeat the process, performing carries from the second digit to the third, and so on. Carries from

the last digit go to the first digit, and after that a second round of carries from the first to second digits, second to third, etc., may occur. This process terminates because each time we have a carry from the last digit to the first, the total degree decreases by $2^t - 1$. To refine this idea we define, for $0 \leq e \leq t - 1$, the e th twisted degree of (3.1) by

$$\text{deg}_e(f) = \sum_{j=0}^{t-1} 2^{[j-e]} \text{deg}(f_j),$$

where $[j - e]$ means the remainder modulo t . The effect of a carry from the $(j - 1)$ th digit to the j th digit is to lower the j th twisted degree by $2^t - 1$, leaving other twisted degrees unchanged. For those functions f which have a well-defined \mathcal{H} -type the relation between the \mathcal{H} -type (s_0, \dots, s_{t-1}) and the twisted degrees is simple: $(2^t - 1)s_e = \text{deg}_e(f)$. We conclude that in the process of rewriting gb , whenever a carry is performed on a monomial the basis monomial which results will have a strictly lower \mathcal{H} -type than our original function b .

4. Submodules of the exterior powers

The ring $k[X_1, \dots, X_m, Y_1, \dots, Y_m]/(X_i^2, Y_i^2)_{i=1}^m$ can be viewed as the exterior algebra $\bigwedge(V^*)$, with X_i and Y_i corresponding to the basis elements x_i and y_i of V^* , respectively. Therefore, we can think of elements of $\bigwedge(V^*)$ as square-free polynomials in the X_i and Y_i and identify $\bigwedge^\lambda(V^*)$ with S^λ , for $0 \leq \lambda \leq 2m$.

For each λ , we define S_ℓ^λ to be the k -span of P_ℓ^λ , which is also the k -span of B_ℓ^λ . Then the levels of functions define a filtration of subspaces

$$0 \subset S_0^\lambda \subseteq \dots \subseteq S_{\lfloor \frac{\lambda}{2} \rfloor}^\lambda = S^\lambda.$$

Note that if $\lambda \geq m$ then $S_\ell^\lambda = 0$ for $\ell < \lambda - m$.

Lemma 4.1. Each S_ℓ^λ is a $k\text{Sp}(V)$ -submodule of S^λ .

We delay the proof of Lemma 4.1 until after the similar proof of Lemma 5.3.

Lemma 4.2. Assume $\lambda \leq m$. Then S_0^λ has dimension $\binom{2m}{\lambda} - \binom{2m}{\lambda-2}$. A basis of S_0^λ consists of all elements of P_0^λ of the form $\prod_{i=1}^\rho (W_{r_i} + W_{r'_i}) \prod_{t \in T} Z_t$, where $R = \{r_1, r_2, \dots, r_\rho\}$, $R' = \{r'_1, r'_2, \dots, r'_\rho\}$ and T are disjoint subsets of $\{1, \dots, m\}$ such that $2|R| + |T| = \lambda$ and the following conditions hold:

- (1) $r_1 < r_2 < \dots < r_\rho$ and $r'_1 < r'_2 < \dots < r'_\rho$;
- (2) r_i is the smallest element of $\{1, \dots, m\} \setminus T$ which is not in the set $\{r_j, r'_j \mid j < i\}$.

Proof. A proof can be found in [2, Theorem 1.1]; an earlier proof appears in the examples of [6, p. 39]. (We only need the statements in the characteristic 2 case, but these references treat all characteristics.) \square

Remark 4.3. The module S_0^λ or, more precisely, their scalar extensions to an algebraic closure \bar{k} are examples of Weyl modules for the algebraic group $\text{Sp}(\bar{k} \otimes_k V)$ and it is in this context that they were studied in [6] and [2]. We refer the interested reader to [7] for the definitions and properties of this important class of modules.

Lemma 4.4. Assume $\lambda \leq m$.

(1) We have an isomorphism

$$S_\ell^\lambda / S_{\ell-1}^\lambda \cong S_0^{\lambda-2\ell}. \tag{4.1}$$

(2) The dimension of S_ℓ^λ is $\binom{2m}{\lambda} - \binom{2m}{\lambda-2\ell-2}$.

Proof. We shall define a linear map

$$\alpha : S_0^{\lambda-2\ell} \rightarrow S_\ell^\lambda.$$

Let f be a basis element of $S_0^{\lambda-2\ell}$ given by Lemma 4.2, with index sets (R, R', T) . Set $M = \{1, \dots, m\}$. Then the number of indices involved in f is $2\rho + \tau = \lambda - 2\ell$, and the number of unused indices is $(m - \lambda) + 2\ell \geq 2\ell$. We choose an arbitrary ℓ -subset S of these unused indices and define

$$\alpha(f) = f \prod_{s \in S} W_s.$$

Composing with the natural projection gives a map

$$\bar{\alpha} : S_0^{\lambda-2\ell} \rightarrow S_\ell^\lambda / S_{\ell-1}^\lambda.$$

We claim that the map $\bar{\alpha}$ is surjective.

Now S_ℓ^λ is spanned by polynomials of the form

$$\prod_{r_i \in R, r'_i \in R'} (W_{r_i} + W_{r'_i}) \prod_{s_i \in S} W_{s_i} \prod_{t_i \in T} Z_{t_i}, \tag{4.2}$$

where R, R', S and T are disjoint subsets of $M = \{1, \dots, m\}$, with $|R| = |R'|$, and $2|R| + 2|S| + |T| = \lambda$, and $|S| \leq \ell$. The classes of these elements such that $\sigma = \ell$ span the quotient space $S_\ell^\lambda / S_{\ell-1}^\lambda$. We consider what restrictions may be placed on the form of such a polynomial and still have a spanning set. Of course we may assume that the r_i form an increasing sequence and that $r_i < r'_i$ for all i . Furthermore, the equation

$$(W_1 + W_4)(W_2 + W_3) = (W_1 + W_3)(W_2 + W_4) + (W_1 + W_2)(W_3 + W_4) \tag{4.3}$$

implies that we can assume that the r'_i also form an increasing sequence. Next observe that

$$(W_2 + W_3) = (W_1 + W_2) + (W_1 + W_3) \tag{4.4}$$

and

$$(W_2 + W_3)W_1 = (W_1 + W_3)W_2 + (W_1 + W_2)W_3. \tag{4.5}$$

Eqs. (4.4) and (4.5) respectively allow us to assume in addition that no element of U or S precedes any element of R , or in other words that for each i , r_i is the smallest index in $(\{1, \dots, m\} \setminus T) \setminus \{r_j, r'_j \mid j < i\}$. Thus the sets R and R' can be assumed to satisfy the same conditions as the corresponding sets for the basis elements of $S_0^{\lambda-2\ell}$ stated in Lemma 4.2. We have proved that there is a spanning set of S_ℓ^λ such that for each element for which $\sigma = \ell$, the subsets R and R' and T are exactly the same as those of some $\alpha(f)$, where f is a basis element of $S_0^{\lambda-2\ell}$. Finally, we note that two elements of P_ℓ^λ

with the same sets R, R' and T are equal modulo $S_{\ell-1}^\lambda$. To see this, suppose the index sets of the two elements are (R, R', S, T, U) and (R, R', S^*, T, U^*) and assume that the symmetric difference of S and S^* is $\{s, s^*\}$. Then the sum of the two polynomials has index sets $(R \cup \{s\}, R' \cup \{s^*\}, S \cap S^*, T, (U \cup U^*) \setminus \{s, s^*\})$; so it belongs to $S_{\ell-1}^\lambda$. Therefore the map $\bar{\alpha}$ is surjective (and, incidentally, independent of the choice of ℓ -subset S in the definition of α).

Thus, since Lemma 4.2 gives the dimension of $S_0^{\lambda-2\ell}$, we have

$$\begin{aligned} \binom{2m}{\lambda} &= \dim S^\lambda = \sum_{\ell=0}^{\lfloor \frac{\lambda}{2} \rfloor} \dim(S_\ell^\lambda / S_{\ell-1}^\lambda) \\ &\leq \sum_{\ell=0}^{\lfloor \frac{\lambda}{2} \rfloor} \dim S_0^{\lambda-2\ell} \\ &= \sum_{\ell=0}^{\lfloor \frac{\lambda}{2} \rfloor} \binom{2m}{\lambda-2\ell} - \binom{2m}{\lambda-2\ell-2} \\ &= \binom{2m}{\lambda}, \end{aligned} \tag{4.6}$$

so equality holds throughout. Both parts of the lemma now follow. \square

The case where $\lambda \geq m$ will be treated by using various dualities, which we now proceed to discuss. We do not assume $\lambda \geq m$ yet. We start with the $kGL(V)$ -isomorphism $\bigwedge^\lambda(V^*) \cong (\bigwedge^\lambda(V))^*$. Let $\lambda^* = 2m - \lambda$. Then there is a natural pairing

$$\bigwedge^\lambda(V) \times \bigwedge^{\lambda^*}(V) \rightarrow \bigwedge^{2m}(V) = k$$

given by exterior multiplication, which defines a $kGL(V)$ -isomorphism $(\bigwedge^\lambda(V))^* \cong \bigwedge^{\lambda^*}(V)$.

