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Semi-regular relative difference sets with large forbidden subgroups

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ABSTRACT

Motivated by a connection between semi-regular relative difference sets and mutually unbiased bases, we study relative difference sets with parameters $(m, n, m, m/n)$ in groups of non-prime-power orders. Let p be an odd prime. We prove that there does not exist a $(2p, p, 2p, 2)$ relative difference set in any group of order $2p^2$, and an abelian $(4p, p, 4p, 4)$ relative difference set can only exist in the group $\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$. On the other hand, we construct a family of non-abelian relative difference sets with parameters $(4q, q, 4q, 4)$, where q is an odd prime power greater than 9 and $q \equiv 1 \pmod 4$. When $q = p$ is a prime, $p > 9$, and $p \equiv 1 \pmod 4$, the $(4p, p, 4p, 4)$ non-abelian relative difference sets constructed here are genuinely non-abelian in the sense that there does not exist an abelian relative difference set with the same parameters.

1. Introduction

Let G be a finite (multiplicative) group of order mn, and let N be a subgroup of G of order n. A k-subset R of G is called an (m, n, k, λ) *relative difference set* (RDS) in G relative to N if every element $g \in G \setminus N$ has exactly λ representations $g = r_1 r_2^{-1}$ with $r_1, r_2 \in R$, and no non-identity element of N has such a representation. The subgroup N is usually called *the forbidden subgroup*. If the group G is abelian (respectively non-abelian), then D is called an *abelian* (*respectively non-abelian*) *relative difference set*. When $n = 1$, R is an (m, k, λ) difference set in the usual sense. If $k = n\lambda$, then R is said to be *semi-regular*.

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For a subset X of G, we set $X^{(-1)} = \{x^{-1} \mid x \in X\}$; also we use the same X to denote the group ring element $\sum_{x\in X} x \in \mathbb{Z}[G]$. Then, a k-subset R of G is an (m,n,k,λ) relative difference set in G relative to N if and only if it satisfies the following equation in the group ring $\mathbb{Z}[G]$:

$$RR^{(-1)} = k + \lambda(G - N).$$

Character theory is a very useful tool in the study of difference sets and relative difference sets in abelian groups. We state the Fourier inversion formula below, which will be used many times in the paper.

Inversion formula. Let G be an abelian group of order v. If $A = \sum_{g\in G} a_g g \in \mathbb{Z}[G]$, then $a_h = \frac{1}{v}\sum_{\chi\in\hat{G}} \chi(Ah^{-1})$, for all $h \in G$, where $\hat{G}$ is the group of (complex) characters of G and $\chi(Ah^{-1}) = \sum_{g\in G} a_g \chi(gh^{-1})$.

One consequence of the inversion formula is as follows. Let G be a finite abelian group, and let A and B be two elements of $\mathbb{Z}[G]$. Then we have $A = B$ if and only if $\chi(A) = \chi(B)$ for all characters χ of G. The following result is a standard characterization of relative difference sets by using their character values (cf. [4, p. 374]).

Proposition 1.1. *Let G be an abelian group of order mn with a subgroup N of order n. Let k and λ be positive integers satisfying $k(k-1) = \lambda n(m-1)$. Then a k-subset D of G is an (m,n,k,λ) difference set in G relative to N if and only if for every non-principal character χ of G,*

$$\chi(D)\overline{\chi(D)} = \begin{cases} k & \text{if } \chi|_N \neq 1, \\ k - \lambda n & \text{if } \chi|_N = 1, \end{cases} \tag{1.1}$$

where $\chi|_N$ is the restriction of χ to N.

Recently a connection between semi-regular abelian RDS and mutually unbiased bases is established in [10]. To explain the connection, we first give the definition of mutually unbiased bases. Let $\mathbb{C}$ be the field of complex numbers. A pair of bases $x_1, x_2, \dots, x_d$ and $y_1, y_2, \dots, y_d$ of $\mathbb{C}^d$ is said to be *mutually unbiased* if they are both orthonormal and