Finally, the $kSp(V)$ -isomorphism $V \cong V^*$ given by $v \mapsto b(v, \cdot)$ induces a $kSp(V)$ -isomorphism $\bigwedge^{\lambda^*}(V) \cong \bigwedge^{\lambda^*}(V^*)$. Combining these isomorphisms yields a $kSp(V)$ -module isomorphism

$$S^\lambda \rightarrow S^{\lambda^*}, \quad X_I Y_J \mapsto X_{M \setminus J} Y_{M \setminus I},$$

for each λ . Putting these isomorphisms together for all λ yields a $kSp(V)$ -automorphism δ of $\bigwedge(V^*)$ of order 2. A straightforward calculation shows that a basis polynomial of degree λ^* and class $(\rho, \sigma, \tau, \nu)$ is mapped under δ to a basis polynomial of degree λ and class $(\rho, \nu, \tau, \sigma)$. (Actually, the map δ is very simple; a basis polynomial of S^{λ^*} with index sets (R, R', S, T, U) is mapped to basis polynomial of S^λ with index sets (R, R', U, T, S) ; δ just swaps U and S .) Now assume $\lambda \geq m$. Since $m = 2\rho + \sigma + \tau + \nu$ and $\lambda^* = 2\rho + 2\sigma + \tau$, we have

$$\nu = m - \lambda^* + \sigma = (\lambda - m) + \sigma,$$

which shows that

$$\delta(S_{\ell^*}^{\lambda^*}) = S_{(\lambda-m)+\ell^*}^\lambda,$$

for $0 \leq \ell^* \leq \lfloor \frac{\lambda^*}{2} \rfloor$.

We have already computed the left-hand side, so if we set $\ell = (\lambda - m) + \ell^*$ then

$$\dim S_\ell^\lambda = \binom{2m}{\lambda^*} - \binom{2m}{\lambda^* - 2\ell^* - 2} = \binom{2m}{\lambda} - \binom{2m}{\lambda - 2\ell - 2}.$$

We have proved:

Theorem 4.5. Let $0 \leq \lambda \leq 2m$ and $\max\{0, \lambda - m\} \leq \ell \leq \lfloor \frac{\lambda}{2} \rfloor$. Then S_ℓ^λ has dimension

$$\binom{2m}{\lambda} - \binom{2m}{\lambda - 2\ell - 2}.$$

Remark 4.6. The isomorphism in Lemma 4.4 is in fact one of $k\text{Sp}(V)$ -modules, and after extending the field to \bar{k} it is an isomorphism of rational $\text{Sp}(V_{\bar{k}})$ -modules. In fact, it can be shown that the filtration given by the submodules $\bar{k} \otimes_k S_\ell^\lambda$ is the filtration of $\bigwedge^\lambda(V_{\bar{k}}^*)$ by Weyl modules, dual to the good filtration described in [5, Appendix A, p. 71] (for all characteristics).

5. Bases for incidence modules

In this section we assume that $q > 2$. In the next section we show that Theorem 5.2 below also holds for $q = 2$, with a slightly different proof.

Definition 5.1. Let $1 \leq r \leq 2m - 1$ and let $f = f_0 f_1^2 \cdots f_{t-1}^{2^{t-1}} \in k[\mathcal{P}]$ with f_j homogeneous and square-free for all j . If the \mathcal{H} -type of f is $(s_0, s_1, \dots, s_{t-1})$, let

$$\ell_j = (m - r)\delta(r \leq m) + (2m - r - s_j). \tag{5.1}$$

The function f is called r -admissible if f is a constant function, or if

- (a) for each j , $0 \leq j \leq t - 1$, $f_j \in \phi(P_{\ell_j}^{\lambda_j})$, and
- (b) the \mathcal{H} -type of f is less than or equal to $(2m - r, 2m - r, \dots, 2m - r)$.

In particular, if f is constant or if $f_j \in \phi(B_{\ell_j}^{\lambda_j})$ for all j , $0 \leq j \leq t - 1$, we call f an r -admissible SBF.

Note that by construction, r -admissible functions are in the linear span of r -admissible SBFs.

Theorem 5.2. Assume $q > 2$. The r -admissible symplectic basis functions form a basis for the $k\text{Sp}(V)$ -submodule \mathcal{C}_r of $k[\mathcal{P}]$.

The rest of this section is devoted to proving this theorem.

Let \mathcal{M}_r denote the linear span of the r -admissible SBFs, for $1 \leq r \leq 2m - 1$.

Let $\bar{P}_\ell^\lambda = \phi(P_\ell^\lambda)$.

We consider the following group elements, which together generate $\text{Sp}(V)$. Let $\alpha \in k^\times$, and let π be a permutation of $\{1, \dots, m\}$.

$$x_1 \mapsto \alpha x_1, \quad y_1 \mapsto \alpha^{-1} y_1, \tag{5.2}$$

$$x_k \mapsto x_{\pi(k)}, \quad y_k \mapsto y_{\pi(k)}, \quad 1 \leq k \leq m, \tag{5.3}$$

$$x_1 \mapsto y_1, \quad y_1 \mapsto x_1, \tag{5.4}$$

$$g_1(\alpha) : x_1 \mapsto x_1 + \alpha y_1, \quad y_1 \mapsto y_1, \tag{5.5}$$

$$g_2(\alpha) : x_1 \mapsto x_1 + \alpha x_2, \quad y_1 \mapsto y_1, \quad x_2 \mapsto x_2, \quad y_2 \mapsto \alpha y_1 + y_2. \tag{5.6}$$

Lemma 5.3. \mathcal{M}_r is a $k\text{Sp}(V)$ -submodule of $k[\mathcal{P}]$.

Proof. Since $\text{Sp}(V)$ acts as the identity on a constant function, we only need to consider the action of $\text{Sp}(V)$ on an SBF of \mathcal{H} -type (s_0, \dots, s_{t-1}) . For each fixed λ , $0 \leq \lambda \leq 2m$, and ℓ , $0 \leq \ell \leq \lambda/2$. Since $m = 2\rho + \sigma + \tau + \nu$, we note that if $\ell < \lambda - m$, then \bar{P}_ℓ^λ is empty, while if $\ell > \lambda/2$, then $\bar{P}_\ell^\lambda = \bar{P}_{\lfloor \lambda/2 \rfloor}^\lambda$.

We examine one by one the actions of generators of $\text{Sp}(V)$ on elements of our spanning set of functions. First, since our set is defined symmetrically with respect to the subscripts, it is clear that elements (5.3) of $\text{Sp}(V)$ which permute the subscripts of the standard basis of V^* also permute the members of \bar{P}_ℓ^λ . It is also clear that interchanging x_i and y_i , $1 \leq i \leq m$, (5.4), also interchanges some members of \bar{P}_ℓ^λ . The diagonal group elements (5.2) multiply each z_i by some constant and leave each w_i unchanged. Thus they only multiply members of \bar{P}_ℓ^λ by finite field elements. It remains to consider the actions of the group elements (5.5) and (5.6).

Let $f = f_0 f_1^2 \cdots f_{t-1}^{2^{t-1}}$ be an r -admissible SBF, and let $f_j \in \bar{P}_{\ell_j}^{\lambda_j}$. First we will examine the action of $g_1(\alpha)$ or $g_2(\alpha)$ on the j th digit f_j assuming there is no carry from the $(j - 1)$ th digit to the j th digit. Then we consider the effect of such a carry.

In calculating $g_1(\alpha)f_j$, it is sufficient to write just the part of f_j involving x_1 or y_1 , except when $1 \in R$. Whenever we get a square term, y_1^2 or x_2^2 , we replace it by 1, reflecting the carry to f_{j+1} . Thus we compute $g_1(\alpha)f_j$ modulo $(y_1^2 - 1)$. Recall $w_1 = x_1 y_1$. We have

$$g_1(\alpha)w_1 = w_1 + \alpha y_1^2 \equiv w_1 + \alpha, \tag{5.7}$$

$$\begin{aligned} g_1(\alpha)(w_1 + w_2) &= (w_1 + w_2) + \alpha y_1^2 \equiv (w_1 + w_2) + \alpha, \\ g_1(\alpha)x_1 &= x_1 + \alpha y_1, \\ g_1(\alpha)y_1 &= y_1, \\ g_1(\alpha)(1) &= 1. \end{aligned} \tag{5.8}$$

In each case we get a linear combination of elements of $\bar{P}_{\ell_j}^{\lambda_j}$; that is, σ_j did decrease in one term of (5.7), but it never increased. Similarly, we present the following calculations, exhaustively showing the action of $g_2(\alpha)$ on the portion of f_j involving x_1 , y_1 , x_2 , and y_2 . The calculation is modulo $((y_1^2 - 1), (x_2^2 - 1))$:

$$\begin{aligned} g_2(\alpha)w_1 w_2 &= (x_1 + \alpha x_2)y_1 x_2 (y_2 + \alpha y_1) \\ &= w_1 w_2 + \alpha y_1 y_2 x_2^2 + \alpha x_1 x_2 y_1^2 + \alpha^2 y_1^2 x_2^2 \\ &\equiv w_1 w_2 + \alpha y_1 y_2 + \alpha x_1 x_2 + \alpha^2, \end{aligned} \tag{5.9}$$

$$\begin{aligned} g_2(\alpha)(w_1 + w_2) &= (w_1 + w_2) + \alpha x_2 y_1 + \alpha x_2 y_1 \\ &= (w_1 + w_2), \end{aligned} \tag{5.10}$$

$$\begin{aligned} g_2(\alpha)(w_1 + w_3)(w_2 + w_4) &= (w_1 + w_3)(w_2 + w_4) + \alpha x_2 y_1 w_4 \\ &\quad + \alpha w_3 x_2 y_1 + \alpha y_1 y_2 x_2^2 + \alpha x_1 x_2 y_1^2 + \alpha^2 y_1^2 x_2^2 \\ &\equiv (w_1 + w_3)(w_2 + w_4) + \alpha y_1 y_2 \\ &\quad + \alpha x_1 x_2 + \alpha x_2 y_1 (w_3 + w_4) + \alpha^2, \end{aligned} \tag{5.11}$$

$$g_2(\alpha)(w_1 + w_3)w_2 \equiv (w_1 + w_3)w_2 + \alpha w_3 x_2 y_1 + \alpha y_1 y_2 + \alpha x_1 x_2 + \alpha^2, \tag{5.12}$$

$$g_2(\alpha)(w_1 + w_3) = (w_1 + w_3) + \alpha x_2 y_1, \tag{5.13}$$

$$g_2(\alpha)(w_1 + w_3)x_2 \equiv (w_1 + w_3)x_2 + \alpha y_1, \tag{5.14}$$

$$\begin{aligned} g_2(\alpha)(w_1 + w_3)y_2 &\equiv (w_1 + w_3)y_2 + \alpha y_1 x_2 y_2 + \alpha x_1 + \alpha y_1 w_3 + \alpha^2 x_2 \\ &= (w_1 + w_3)y_2 + \alpha y_1(w_2 + w_3) + \alpha x_1 + \alpha^2 x_2, \end{aligned} \tag{5.15}$$

$$g_2(\alpha)w_1 = w_1 + \alpha x_2 y_1, \tag{5.16}$$

$$g_2(\alpha)w_1 x_2 \equiv w_1 x_2 + \alpha y_1, \tag{5.17}$$

$$g_2(\alpha)w_1 y_2 = w_1 y_2 + \alpha y_1 w_2 + \alpha x_1 + \alpha^2 x_2, \tag{5.18}$$

$$g_2(\alpha)x_1 x_2 \equiv x_1 x_2 + \alpha, \tag{5.19}$$

$$g_2(\alpha)x_1 y_2 = x_1 y_2 + \alpha(w_1 + w_2) + \alpha^2 y_1 x_2, \tag{5.20}$$

$$g_2(\alpha)x_2 y_1 = x_2 y_1. \tag{5.21}$$

We see that in no term does σ_j increase. Notice that s_j never increases under the action of $\text{Sp}(V)$, but decreases in the case of a carry from the $(j - 1)$ th to the j th digit. Thus, ℓ_j never decreases, where $\ell_j = (m - r)\delta(r \leq m) + 2m - r - s_j$. Therefore, not considering any carries into a digit, $\bar{P}_{\ell_j}^{\lambda_j}$ is stable under the actions of $g_1(\alpha)$ and $g_2(\alpha)$.

We also take into account that in some terms, $y_1^{2^j}$ might be carried from f_{j-1} to f_j , and possibly also $x_2^{2^j}$ in the case of $g_2(\alpha)$. The carry is caused by the action of $g_1(\alpha)$ or $g_2(\alpha)$ on f_{j-1} . In these cases, σ_j might increase. We consider a carry of the exponent b_1 of y_1 , as the case for x_2 is similar. The carry reduces λ_{j-1} by 2 and increases λ_j by 1; therefore s_j is decreased by 1, and ℓ_j is increased by 1. The following illustrates that σ_j increases by at most 1 when a factor of y_1 is carried from f_{j-1} to f_j :

$$x_1 \rightarrow w_1,$$

$$y_1 \rightarrow 1,$$

$$1 \rightarrow y_1,$$

$$w_1 \rightarrow x_1,$$

$$w_1 + w_3 \rightarrow x_1 + y_1 w_3.$$

In summary, ℓ_j increased by 1 in each case, while σ_j increased, only in the first case and the second term of the last case. Thus $\sigma_j \leq \ell_j$ is still true. \square

Proof of Lemma 4.1. The calculations here are similar to those in the proof of Lemma 5.3. In fact the calculations are easier since they are done modulo X_i^2 and Y_i^2 , which means that there are no carries to consider. We will not repeat the detailed computations here. \square

Lemma 5.4. $\mathcal{C}_r \subseteq \mathcal{M}_r$.

Proof. Since \mathcal{C}_r is generated as a $k\text{Sp}(V)$ -submodule of $k[\mathcal{P}]$ by the characteristic function of any element of \mathcal{I}_r , we pick one such element and write its characteristic function as

$$\begin{aligned} (1 - x_1^{q-1}) \cdots (1 - x_m^{q-1})(1 - y_1^{q-1}) \cdots (1 - y_{m-r}^{q-1}) &\text{ if } r < m, \\ (1 - x_1^{q-1}) \cdots (1 - x_{2m-r}^{q-1}) &\text{ if } r \geq m. \end{aligned} \tag{5.22}$$

Each monomial in the expansion of this function is in \mathcal{M}_r . Indeed, take any nonconstant monomial f in the expansion of either characteristic function above, and assume that its \mathcal{H} -type is $(s_0, s_1, \dots, s_{t-1})$. Then $(s_0, s_1, \dots, s_{t-1}) \leq (2m - r, 2m - r, \dots, 2m - r)$. Also, for each digit of f , $\sigma_j = 0$ if $r \geq m$, and $\sigma_j \leq m - r$ if $r \leq m$. Since $2m - r - s_j$ is nonnegative, it is clear that $f_j \in \overline{P}_{\ell_j}^{\lambda_j}$, where $\ell_j = (m - r)\delta(r \leq m) + (2m - r - s_j)$. Since also the \mathcal{H} -type is at most $(2m - r, \dots, 2m - r)$, the monomial is r -admissible. Therefore $C_r \subseteq \mathcal{M}_r$. \square

In order to prove Theorem 5.2, we will show that every r -admissible SBF can be obtained from an element of C_r by applying operators from the group ring $k\text{Sp}(V)$.

The next two lemmas provide us with operators from the group ring which allow us to control the shape of a polynomial. The utility of these operators lies in the fact that they modify factorizable functions in only one digit.

Lemma 5.5. For $0 \leq j \leq t - 1$, let

$$g(j) = \sum_{\mu \in k^\times} \mu^{2^j} g_1(\mu^{-1}) \in k[\text{Sp}(V)]. \tag{5.23}$$

Given any basis monomial $f = x_1^{a_1} y_1^{b_1} \cdots x_m^{a_m} y_m^{b_m}$ of $k[V]$, we have

$$g(j)f = \begin{cases} 0, & \text{if the } j\text{th digit of } a_1 \text{ is } 0, \\ x_1^{a_1-2^j} y_1^{b_1+2^j} x_2^{a_2} y_2^{b_2} \cdots x_m^{a_m} y_m^{b_m}, & \text{if the } j\text{th digit of } a_1 \text{ is } 1. \end{cases}$$

Proof. We first prove the lemma for the $j = 0$ case. If $a_1 = 0$, then clearly we have $g(0)f = (\sum_{\mu \in k^\times} \mu) f = 0$. So we assume that $a_1 > 0$,

$$\begin{aligned} g(0)f &= \sum_{\mu \in k^\times} \mu (x_1 + \mu^{-1} y_1)^{a_1} y_1^{b_1} x_2^{a_2} y_2^{b_2} \cdots x_m^{a_m} y_m^{b_m} \\ &= \sum_{\mu \in k^\times} \mu \left(x_1^{a_1} + \binom{a_1}{1} \mu^{-1} x_1^{a_1-1} y_1 + \binom{a_1}{2} \mu^{-2} x_1^{a_1-2} y_1^2 + \cdots \right) y_1^{b_1} x_2^{a_2} y_2^{b_2} \cdots x_m^{a_m} y_m^{b_m} \\ &= a_1 x_1^{a_1-1} y_1^{b_1+1} x_2^{a_2} y_2^{b_2} \cdots x_m^{a_m} y_m^{b_m}. \end{aligned}$$

Therefore the lemma is proved in the case where $j = 0$. The general case follows from the $j = 0$ case by applying the Frobenius automorphism. \square

Lemma 5.6. Let $1 \leq i \leq m$, let $0 \leq j \leq t - 1$, and let $f = x_1^{a_1} y_1^{b_1} \cdots x_m^{a_m} y_m^{b_m}$ be a basis monomial of $k[V]$. Then there exist projectors $p_{i,j}^{(1)}$, $p_{i,j}^{(2)}$, and $p_{i,j}^{(3)}$ such that

$$\begin{aligned} p_{i,j}^{(1)}(f) &= \begin{cases} f, & \text{if the } j\text{th digits of } a_i \text{ and } b_i \text{ are } 1 \text{ and } 0 \text{ respectively,} \\ 0, & \text{otherwise.} \end{cases} \\ p_{i,j}^{(2)}(f) &= \begin{cases} f, & \text{if the } j\text{th digits of } a_i \text{ and } b_i \text{ are } 0 \text{ and } 1 \text{ respectively,} \\ 0, & \text{otherwise.} \end{cases} \\ p_{i,j}^{(3)}(f) &= \begin{cases} f, & \text{if the } j\text{th digits of } a_i \text{ and } b_i \text{ are both } 0 \text{ or both } 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. For $i = 1$, we use the operator $g(j)$ from Lemma 5.5 and let $g'(j)$ denote the analogous operator which shifts 2^j from b_1 to a_1 . Then it is easy to see that

$$\begin{aligned} p_{1,j}^{(1)} &= g'(j)g(j), \\ p_{1,j}^{(2)} &= g(j)g'(j), \\ p_{1,j}^{(3)} &= 1 - p_{1,j}^{(1)} - p_{1,j}^{(2)}. \end{aligned}$$

The same proof works for the cases where $i > 1$. \square

We will sometimes drop the second subscript when it is clear from the context that it is j .