$$\left|\langle x_i, y_j\rangle\right| = \frac{1}{\sqrt{d}},$$

for all i and j, $1 \leqslant i, j \leqslant d$, where $\langle\cdot,\cdot\rangle$ is the standard inner product of $\mathbb{C}^d$. The notion of mutually unbiased bases first appeared in [20]. It has received a lot of attention in recent years because of their applications in quantum state determination [14,24] and quantum cryptography [2]. Let $N_{\mathrm{MUB}}(d)$ denote the maximum size of any set containing pairwise mutually unbiased bases (MUB) of $\mathbb{C}^d$. The main open problem about MUB is to determine $N_{\mathrm{MUB}}(d)$ for non-prime-power integers d. There are some similarities between $N_{\mathrm{MUB}}(d)$ and $N_{\mathrm{MOLS}}(d)$, the maximum number of mutually orthogonal Latin squares of size d. For example, it is known [7] that $N_{\mathrm{MUB}}(d) \leqslant d+1$; and when $d = p^e$ is a prime power it was shown [14,24] that $N_{\mathrm{MUB}}(p^e) = p^e + 1$. Also if $d = st$, then we have

$$N_{\mathrm{MUB}}(d) \geqslant \min\bigl\{N_{\mathrm{MUB}}(s), N_{\mathrm{MUB}}(t)\bigr\}. \tag{1.2}$$

For an arbitrary positive integer d and a prime p, we use $\nu_p(d)$ to denote p^α, where $p^\alpha | d$ but $p^{\alpha+1} \nmid d$. We also use $\pi(d)$ to denote the set of prime divisors of d. Then by (1.2), we have

$$N_{\mathrm{MUB}}(d) \geqslant \min_{p\in\pi(d)}\bigl\{N_{\mathrm{MUB}}\bigl(\nu_p(d)\bigr)\bigr\} = \min_{p\in\pi(d)}\bigl\{\nu_p(d) + 1\bigr\}. \tag{1.3}$$

We will refer to this construction as the *reduce to prime power construction*. For more information on $N_{\mathrm{MUB}}(d)$, we refer the reader to [1,10,14,24].

We now state a theorem in [10] which establishes a connection between semi-regular abelian RDS and mutually unbiased bases.

Theorem 1.2. *(See [10].) The existence of a semi-regular $(m,n,m,m/n)$ RDS in an abelian group implies the existence of a set of $n+1$ mutually unbiased bases of $\mathbb{C}^m$.*

We refer the reader to [10] for the proof of Theorem 1.2. A *semifield* $(S,+,*)$ is a ring with no zero-divisor, a multiplicative identity and left and right distributivity. It is known [12] that a finite commutative semifield of order q (necessarily a prime power) gives rise to an abelian $(q,q,q,1)$ relative difference set. In the case where q is an odd prime power, there is a very simple description of such an RDS. Let $(S,+,*)$ be a commutative semifield of odd order q. Let

$$R=\{(x,x*x)\mid x\in S\}.$$

Then R is a $(q,q,q,1)$ RDS in $(S\times S,+)$ relative to $\{0\}\times S$. By Theorem 1.2, one obtains $(q+1)$ MUB in $\mathbb{C}^q$ from any commutative semifield of order q. It was shown in [10] that the MUB coming from commutative semifields are equivalent to those coming from a construction by Calderbank et al. [5]. Moreover it was shown in [10] that all known sets of MUB of size $q+1$ (q a prime power) fit into this framework.

Motivated by the desire to use Theorem 1.2 to construct more MUB than the minimum in (1.3) given by the reduce to prime power construction, Wocjan [23] asked the following question: Does there exist an abelian semi-regular relative difference set with parameters $(m,n,m,m/n)$ satisfying

$$n>\min_{p\in\pi(m)}\{\nu_p(m)\}? \tag{1.4}$$

We make some preliminary observations regarding this question. First of all, most known semi-regular RDS have parameters (p^a,p^b,p^a,p^{a-b}), where p is a prime. The parameters of these RDS will not satisfy (1.4). The reason is quite straightforward. Note that if m is a prime power, then (1.4) simply becomes $n>m$. For RDS with parameters (p^a,p^b,p^a,p^{a-b}), where p is a prime, we have $p^{a-b}\geqslant 1$; hence $n=p^b\leqslant p^a=m$. Therefore to answer the question of Wocjan we have to consider semi-regular $(m,n,m,m/n)$ RDS with m not a prime power. As far as we know, prior to 2007 there are only two general constructions [6,16] of such semi-regular RDS with $n>2$. The RDS constructed in these papers have parameters