The next five lemmas describe in detail the generation of new functions from existing ones in $k\text{Sp}(V)$ -submodules of $k[\mathcal{P}]$. In the proofs of some of these lemmas, we will use the element $g_2 = g_2(1)$ of $\text{Sp}(V)$ defined in (5.6) and the element

$$g'_2 : x_1 \mapsto x_1, \quad x_2 \mapsto x_1 + x_2, \quad y_1 \mapsto y_1 + y_2, \quad y_2 \mapsto y_2.$$

Lemma 5.7. *Let*

$$f = f_0 f_1^2 f_2^{2^2} \dots f_j^{2^j} \dots f_{t-1}^{2^{t-1}}$$

be an SBF of \mathcal{H} -type $(s_0, \dots, s_j, \dots, s_{t-1})$, where f_j is of the class $(\rho, \sigma, \tau, \nu)$. Let f'_j be any other function of the class $(\rho, \sigma, \tau, \nu)$. Then there is a group ring element $g \in k\text{Sp}(V)$ such that

$$gf \equiv f_0 f_1^2 f_2^{2^2} \dots f'_j{}^{2^j} \dots f_{t-1}^{2^{t-1}}$$

modulo the span of SBFs of lower \mathcal{H} -types.

Proof. Clearly if $i \in T$ for f_j , then we can use a shift operator to exchange x_i and y_i in that digit, leave all other digits unchanged. We need to show further that we can permute the indices $1, \dots, m$ of the variables in f_j , while keeping the functions for all the other digits unchanged. It suffices to show that we can interchange the subscripts 1 and 2, if they belong to any two of $\{R, S, T, U\}$, or if they both belong to R . In the calculations below there are several notational points to keep in mind. First, we use the projectors $p_{i,j}^{(\alpha)}$, $\alpha = 1, 2, 3$, from Lemma 5.6, dropping the second subscript when it is understood to be j . Next, since we are calculating modulo the span of SBFs of lower \mathcal{H} -types, we can ignore any term in which there is a carry from one digit to another. Most importantly, we shall write only the part of the digit f_j involving x_1, y_1, x_2 , and y_2 , except that when $1 \in R$ we assume it is paired with $3 \in R'$ and include x_3 and y_3 . We calculate:

$$\begin{aligned} (1 \in S, 2 \in U) \quad g_2(x_1 y_1) &= x_1 y_1 + y_1 x_2, \\ p_1^{(2)} g_2(x_1 y_1) &= y_1 x_2, \\ g'_2 p_1^{(2)} g_2(x_1 y_1) &= (x_1 + x_2)(y_1 + y_2), \\ (p_1^{(3)} g'_2 p_1^{(2)} g_2 - 1)(x_1 y_1) &= x_2 y_2, \\ (1 \in T, 2 \in U) \quad g_2(x_1) &= x_1 + x_2, \\ (g_2 - 1)(x_1) &= x_2, \end{aligned}$$

$$(1 \in S, 2 \in T) \quad g_2(x_1 y_1 y_2) = (x_1 + x_2) y_1 (y_1 + y_2),$$

$$p_1^{(2)} g_2(x_1 y_1 y_2) = y_1 x_2 y_2,$$

$$(1 \in R, 2 \in U, 3 \in R') \quad g_2(x_1 y_1 + x_3 y_3) = (x_1 + x_2) y_1 + x_3 y_3,$$

$$p_1^{(2)} g_2(x_1 y_1 + x_3 y_3) = y_1 x_2,$$

$$g_2' p_1^{(2)} g_2(x_1 y_1 + x_3 y_3) = (x_1 + x_2)(y_1 + y_2),$$

$$p_1^{(3)} g_2' p_1^{(2)} g_2(x_1 y_1 + x_3 y_3) = x_1 y_1 + x_2 y_2.$$

We interchanged 1 and 2.

$$(1 \in R, 2 \in T, 3 \in R') \quad g_2((x_1 y_1 + x_3 y_3) y_2) = ((x_1 + x_2) y_1 + x_3 y_3)(y_1 + y_2)$$

$$= y_1 x_3 y_3 + (x_1 + x_2) y_1 y_2 + y_2 x_3 y_3,$$

$$p_2^{(3)} g_2((x_1 y_1 + x_3 y_3) y_2) = y_1 x_3 y_3 + y_1 x_2 y_2 = y_1 (w_2 + w_3),$$

$$(1 \in R, 2 \in S, 3 \in R') \quad g_2(x_2 y_2 (x_1 y_1 + x_3 y_3)) = x_2 (y_1 + y_2) ((x_1 + x_2) y_1 + x_3 y_3)$$

$$= x_1 y_1 x_2 y_2 + y_1 x_2 x_3 y_3 + x_2 y_2 x_3 y_3,$$

$$p_1^{(2)} g_2(x_2 y_2 (x_1 y_1 + x_3 y_3)) = y_1 x_2 x_3 y_3,$$

$$g_2' p_1^{(2)} g_2(x_2 y_2 (x_1 y_1 + x_3 y_3)) = (y_1 + y_2) (x_1 + x_2) x_3 y_3,$$

$$p_1^{(3)} g_2' p_1^{(2)} g_2(x_2 y_2 (x_1 y_1 + x_3 y_3)) = (x_1 y_1 + x_2 y_2) x_3 y_3.$$

We interchanged 2 and 3.

We can also interchange two subscripts in R . Let

$$f_j = (x_1 y_1 + x_3 y_3)(x_2 y_2 + x_4 y_4).$$

Then

$$g_2(f_j) = ((x_1 + x_2) y_1 + x_3 y_3)(x_2 (y_1 + y_2) + x_4 y_4),$$

$$p_2^{(1)} g_2(f_j) = y_1 x_2 (x_3 y_3 + x_4 y_4),$$

$$p_1^{(3)} g_2' p_2^{(1)} g_2(f_j) = (x_1 y_1 + x_2 y_2)(x_3 y_3 + x_4 y_4).$$

It is important to remember that when g_2 or g_2' is applied, substitutions take place in all digits, as in (3.2), not just in f_j , and carries must be performed. In the above we have only shown the effect of substitution on f_j and, since we are calculating modulo the span of SBFs of lower \mathcal{H} -type, we have set equal to zero the terms of the resulting j th digit in which there is a carry to the $(j + 1)$ th digit. To see that this calculation gives the correct j th digit of the function obtained by the substitution in all digits, as in (3.2), we recall the rewriting process described in Section 3.1. After distributing the product into a sum of monomials there are possibly carries to be performed. The result of a carry will be a monomial of lower \mathcal{H} -type, which we may disregard since we are working modulo functions of lower \mathcal{H} -type. Therefore, when computing the j th digit of the function obtained by substitution modulo functions of lower \mathcal{H} -types, we can treat as zero any contributions from carries between the $(j - 1)$ th and j th digits.

We also use the projectors to remove those terms for which the k th digit, $k \neq j$, is not equal to f_k . Let f' be a factorizable function of the same \mathcal{H} -type as f and with k th digit equal to f_k . After application of g_2 to f' we end up, modulo functions of lower \mathcal{H} -type, with a certain sum of factorizable functions of the same \mathcal{H} -type as f , including one term equal to f' . If neither x_1 nor y_2 occurs in f_k then the k th digit of every term will be equal to f_k . Suppose either x_1 or y_2 or x_1y_2 is a factor of f_k . We treat as zero any term involving a carry from the $(k - 1)$ th to the k th digit or from the k th to the $(k + 1)$ th digit. Then there is some choice of α_k , $1 \leq \alpha_k \leq 3$, such that $p_{1k}^{(\alpha_k)}$ acts as the identity on f' and kills any function in the above sum whose k -th digit differs from f_k in the variables x_1 and y_1 . After applying these projectors for all $k \neq j$, we end up with a sum of factorizable functions whose k th digits are the same as those of f , for all $k \neq j$. Similarly, every function arising from the application of g'_2 whose k th digit is unequal to f_k , for some $k \neq j$, can be removed by a suitable projector. We have shown that we can change f_j to any other SBF of the same class, while keeping every other digit the same. \square

Lemma 5.8. *As in Lemma 5.7, let f_j be the j th digit of an SBF f , and let the class of f_j be $(\rho, \sigma, \tau, \nu)$, $\tau \geq 2$. Then there exists a group ring element g such that gf is an SBF identical to f (modulo the span of SBFs of lower \mathcal{H} -types), except that f_j is replaced by a function of class $(\rho + 1, \sigma, \tau - 2, \nu)$.*

Proof. By Lemma 5.7, we may assume that $\{1, 2\} \subseteq T$ and $f_j = x_1y_2 \cdots$. Also by Lemma 5.7, we can apply a group ring element of $k\text{Sp}(V)$ to f to obtain a function f' which is identical to f except that f_j is replaced by $f'_j = y_1x_2 \cdots$, where f_j and f'_j differ only in the first two subscripts. Then we have

$$g_2f_j - f_j - f'_j = [(x_1 + x_2)(y_1 + y_2) - f_j - f'_j] \cdots = (x_1y_1 + x_2y_2) \cdots.$$

We apply projectors to the other digits, as in the proof of Lemma 5.7, to kill any functions which are not the same as the original ones in those digits. \square

Lemma 5.9. *Let f_j be the j th digit of an SBF f , and let the class of f_j be $(\rho, \sigma, \tau, \nu) \neq (1, 0, m - 2, 0)$, $\rho > 0$. Then there exists a group ring element g such that gf is an SBF identical to f modulo the span of SBFs of lower \mathcal{H} -types, except that f_j is replaced by a function of class $(\rho - 1, \sigma, \tau + 2, \nu)$.*

Proof. Since $(\rho, \sigma, \tau, \nu) \neq (1, 0, m - 2, 0)$, and $2\rho + \sigma + \tau + \nu = m$, we will consider three cases: $\rho > 1$, $\sigma > 0$, and $\nu > 0$. We omit the part of f_j which is not acted upon by g . Note that by Lemma 5.7, we may take f_j to have some special form, as long as it belongs to the class $(\rho, \sigma, \tau, \nu)$.

If $\rho > 1$, we take $f_j = (x_1y_1 + x_3y_3)(x_2y_2 + x_4y_4)$ and we get (keeping only the square-free terms):

$$\begin{aligned} (g_2 - 1)f_j &= ((x_1 + x_2)y_1 + x_3y_3)(x_2(y_1 + y_2) + x_4y_4) - f_j \\ &\equiv y_1x_2(x_3y_3 + x_4y_4). \end{aligned}$$

If $\rho > 0$, $\sigma > 0$, then we take $f_j = x_1y_1(x_2y_2 + x_3y_3)$ and compute

$$\begin{aligned} (g_2 - 1)f_j &= (x_1 + x_2)y_1(x_2(y_1 + y_2) + x_3y_3) - f_j \\ &\equiv (x_1y_1x_2y_2 + (x_1 + x_2)y_1x_3y_3) - f_j \\ &= y_1x_2x_3y_3. \end{aligned}$$

If $\rho > 0$, $\nu > 0$, we take $f_j = (x_1y_1 + x_3y_3)$ and similarly get

$$(g_2 - 1)f_j = ((x_1 + x_2)y_1 + x_3y_3) - f_j = y_1x_2.$$

We again keep the functions for the other digits from changing as in the proof of Lemma 5.7. \square

Lemma 5.10. *Let f be an SBF whose j th digit is f_j of class $(0, 1, m - 2, 1)$. Then there is a group ring element g such that gf is identical to f , modulo the span of SBFs of lower \mathcal{H} -types, except that f_j is replaced by a function of class $(0, 0, m, 0)$.*

Proof. By Lemma 5.7, we may take $1 \in S, 2 \in U$, so that $f_j = x_1 y_1 (x_2 y_2)^0 \dots$. Then $(g_2 - 1)f_j = y_1 x_2 \dots$, and we proceed as before. \square

Lemma 5.11. *Let f be an SBF whose j th digit is f_j of class $(\rho, \sigma, \tau, \nu)$, $\sigma > 0, \nu > 0$. Then there is a group ring element g such that gf is identical to f , modulo the span of SBFs of lower \mathcal{H} -types, except that f_j is replaced by a function of class $(\rho + 1, \sigma - 1, \tau, \nu - 1)$.*

Proof. By Lemma 5.7, we may assume $1 \in S$ and $2 \in U$. Let h be the element from Lemma 5.7 which interchanges 1 and 2 in f_j , while keeping the rest of f the same. Then $g = 1 + h$ is the desired element. \square

The next lemma, whose precise statement is rather complicated, relates SBFs of a given \mathcal{H} -type with SBFs whose \mathcal{H} -types are one step down in the partial order on \mathcal{H} . A fuller discussion of its meaning will be given after the proof.

Lemma 5.12. *Let $(s_0, s_1, \dots, s_j, \dots, s_{t-1})$ and $(s_0, s_1, \dots, s_{j-1}, \dots, s_{t-1})$ be a pair of \mathcal{H} -types, with $s_k \leq 2m - r, 0 \leq k \leq t - 1$, and let $\lambda_j = 2s_{j+1} - s_j$ for $0 \leq j \leq t - 1$. Suppose f is an SBF of type $(s_0, s_1, \dots, s_j, \dots, s_{t-1})$. Assume that the j th digit f_j of f is of class $(0, \sigma_j, \tau_j, \nu_j)$, where*

$$\begin{aligned} \sigma_j &= \min\{2m - r - s_j + \delta(r \leq m)(m - r), \lfloor \lambda_j/2 \rfloor\}, \\ \tau_j &= \lambda_j - 2\sigma_j, \quad \text{and} \\ \nu_j &= m - \sigma_j - \tau_j. \end{aligned}$$

Case 1. $\sigma_j \neq \lambda_j/2$ (so that $\tau_j > 0$).

(a) If $\sigma_{j-1} < \lfloor \frac{\lambda_{j-1}}{2} \rfloor$, let f_{j-1} be of class $(1, \sigma_{j-1}, \tau_{j-1}, \nu_{j-1})$, where

$$\begin{aligned} \sigma_{j-1} &= 2m - r - s_{j-1} + \delta(r \leq m)(m - r), \\ \tau_{j-1} &= \lambda_{j-1} - 2\sigma_{j-1} - 2, \\ \nu_{j-1} &= m - \sigma_{j-1} - \tau_{j-1} - 2. \end{aligned} \tag{5.24}$$

Then the following is true modulo the span of SBFs of lower \mathcal{H} -type. There is a group ring element g such that the j th digit of gf is of class $(0, \sigma_j + 1, \tau_j - 1, \nu_j)$ and the $(j - 1)$ th digit of gf is of class $(0, \sigma_{j-1}, \tau_{j-1}, \nu_{j-1} + 2)$, while all the other digits remain the same.

(b) If $\sigma_{j-1} = \lfloor \frac{\lambda_{j-1}}{2} \rfloor$, let f_{j-1} be of class $(0, \lambda_{j-1}/2, 0, m - \lambda_{j-1}/2)$ or $(0, (\lambda_{j-1} - 1)/2, 1, m - (\lambda_{j-1} + 1)/2)$, depending on whether λ_{j-1} is even or odd. Then the following is true modulo the span of SBFs of lower \mathcal{H} -type. There is a group ring element g such that gf has j th digit as in (a) and $(j - 1)$ th digit of class $(0, \sigma_{j-1} - 1, \tau_{j-1}, \nu_{j-1} + 1)$, while all the other digits remain the same.

Case 2. $\sigma_j = \lambda_j/2$.

(a) If $\sigma_{j-1} < \lfloor \frac{\lambda_{j-1}}{2} \rfloor$, let f_{j-1} be of class $(1, \sigma_{j-1}, \tau_{j-1}, \nu_{j-1})$, where $\sigma_{j-1}, \tau_{j-1}, \nu_{j-1}$ are defined as in (5.24). Then the following is true modulo the span of SBFs of lower \mathcal{H} -type. There is a group ring element g such that the j th digit of gf is of class $(0, \frac{\lambda_j}{2}, 1, m - \frac{\lambda_j}{2} - 1)$ and the $(j - 1)$ th digit of gf is of class $(0, \sigma_{j-1}, \tau_{j-1}, \nu_{j-1} + 2)$, while all the other digits remain the same.

- (b) If $\sigma_{j-1} = \lfloor \frac{\lambda_{j-1}}{2} \rfloor$, let f_{j-1} be of class $(0, \lambda_{j-1}/2, 0, m - \lambda_{j-1}/2)$ or $(0, (\lambda_{j-1} - 1)/2, 1, m - (\lambda_{j-1} + 1)/2)$, depending on whether λ_{j-1} is even or odd. Then the following is true modulo the span of SBFs of lower \mathcal{H} -type. There is a group ring element g such that gf has j th digit as in (a) and $(j - 1)$ th digit of class $(0, \sigma_{j-1} - 1, \tau_{j-1}, \nu_{j-1} + 1)$, while all the other digits remain the same.

In both Cases 1 and 2, the elements g can be chosen so that if f' is a monomial of \mathcal{H} -type lower than that of f , then gf' is either zero or has \mathcal{H} -type lower than that of gf .

Proof. For \mathcal{H} -type $(s_0, \dots, s_{j-1}, s_j - 1, \dots, s_{t-1})$, we let $(\lambda'_0, \lambda'_1, \dots, \lambda'_{t-1})$ be the corresponding type in \mathbf{A} . Since $(s_0, \dots, s_{j-1}, s_j - 1, \dots, s_{t-1})$ is the \mathcal{H} -type that results when there is a carry from the $(j - 1)$ th digit to the j th digit of f , we have $\lambda'_{j-1} = \lambda_{j-1} - 2 \geq 0$, $\lambda'_j = \lambda_j + 1$, and $\lambda'_k = \lambda_k$ for all $k \notin \{j - 1, j\}$.

We deal with Case 1 first. We may assume that $1 \in R$ for f_{j-1} in Case 1(a), or that $1 \in S$ for f_{j-1} in Case 1(b). In either case, we may assume that $1 \in T$ for $f_j = x_1 y_1^0 \dots$, using Lemma 5.7, at the cost of adding some terms in the span of SBFs of lower \mathcal{H} -type than f . The shift operator $g(j - 1)$ shifts 2^{j-1} from a_1 , the exponent of x_1 , to b_1 , the exponent of y_1 , or returns 0 if a monomial has a 0 in the $(j - 1)$ th digit of a_1 . It is clear that $g(j - 1)f$ has the desired properties.