$$\left(p^{2t}(p+1),p+1,p^{2t}(p+1),p^{2t}\right), \tag{1.5}$$

where t is a positive integer, and $p=2$ or p is a Mersenne prime. Note that the parameters in (1.5) do not satisfy (1.4) either since $n=p+1$ and $\min_{r\in\pi(m)}\{\nu_r(m)\}=p+1$ (here $p=2$ or p is a Mersenne prime). Very recently the first author [8] gave a construction of $(p(p+1),p,p(p+1),p+1)$ abelian RDS, where p is a Mersenne prime. But the parameters of these RDS still do not satisfy (1.4).

Therefore we are motivated to search for semi-regular RDS with parameters $(m,n,m,m/n)$ not of the form (1.5) and m not a prime power. The simplest case to consider is when $(m,n,m,m/n)=(2p,p,2p,2)$, p an odd prime. We prove in Section 3 that there does not exist a $(2p,p,2p,2)$ RDS in any group of order $2p^2$. Next we prove that an abelian $(4p,p,4p,4)$ RDS with p an odd prime can only exist in the group $\mathbb{Z}_2^2\times\mathbb{Z}_3^2$. These non-existence results suggest that in order to use Theorem 1.2 to construct more MUB than the minimum in (1.3), one has to go beyond the simple parameter sets considered above. Some investigations in this direction were carried out by Feng [8]. On the construction side, we construct a family of $(4q,q,4q,4)$ non-abelian RDS, where q is an odd prime power greater than 9, $q\equiv 1 \pmod 4$. Since Theorem 1.2 requires the RDS to be abelian, it is not clear what implications of these non-abelian RDS have on MUB. When $q=p$ is a prime (also $p>9$ and $p\equiv 1 \pmod 4$), by the above non-existence result on abelian $(4p,p,4p,4)$ RDS, we see that the RDS constructed here are genuinely non-abelian in the sense that there does not exist an abelian RDS with the same parameters. As far as we know, this is the first infinite family of genuinely non-abelian $(m,n,m,m/n)$ relative difference sets with $n>2$.

We give some preparation results in the rest of this section. For any group G with a subgroup N, we use $C_G(N)$ to denote the *centralizer* of N in G, namely, $C_G(N)=\{x\in G\colon\ xy=yx,\ \forall y\in N\}$. Also we use $\exp(G)$ to denote the exponent of G. The following lemma on RDS is implicitly contained in [9], and has its origin in [19].

Lemma 1.3. *Let G be a group of order mn with an abelian normal subgroup N of order n, and let R be an $(m,n,m,m/n)$ RDS in G relative to N. Then $\exp(C_G(N))$ divides $2m$. Furthermore if the Sylow 2-subgroup of N is not cyclic or m/n is even, then $\exp(C_G(N))$ divides m.*

Proof. Since N is abelian, we have $C_G(N) \geqslant N$. If $C_G(N) = N$, then of course $|C_G(N)| = |N|$. Hence $\exp(C_G(N))$ divides $|N| = n$, which in turn divides m since m/n is an integer. So we will assume that $C_G(N) \neq N$ from now on. Given an element $g \in G$, we use $\bar{g}$ to denote its image in G/N. Also we use $r_{\bar{g}}$ to denote the unique element in $R \cap gN$. Now for any given $g \in C_G(N) \setminus N$, we set

$$S = \left\{(r_{\overline{gh}}, r_{\bar{h}}):\ \bar{h} \in G/N\right\}.$$

We have $|S| = m$. Since N is normal in G, we see that for any pair $(r_1, r_2) \in S$, $r_1 r_2^{-1} \in gN$. Next we claim that each gu, where $u \in N$, can be represented as $gu = r_1 r_2^{-1}$, for m/n pairs $(r_1, r_2) \in S$. This claim can be seen as follows. Since R is an $(m,n,m,m/n)$ RDS in G relative to N, each gu, $u \in N$, can be represented as $gu = xy^{-1}$, for m/n pairs $(x, y) \in R \times R$. Let $y = hu'$, where $u' \in N$. Then $x = guhu' = gh(h^{-1}uh)u'$. Since N is normal in G, we have $h^{-1}uh \in N$. Hence $x \in R \cap ghN$. The claim is proved. It follows that,

$$g^m\left(\prod_{u \in N} u\right)^{m/n} = \prod_{u \in N}(gu)^{m/n} = \prod_{(r_1,r_2)\in S} r_1 r_2^{-1}.$$