For Case 2, we may make the same assumptions on f_{j-1} as in Case 1. Note that $\sigma_j = \frac{\lambda_j}{2}$ and $m = \sigma_j + \nu_j$, we have $\nu_j > 0$ (otherwise $\lambda_j = 2m$, which implies $\lambda'_j = 2m + 1$, impossible). Therefore for f_j , we can assume that $1 \in U$. Let $g(j - 1)$ be the same shift operator as above. We see that $g(j - 1)f$ has the desired properties.

Suppose f' is a monomial of \mathcal{H} -type \mathbf{s}' strictly below $\mathbf{s} = (s_0, \dots, s_{t-1})$. Then $p_{1,j-1}^{(3)} f'$ is equal to 0, if the $(j - 1)$ th digits of the exponents of x_1 and y_1 are different, or to f' if these exponents are the same. In the latter case either $g(j - 1)f'$ is zero or else its \mathcal{H} -type \mathbf{s}'' is strictly less than \mathbf{s}' due to carry in its computation. Since the ordering on \mathcal{H} reflects the $k\text{Sp}(V)$ -module structure (Section 2.3), we also know $\mathbf{s}'' \leq (s_0, \dots, s_j - 1, \dots, s_{t-1})$. We see that this inequality must actually be strict, because the sum of the entries of \mathbf{s}'' is at least 2 less than $\sum_{j=0}^{t-1} s_j$. Thus, since $p_{1,j-1}^{(3)} f = f$, the group ring element $g(j - 1)p_{1,j-1}^{(3)}$ has the required properties in the last statement of the lemma. \square

We would like to explain the meaning of this rather technical lemma. We first focus on the $(j - 1)$ th digit of f and gf . Note that the maximum level for the $(j - 1)$ th digit of an r -admissible function of \mathcal{H} -type $(s_0, \dots, s_{j-1}, s_j, \dots, s_{t-1})$ is $L_{j-1} := \min\{\ell_{j-1}, \lfloor \frac{\lambda_{j-1}}{2} \rfloor\}$, where $\ell_{j-1} = 2m - r - s_{j-1} + \delta(r \leq m)(m - r)$. In the lemma, we have set σ_{j-1} equal L_{j-1} . So the level σ_{j-1} of f_{j-1} has its maximum value for the given \mathcal{H} -type. The maximum level for the $(j - 1)$ th digit of an r -admissible function of \mathcal{H} -type $(s_0, \dots, s_{j-1}, s_j - 1, \dots, s_{t-1})$ is $L'_{j-1} := \min\{\ell_{j-1}, \lfloor \frac{\lambda'_{j-1}}{2} \rfloor\}$. Since $\lambda'_{j-1} = \lambda_{j-1} - 2$, we have

$$L'_{j-1} = \begin{cases} \sigma_{j-1}, & \text{if } \sigma_{j-1} < \lfloor \frac{\lambda_{j-1}}{2} \rfloor, \\ \sigma_{j-1} - 1, & \text{if } \sigma_{j-1} = \lfloor \frac{\lambda_{j-1}}{2} \rfloor. \end{cases}$$

So the level of the $(j - 1)$ th digit of gf also has its maximum value for that given \mathcal{H} -type.

In the lemma, we also specified σ_j to have its maximum value. The maximum level for the j th digit of an r -admissible function of \mathcal{H} -type $(s_0, \dots, s_{j-1}, s_j - 1, \dots, s_{t-1})$ is

$$L'_j := \min \left\{ 2m - r - s_j + 1 + \delta(r \leq m)(m - r), \left\lfloor \frac{\lambda'_j}{2} \right\rfloor \right\}.$$

In Case 1 of the lemma, since $2\sigma_j < \lambda_j$ and $\lambda'_j = \lambda_j + 1$, we have $L'_j = \sigma_j + 1$. So the level $\sigma_j + 1$ of the j th digit of gf in Case 1 of the lemma is maximum for that given \mathcal{H} -type. In Case 2 of the

lemma, since $\lambda_j = 2\sigma_j$ and $\lambda'_j = \lambda_j + 1$, we have $L'_j = \sigma_j$. So the level σ_j of the j th digit of gf in Case 2 of the lemma is also maximum for that given \mathcal{H} -type.

Thus, assuming we picked σ_k also to be maximum for each of the other digits, $0 \leq k \leq t - 1$, gf will be an SBF of \mathcal{H} -type $(s_0, \dots, s_j - 1, \dots, s_{t-1})$ which has the maximum r -admissible level for each digit, for that \mathcal{H} -type.

Lemma 5.13. *The set of r -admissible SBFs forms a basis for \mathcal{C}_r .*

Proof. The set of SBFs is linearly independent by construction. By Lemma 5.4, it is sufficient to prove that every r -admissible SBF lies in \mathcal{C}_r . We shall proceed by induction on the partial order on \mathcal{H} . To start the induction, we observe that $\mathcal{C}_r \supseteq \mathcal{C}_{2m-1}$, by (1.1), and that \mathcal{C}_{2m-1} contains the constant functions and all r -admissible SBFs of the lowest \mathcal{H} -type $(1, 1, \dots, 1)$, since it contains all basis monomials of this type. Next we wish to explain how Lemmas 5.7–5.12 will be applied in our inductive proof. They are all of the same form: from a given function f and a target function f' of a certain \mathcal{H} -type, each lemma yields a group ring element g such that $f' - gf$ lies in the span of SBFs whose \mathcal{H} -types are lower than that of f' . If the functions f and f' belong to \mathcal{M}_r , then by Lemma 5.3 the conclusions of Lemmas 5.7–5.12 can be strengthened to say that the error term $f' - gf$ is in the span of r -admissible SBFs of lower \mathcal{H} -type than f' . With this in mind, we can now continue with the proof.

Let f' be an r -admissible SBF. We will show that $f' \in \mathcal{C}_r$. We can assume that f' is not constant and has \mathcal{H} -type strictly higher than $(1, 1, \dots, 1)$. The inductive hypothesis states that every r -admissible SBF of strictly lower \mathcal{H} -type belongs to \mathcal{C}_r .

Suppose \mathcal{C}_r contains any SBF whose digits belong to the same classes as the corresponding digits of f' . By Lemma 5.4 such an SBF belongs to \mathcal{M}_r . Then Lemma 5.7 shows, together with the inductive hypothesis, that $f' \in \mathcal{C}_r$. So it suffices to show that \mathcal{C}_r contains some SBF with the same digit classes as f' .

Next, let f be an SBF of the same \mathcal{H} -type as f' , but having maximum r -admissible levels in all digits. We will show that the f' lies in the $k\text{Sp}(V)$ -submodule generated by f and \mathcal{C}_r . This will go smoothly unless very special conditions hold, in which case more work is required to obtain the result. By the inductive hypothesis, it is enough to show that the above submodule contains f' modulo the span of r -admissible SBFs of lower \mathcal{H} -type. We do this by applying the group ring operators given by Lemmas 5.8–5.11. These operators allow us to modify the digits of f one at a time while leaving the other digits unchanged until we obtain, modulo the span of r -admissible of lower \mathcal{H} -type, an SBF with the same digit classes as f' . Then we apply the preceding paragraph.

Of course, we only need to modify a digit of f if it belongs to a different class from the corresponding digit of f' . If the j th digit of f' is not of class $(0, 0, m, 0)$, then using Lemmas 5.8, 5.9 and 5.11 we can modify the j th digit of f successfully. If the j th digit of f' is $(0, 0, m, 0)$ but the maximum r -admissible level for the j th digit in an SBF of this \mathcal{H} -type is nonzero, then the same lemmas, together with Lemma 5.10 allow us to modify the j th digit f to the desired form. Finally, suppose the maximum r -admissible level ℓ_j for the j th digit is equal to 0. Let \mathcal{H} -type of f' and f be (s_0, \dots, s_{t-1}) . Then the conditions (5.1),

$$s_j \leq 2m - r, \quad \ell_j = (m - r)\delta(r \leq m) + (2m - r - s_j)$$

force $r \geq m$ and $s_j = 2m - r$. Then

$$m = \lambda_j = 2s_{j+1} - s_j \leq 2(2m - r) - (2m - r) = 2m - r \leq m; \tag{5.25}$$

so equality holds throughout, and $s_j = s_{j+1} = 2m - r = m$, which means $r = m$. If $r = m$ and some digits of f' have class $(0, 0, m, 0)$ and 0 is the maximum r -admissible level for those digits, we shall refer to f' as *exceptional*. Thus, unless f' is exceptional, the desired conclusion holds.

Suppose the \mathcal{H} -type \mathbf{s}' of f' is not $(2m - r, 2m - r, \dots, 2m - r)$, the highest possible \mathcal{H} -type for r -admissible SBFs. Then Lemma 5.12 says that for any \mathcal{H} -type \mathbf{s}'' which is one step higher than that of f' in the partial order, and any r -admissible SBF f'' of type \mathbf{s}'' which has maximum r -admissible

levels in all digits, there exists a group ring element g'' , such $g''f''$ is an SBF of type \mathbf{s}' and maximum r -admissible levels in all digits, modulo the span of r -admissible SBFs of \mathcal{H} -types lower than \mathbf{s}' . Moreover, g'' sends monomials of \mathcal{H} -type lower than \mathbf{s}'' to monomials of \mathcal{H} -type lower than \mathbf{s}' .

Given two \mathcal{H} -types, $\mathbf{s} = (s_0, \dots, s_{t-1})$ and $\mathbf{s}' = (s'_0, \dots, s'_{t-1})$, $2m - r \geq s_j \geq s'_j$, $0 \leq j \leq t - 1$, it is easy to see that there exists a sequence of \mathcal{H} -types connecting \mathbf{s} and \mathbf{s}' , such that two consecutive \mathcal{H} -types differ in only one component, with that component of the second being reduced by 1 (see [1]).

Consider the characteristic function (5.22), which belongs to \mathcal{C}_r by definition. It is the sum of its leading monomial

$$m_+ = \begin{cases} x_1^{q-1} \cdots x_m^{q-1} y_1^{q-1} \cdots y_{m-r}^{q-1}, & \text{if } r < m, \\ x_1^{q-1} \cdots x_{2m-r}^{q-1}, & \text{if } r \geq m, \end{cases}$$

and some other r -admissible SBFs of lower \mathcal{H} -type. Now m_+ is an SBF of type $(2m - r, \dots, 2m - r)$ and has maximum r -admissible levels in all of its digits. Thus, by repeated application of Lemma 5.12 down a sequence from $(2m - r, \dots, 2m - r)$ to \mathbf{s}' , we see that \mathcal{C}_r contains an r -admissible SBF f of type \mathbf{s}' , modulo the span of r -admissible SBFs of \mathcal{H} -types lower than \mathbf{s}' . If f is not exceptional, then we have seen that \mathcal{C}_r also contains f' , modulo the span of r -admissible SBFs of lower \mathcal{H} -type.

If f is exceptional, then $r = m$ and each exceptional digit of f belongs to the same class $(0, 0, m, 0)$ as the same digit of m_+ . We can use Lemmas 5.8, 5.9, 5.11 and 5.12 to modify the other digits of m_+ , without doing anything to the exceptional digits, until we obtain f' modulo the span of r -admissible SBFs of lower \mathcal{H} -type.

In both cases, we apply the inductive hypothesis to deduce $f' \in \mathcal{C}_r$, which completes the proof. \square

6. The case $q = 2$

In this section we modify the arguments of Section 5 to show that Theorem 5.2 also holds when $q = 2$. We will use the $\text{Sp}(V)$ elements $g_1 = g_1(1)$ and $g_2 = g_2(1)$ defined in (5.5) and (5.6).

Lemma 6.1. *Lemma 5.3 holds when $q = 2$.*

Proof. We again examine the actions of g_1 and g_2 , since the result is trivial for the other generators of $\text{Sp}(2m, 2)$. We now must do the calculations modulo $(x_2^2 - x_2, y_1^2 - y_1)$, instead of modulo $(x_2^2 - 1, y_1^2 - 1)$. Instead of redoing the calculations, we make the following observation: when $q = 2$, replacing the squared variable in a function by the unsquared one is equivalent to the situation, in the $q = 2^t > 2$ case, of a carry from the 0th digit to the 1st digit, replacing the squared variable in the 0th digit by 1, followed by a carry from the $(t - 1)$ th digit into the 0th digit. Therefore, the calculations in the proof of Lemma 5.3 also prove this lemma for $q = 2$. \square

Here we comment that Lemma 5.5 does not hold if $q = 2$, because the elements of \mathbf{F}_2 sum to 1, while the elements of any other finite field sum to 0. Thus the projectors developed in Lemma 5.6 are not available, and we have the need for this section.

Lemma 6.2. *The following lemmas from Section 5 hold when $q = 2$: (a) Lemma 5.7, (b) Lemma 5.8, (c) Lemma 5.10, (d) Lemma 5.11. Furthermore, when $q = 2$, we may drop the condition, “modulo the span of SBFs of lower \mathcal{H} -type” in all cases.*

Proof. For Lemma 5.7, since there is only one digit, we can just permute the subscripts directly using group elements of the form (5.3), and we can interchange x_i and y_i , $1 \leq i \leq m$, by combining elements of the form (5.3) with (5.4). In the cases of Lemmas 5.8, 5.10, and 5.11, we do not use any shift operators in the proofs. Also, the projectors are used only to keep digits other than the j th digit from changing, but now we have only one digit. Thus the proofs are still valid for the case $q = 2$. \square

The $q = 2$ version of Lemma 5.9 requires a little more work to prove.

Lemma 6.3. *Let f be an SBF, and let the class of f be $(\rho, \sigma, \tau, \nu) \neq (1, 0, m - 2, 0)$, $\rho > 0$. Then there exists a group ring element g such that gf is of class $(\rho - 1, \sigma, \tau + 2, \nu)$.*

Proof. Since $(\rho, \sigma, \tau, \nu) \neq (1, 0, m - 2, 0)$, and $2\rho + \sigma + \tau + \nu = m$, we will consider three cases: $\rho > 1$, $\sigma > 0$, and $\nu > 0$. In each case, by Lemma 6.2(a), we may take f to have the form indicated. We omit the part of f which is not acted on by g .

If $\rho > 1$, we may assume $f = (x_1y_1 + x_3y_3)(x_2y_2 + x_4y_4)$ and we get:

$$\begin{aligned} (g_2 - 1)f &= ((x_1 + x_2)y_1 + x_3y_3)(x_2(y_1 + y_2) + x_4y_4) - f \\ &= y_1x_2(w_3 + w_4) + y_1x_2 + y_1w_2 + w_1x_2. \end{aligned}$$

Next we interchange x_1 and y_1 and apply $(g_1 - 1)$ to get

$$(g_1 - 1)(x_1x_2(w_3 + w_4) + x_1x_2 + x_1w_2 + w_1x_2) = y_1x_2(w_3 + w_4) + y_1w_2.$$

Then, interchanging subscripts 1 and 2 and again applying $(g_1 - 1)$, we get

$$(g_1 - 1)(x_1y_2(w_3 + w_4) + w_1y_2) = y_1y_2(w_3 + w_4) + y_1y_2.$$

If $\rho > 0$, $\sigma > 0$, then we may assume $f = x_1y_1(x_2y_2 + x_3y_3)$ and compute

$$\begin{aligned} (g_2 - 1)f &= (x_1 + x_2)y_1(x_2(y_1 + y_2) + x_3y_3) - f \\ &= (w_1w_2 + (x_1 + x_2)y_1w_3) + w_1x_2 + y_1w_2 + y_1x_2 - f \\ &= y_1x_2w_3 + w_1x_2 + y_1w_2 + y_1x_2. \end{aligned}$$

If $\rho > 0$, $\nu > 0$, we may assume $f = (x_1y_1 + x_3y_3)$, $2 \in U$ and similarly get

$$\begin{aligned} (g_2 - 1)f &= ((x_1 + x_2)y_1 + x_3y_3) - f \\ &= y_1x_2. \end{aligned}$$

Now we notice that for $1 < i \leq m$, $(g_1 - 1)(w_1 + w_i) = y_1$. Therefore, we can always go from a function of class $(\rho, \sigma, \tau, \nu)$, $\rho \neq 0$, to one of class $(\rho - 1, \sigma, \tau + 1, \nu + 1)$, reducing the degree λ by 1. Similarly, since $(g_1 - 1)w_1 \equiv y_1$, we can go from a function of class $(\rho, \sigma, \tau, \nu)$, $\sigma > 0$, to one of class $(\rho, \sigma - 1, \tau + 1, \nu)$. Therefore, in the first case, where $\rho > 1$, by Lemma 6.2(a), we may subtract the extra term of degree 2. In the second case, where $\sigma > 0$, we may subtract the extra term of degree 2 and the two terms of degree 3. We have exactly the functions we want. \square

The following lemma takes the place of Lemma 5.12. It shows that given an SBF of degree $\lambda = s > 1$ and of a class having maximum σ for that degree, we can find a group ring element which will give us an SBF with λ reduced by 1 and σ maximum for the new degree.

Lemma 6.4. *Let $s \leq 2m - r$ and let f be an SBF of class $(0, \sigma, \tau, \nu)$ of degree $\lambda = s > 1$, where $\sigma = \min\{2m - r - s + \delta(m > r)(m - r), \lfloor s/2 \rfloor\}$, $\tau = s - 2\sigma$, and $\nu = m - \sigma - \tau$.*

- (a) *If $\sigma < s/2 - 1$, then there exists $g \in k[\text{Sp}(V)]$ such that gf is of class $(0, \sigma + 1, \tau - 3, \nu + 2)$.*
- (b) *If $\sigma = s/2 - 1$ or $\sigma = (s - 1)/2$, then there is a $g \in k[\text{Sp}(V)]$ such that gf is of class $(0, \sigma, \tau - 1, \nu + 1)$.*
- (c) *If $\sigma = s/2$, then there is a $g \in k[\text{Sp}(V)]$ such that gf is of class $(0, \sigma - 1, \tau + 1, \nu)$.*

Proof. (a) In the first case, by Lemma 6.2(b), we instead consider f' and f'' of class $(1, \sigma, \tau - 2, \nu)$ of the forms $f' = (w_1 + w_3)y_2 \cdots$ and $f'' = (w_1 + w_3)x_2 \cdots$. We first calculate:

$$(g_2 - 1)f' = y_1x_2 \cdots.$$

We also have (omitting variables with subscripts higher than 3):

$$\begin{aligned} (g_1 + g_2)f' &= ((x_1 + y_1)y_1 + x_3y_3)y_2 + ((x_1 + x_2)y_1 + x_3y_3)(y_1 + y_2) \\ &= y_1(w_2 + w_3) + w_1 + y_1y_2 + x_2y_1. \end{aligned}$$

Since the first term is of the same class we started with, and the last two are of the same class we obtained from f'' , we may subtract them, leaving only w_1 , the term we want.

(b) If $\sigma = s/2 - 1$, $\tau = 2$, by Lemma 6.2(a), we may take $f = x_1x_2 \cdots$, and get $(g_2 - 1)f = x_2 \cdots$.