Now using the assumption that $g \in C_G(N)$, we can arrange the terms in the last product above in such a way that $r_1 r_2^{-1}$ is followed by $r_2 r_3^{-1}$, and so on. Therefore we have

$$g^m\left(\prod_{u \in N} u\right)^{m/n} = 1.$$

The element $a := \prod_{u \in N} u$ has order at most 2. So $g^{2m} = 1$. Hence $\exp(C_G(N))$ divides $2m$. If the Sylow 2-subgroup of N is not cyclic, then N has at least two elements of order 2; hence $a = 1$. Therefore we have $g^m = 1$ and $\exp(C_G(N))$ divides m. If m/n is even, then clearly we have $g^m = 1$ and $\exp(C_G(N))|m$. The proof is complete. □

Let p be a prime and $f : \mathbb{Z}_p^n \to \mathbb{Z}_p$ be a function. The *Fourier transform* $\hat{f}$ of f is defined by

$$\hat{f}(\mathbf{b}) = \sum_{\mathbf{x} \in \mathbb{Z}_p^n} \xi_p^{f(\mathbf{x}) + \mathbf{b}\cdot\mathbf{x}}, \quad \forall \mathbf{b} \in \mathbb{Z}_p^n,$$

where $\mathbf{b}\cdot\mathbf{x}$ is the standard dot product and ξ_p is a primitive pth root of unity in $\mathbb{C}$. The function f is said to be *p-ary bent* if $|\hat{f}(\mathbf{b})| = p^{n/2}$ for all $\mathbf{b} \in \mathbb{Z}_p^n$. In Section 4, we will need the following theorem from [11].

Theorem 1.4. *(See [11].) Let p be an odd prime. Then a function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is p-ary bent if and only if* $\deg(f) = 2$.

Throughout this paper, we fix the following notation: For a multiplicative group G, we denote its identity by 1_G, or simply by 1 if there is no confusion. For a positive integer m, ξ_m denotes a primitive mth root of unity in $\mathbb{C}$. For an odd prime p, $(\frac{\cdot}{p})$ is the Legendre symbol; also we let

$$\Delta = \sum_{x \in \mathbb{Z}_p} \xi_p^{x^2} = \sum_{i=0}^{p-1}\left(\frac{i}{p}\right)\xi_p^i.$$

It is well known [15] that $\Delta\bar{\Delta} = p$ and $\Delta = \pm\sqrt{p^*}$, where $p^* = (-1)^{\frac{p-1}{2}} p$. For an integer t such that $p \nmid t$, we use σ_t to denote the element in $Gal(\mathbb{Q}(\xi_p)/\mathbb{Q})$ that maps ξ_p to ξ_p^t. We have $\sigma_t(\Delta) = (\frac{t}{p})\Delta$. We will use standard facts on prime ideal decompositions of rational integers in cyclotomic fields freely. The readers are referred to [15,18,22] for proofs of these facts.

2. A construction of $(4q, q, 4q, 4)$ RDS in non-abelian groups

In this section, we construct a family of $(4q, q, 4q, 4)$ RDS in certain non-abelian groups of order $4q^2$, where q is an odd prime power, $q \equiv 1 \pmod 4$, and $q > 9$.

For prime power $q = p^n$, $n \geqslant 1$, p an odd prime, let $K := \mathbb{F}_q$ be the finite field of order q, $K^* = K \setminus \{0\}$, and $\operatorname{tr}: K \to \mathbb{F}_p$ be the absolute trace function. The quadratic character η on K is defined by

$$\eta(x) = \begin{cases} 1 & \text{if } x \text{ is a non-zero square of } K, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x \text{ is a non-square of } K. \end{cases}$$

For $u \in K^*$, we define

$$S(u) := \sum_{x \in K} \xi_p^{\operatorname{tr}(ux^2)}.$$

For simplicity, we write S for $S(1)$. We have $S + S(u) = 2\sum_{x \in K} \xi_p^{\operatorname{tr}(x)} = 0$ if u is a non-square of K. Therefore $S(u) = \eta(u)S$ for every $u \in K^*$.