If $\sigma = (s - 1)/2 > 0$, we may take $f = w_1y_2 \cdots$ and get (omitting variables with subscripts higher than 2) $(g_2 - 1)f = y_1w_2 + w_1 + y_1x_2$. We may subtract the first term, since it is of the same class as f . We subtract the last term, as it is of the same class as $(g_1 - 1)f = y_1y_2$, leaving the term we want, w_1 .

(c) If $\sigma = s/2$, we take $f = w_1 \cdots$ and have $(g_1 - 1)f = y_1 \cdots$. \square

The following lemma gives us an SBF which generates all the other SBFs in the submodule, according to the lemmas of this section.

Lemma 6.5. *If $r \geq m$, there is an SBF of class $(0, 0, 2m - r, r - m)$ in \mathcal{C}_r . If $r \leq m$, there is an SBF of class $(0, m - r, r, 0)$ in \mathcal{C}_r .*

Proof. We may assume $1 < r < 2m - 1$, since the case $r = 1$ is trivial and the case $r = 2m - 1$ is the incidence of points with all hyperplanes [3]. We will pick a particular r -subspace W and show how to obtain the desired monomial SBF by applying a sequence of group ring elements. If $r \leq m$, we take the totally isotropic r -subspace $x_1 = 0, \dots, x_r = 0, x_{r+1} = 0, y_{r+1} = 0, \dots, x_m = 0, y_m = 0$, with characteristic function

$$f = (x_1 - 1)(x_2 - 1) \cdots (x_r - 1)(x_{r+1} - 1)(y_{r+1} - 1) \cdots (x_m - 1)(y_m - 1),$$

while if $r \geq m$, we pick the r -subspace $x_1 = 0, \dots, x_{2m-r} = 0$, which is the complement of a totally isotropic subspace with respect to the form. Its characteristic function is

$$f = (x_1 - 1)(x_2 - 1) \cdots (x_{2m-r} - 1).$$

Let H be the $(r + 2)$ -dimensional subspace of V defined by the same equations as W , but with the two equations $x_1 = 0$ and $x_2 = 0$ omitted and let f_H be its characteristic function. We have

$$f = (x_1 - 1)(x_2 - 1)f_H.$$

By (1.1), we have $f_H \in \mathcal{C}_r$. Next let $f' = (x_1 - 1)(y_2 - 1)f_H$ be the function obtained from f by interchanging x_2 and y_2 . Then $f' \in \mathcal{C}_r$, since \mathcal{C}_r is a $k\text{Sp}(V)$ -module. We calculate:

$$(g_2 - 1)f' = [x_1y_1 + x_2y_2 + 1 + (y_1 - 1)(x_2 - 1)]f_H.$$

We may subtract the last term because it represents the characteristic function of another r -subspace. Then we calculate:

$$(g_1 - 1)(x_1 y_1 + x_2 y_2 + 1) f_H = y_1 f_H.$$

Using these results, we see that $x_1 f_H$, $y_2 f_H$ and f_H lie in C_r . If we subtract these terms from f' , we have $f'' = x_1 y_2 f_H \in C_r$. If $m = 2$, we are done. If $m > 2$, f_H will have factors such as $(x_i - 1)$ or $(x_i - 1)(y_i - 1)$, $i \geq 3$. We now describe group ring operations on f'' which convert these factors to monomials. By permuting the subscripts 1, 2 and i , we can obtain from f'' a function which starts with either $(x_1 - 1)x_2$ or $(x_1 - 1)(y_1 - 1)x_2$. We calculate:

$$\begin{aligned} g_2(x_1 - 1)x_2 &= x_1 x_2, \\ (g_2 - 1)(x_1 - 1)(y_1 - 1)x_2 &= x_1(y_1 - 1)x_2, \\ (g_2 - 1)(x_1 - 1)y_1 x_2 &= x_1 y_1 x_2, \end{aligned}$$

where we interchanged x_1 and y_1 in going from the second line to the third line. By applying these operators to f'' , we obtain a new function in which the factors of f_H involving the index i have been replaced by monomials. This process can be repeated for each $i \geq 3$ which occurs as an index in f_H . The resulting function is the required monomial SBF. \square

7. Proof of Theorem 1.1, examples and comparisons

We are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Theorem 5.2, the r -admissible symplectic basis functions form a basis of C_r . Hence

$$\begin{aligned} \dim(C_r) &= 1 + \text{number of non-constant } r\text{-admissible SBFs} \\ &= 1 + \sum_{(s_0, s_1, \dots, s_{t-1}) \leq (2m-r, \dots, 2m-r)} \prod_{j=0}^{t-1} \dim(S_{\ell_j}^{\lambda_j}), \end{aligned} \tag{7.1}$$

where $\lambda_j = 2s_{j+1} - s_j$ and $\ell_j = (m - r)\delta(r \leq m) + (2m - r - s_j)$. We use $a_{s_j, s_{j+1}}$ to denote $\dim(S_{\ell_j}^{\lambda_j})$. By Lemma 4.4, we have

$$a_{s_j, s_{j+1}} = \binom{2m}{2s_{j+1} - s_j} - \binom{2m}{2s_{j+1} - s_j - 2\ell_j - 2},$$

where ℓ_j is given as above. Let

$$A = (a_{i,j})_{1 \leq i \leq 2m-r, 1 \leq j \leq 2m-r}.$$

Then by (7.1) we see that $\dim(C_r)$ is equal to 1 plus the trace of A^t . The proof is complete. \square

As noted before, the 2-rank formula for the incidence between \mathcal{I}_1 and \mathcal{I}_2 when $m = 2$ and $q = 2^t$ is the same as what one obtains using the formula obtained in [4] for the p -rank of the incidence matrix of \mathcal{I}_1 and \mathcal{I}_2 when $m = 2$ and $q = p^t$, p odd, and substituting $p = 2$. The same is not true when $m > 2$, except when $m = 3$ and $q = 2$. We review the case for odd q .

Theorem 7.1. (See [4].) Let V be a $2m$ -dimensional vector space over \mathbf{F}_q , $q = p^t$, p odd, equipped with a nondegenerate alternating bilinear form, and let $C_{m,1}$ be the incidence matrix between totally isotropic m -dimensional subspaces of V and 1 -dimensional subspaces of V . Let $\lambda = pj - i$ and let

$$d_\lambda = \sum_{k=0}^{\lfloor \lambda/p \rfloor} (-1)^k \binom{2m}{k} \binom{2m-1+\lambda-kp}{2m-1}.$$

Let A' be the $m \times m$ matrix whose (i, j) -entry is

$$a'_{i,j} = \begin{cases} (d_\lambda + p^m)/2, & \text{if } i = j = m, \\ d_\lambda, & \text{otherwise.} \end{cases}$$

Then

$$\text{rank}_p(C_{r,1}) = 1 + \text{Trace}(A'^t).$$

Note that when $p = 2$, the formula for d_λ simplifies to $\binom{2m}{\lambda}$, and the matrix entry $a_{i,j}$ (for $r = m$) is $d_\lambda - \binom{2m}{2j+i-2m-2}$. The explanation that we get the same formula for the $p = 2$ and p odd cases when $m = 2$ is then that the second term (i.e., $\binom{2m}{2j+i-2m-2}$) vanishes except when $i = j = 2$, in which case it is 1, and $a_{2,2} = a'_{2,2} = 5$.

We now compare the matrices A and A' when $m = 3$. Using the formulas, we get

$$A = \begin{pmatrix} 6 & 20 & 6 \\ 1 & 15 & 14 \\ 0 & 6 & 14 \end{pmatrix} \quad A' = \begin{pmatrix} 6 & 20 & 6 \\ 1 & 15 & 15 \\ 0 & 6 & 14 \end{pmatrix}.$$

Since the two matrices differ only off the diagonal, the actual rank will be less than what is given by the p -odd model when $t > 1$. In fact, the eigenvalues of A are $\alpha_1 = 8$, $\alpha_2 = \frac{27}{2} + \frac{\sqrt{473}}{2}$, and $\alpha_3 = \frac{27}{2} - \frac{\sqrt{473}}{2}$, and the rank formula may be given as

$$\text{rank}_2(B_{3,1}(t)) = 1 + \text{Trace}(A^t) = 1 + \alpha_1^t + \alpha_2^t + \alpha_3^t.$$

The 2-ranks of $B_{3,1}(1)$, $B_{3,1}(2)$, and $B_{3,1}(3)$ are 36, 666, and 15012, respectively. The expressions for the eigenvalues of A' are not so simple, and we do not reproduce them here.

When $m > 3$, we note that $a_{m-1,m-1} < a'_{m-1,m-1}$. Thus the actual 2-rank of $B_{m,1}$ is less than that obtained using the p -odd model, even when $q = 2$.

We can also note that $a_{m,m} \leq a'_{m,m}$. The reason is that each function of class $(\rho, 0, 2m - 2\rho, 0)$, $\rho > 0$, is the sum of $2^{\rho-1}$ basis functions of the p -odd model. These have the form $(\prod_{i \in R} x_i y_i + \prod_{j \in R'} x_j y_j) \prod_{k \in T} z_k$, where $z_k = x_k$ or $z_k = y_k$, and R and R' can be chosen $2^{\rho-1}$ different ways, fixing r_1 and r'_1 , but interchanging r_γ and r'_γ at will, for $2 \leq \gamma \leq \rho$. For $\rho = 0$, the basis functions in the two models are identical.

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