The *quadratic Gauss sum* $g(\eta)$ is defined by

$$g(\eta) := \sum_{x \in K} \eta(x)\xi_p^{\operatorname{tr}(x)}.$$

Straightforward computations show that $g(\eta) = S$. Therefore

$$S\bar{S} = g(\eta)\overline{g(\eta)} = q,$$

cf. [3, p. 11].

In the rest of this section we assume that $q \equiv 1 \pmod 4$, e, f are elements of K satisfying $e^4 = 1$, $f^2 = -1$, respectively.

Given an element $s_2 \in K^*$, we define

$$s_1 = \frac{1}{2}\left((1 + s_2) + \frac{f}{e^2}(1 - s_2)\right),$$
$$s_3 = \frac{1}{2}\left((1 + s_2) - \frac{f}{e^2}(1 - s_2)\right).$$

Lemma 2.1. *If $q > 9$, then there exists $s_2 \in K^*$ such that*

$$\eta(s_1 s_2 s_3) = -1.$$

Proof. First, note that if $s_2 \neq \frac{f+1}{f-1}$ or $\frac{f-1}{f+1}$, then $s_1 \neq 0$ and $s_3 \neq 0$. Secondly,

$$s_1 s_2 s_3 = \frac{s_2}{4}\left((1 + s_2)^2 - \frac{f^2}{e^4}(1 - s_2)^2\right) = \frac{s_2}{2}\left(1 + s_2^2\right).$$

Hence the number of $s_2 \in K^*$ satisfying $\eta(s_1 s_2 s_3) = \eta(2s_2(1 + s_2^2)) = -1$ is at least

$$\sum_{x \in K^*} \frac{1 - \eta(2x(1 + x^2))}{2} - 2 = \frac{1}{2}\left(q - 5 - \sum_{x \in K^*} \eta\left(2x + x^3\right)\right). \tag{2.1}$$

By Theorem 5.41 in [17, p. 225], we have

$$\left|\sum_{x \in K^*} \eta\left(2x + x^3\right)\right| \leqslant 2\sqrt{q}.$$

Therefore, if $q > 9$, then the quantity in (2.1) is positive. The lemma now follows. □

Fix $e, f \in K^*$ as above. Let $H = K \times K$, $N = \{0\} \times K \leqslant H$, and

$$G = \big\langle x, H \;\big|\; x^4 = 1,\ (u, v)^x = (eu, fv),\ \forall (u, v) \in H \big\rangle,$$

where $(u, v)^x$ stands for $x^{-1}(u, v)x$. With s_1, s_2, s_3 as given in Lemma 2.1, we define

$$R := R_0 + R_1 x + R_2 x^2 + R_3 x^3 \in \mathbb{Z}[G], \tag{2.2}$$

where $R_0 = \{(y, y^2) \mid y \in K\}$, $R_1 = \{(y, \frac{1}{s_1} y^2) \mid y \in K\}$, $R_2 = \{(y, \frac{1}{s_2} y^2) \mid y \in K\}$, and $R_3 = \{(y, \frac{1}{s_3} y^2) \mid y \in K\}$.

Theorem 2.2. *Let q be a prime power such that $q \equiv 1 \pmod 4$ and $q > 9$. Then R is a $(4q, q, 4q, 4)$ RDS in G relative to N.*

Proof. For $(u, v) \in H$, let $\chi_{u,v}$ be the character of H defined by

$$\chi_{u,v}(u', v') = \xi_p^{\operatorname{tr}(uu' + vv')}, \quad \forall (u', v') \in H.$$

For notational convenience, we set $s_0 = 1$. Let $(u, v) \neq (0, 0)$. For each i, $0 \leqslant i \leqslant 3$, we have the following facts.

Fact 1. If $v \neq 0$, then for any i, $0 \leqslant i \leqslant 3$,

$$\begin{aligned}
\chi_{u,v}(R_i) &= \sum_{y \in K} \xi_p^{\operatorname{tr}(uy + \frac{v}{s_i} y^2)} \\
&= \sum_{y \in K} \xi_p^{\operatorname{tr}(\frac{v}{s_i}(y + \frac{us_i}{2v})^2 - \frac{u^2 s_i}{4v})} \\
&= \eta(v)\eta(s_i) S \xi_p^{-\operatorname{tr}(\frac{u^2 s_i}{4v})}.
\end{aligned}$$

Fact 2. If $u \neq 0$, then $\chi_{u,0}(R_i) = \sum_{y \in K} \xi_p^{\operatorname{tr}(uy)} = 0$.

Fact 3. We have $\chi_{u,v}(R_i^{(-x^k)}) = \overline{\chi_{e^k u, f^k v}(R_i)}$, where $R_i^{(-x^k)} = \sum_{y \in R_i} x^{-k} y^{-1} x^k$, and $k \geqslant 1$.

To prove the theorem, we will show that $RR^{(-1)} = 4q + 4(G - N)$, which is equivalent to the following system of group ring equations in $\mathbb{Z}[H]$:

$$\begin{aligned}
&R_0 R_0^{(-1)} + R_1 R_1^{(-1)} + R_2 R_2^{(-1)} + R_3 R_3^{(-1)} = 4q + 4(H - N), \\
&R_0 R_1^{(-x)} + R_1 R_2^{(-x)} + R_2 R_3^{(-x)} + R_3 R_0^{(-x)} = 4H, \\
&R_0 R_2^{(-x^2)} + R_2 R_0^{(-x^2)} + R_1 R_3^{(-x^2)} + R_3 R_1^{(-x^2)} = 4H, \\
&R_0 R_3^{(-x^3)} + R_1 R_0^{(-x^3)} + R_2 R_1^{(-x^3)} + R_3 R_2^{(-x^3)} = 4H.
\end{aligned}$$

Note that the fourth equation can be obtained from the second one by first applying $h \mapsto h^{-1}$, $\forall h \in H$, to both sides of the second equation and then conjugating both sides of the resulting equation by x^3. Therefore it suffices to show that the first three equations hold in $\mathbb{Z}[H]$. We will do so by proving that the left-hand side and the right-hand side of each of the first three equations have the same character values for all characters of H. This can be checked easily for the principal character of H. Now let $\chi_{u,v}$ be an arbitrary non-principal character of H. For simplicity write $\chi = \chi_{u,v}$, $\chi_1 = \chi_{eu, fv}$, $\chi_2 = \chi_{e^2 u, f^2 v}$. Let

$$\begin{aligned}
(a, b, c, d) &= \big(\chi(R_0), \chi(R_1), \chi(R_2), \chi(R_3)\big), \\
(a', b', c', d') &= \big(\chi_1(R_0), \chi_1(R_1), \chi_1(R_2), \chi_1(R_3)\big), \\
(a'', b'', c'', d'') &= \big(\chi_2(R_0), \chi_2(R_1), \chi_2(R_2), \chi_2(R_3)\big).
\end{aligned}$$

By Fact 3, in order to prove the theorem, it suffices to show that

$$a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}=4q-4\chi(N),$$
$$a\overline{b'}+b\overline{c'}+c\overline{d'}+d\overline{a'}=0,$$
$$a\overline{c''}+c\overline{a''}+b\overline{d''}+d\overline{b''}=0.$$

If $v=0$, then χ is principal on N. Hence $\chi(N)=q$, and $a=b=c=d=0$. We see that all three equations above hold in this case.

If $v\neq 0$, then χ is non-principal on N. Hence $\chi(N)=0$. Using Fact 1, we see that

$$a\bar{a}=b\bar{b}=c\bar{c}=d\bar{d}=S\bar{S}=q.$$

Therefore we have $a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}=4q-4\chi(N)$ in this case. Next we will show that

$$a\overline{b'}+c\overline{d'}=0,$$
$$b\overline{c'}+d\overline{a'}=0,$$

from which it follows that $a\overline{b'}+b\overline{c'}+c\overline{d'}+d\overline{a'}=0$. We compute $a\overline{b'}+c\overline{d'}$ as follows.

$$\begin{aligned}a\overline{b'}+c\overline{d'}&=q\eta(f)\eta(s_1)\xi_p^{-\mathrm{tr}(\frac{u^2}{4v}-\frac{e^2u^2s_1}{4fv})}+q\eta(f)\eta(s_2s_3)\xi_p^{-\mathrm{tr}(\frac{u^2s_2}{4v}-\frac{e^2u^2s_3}{4fv})}\\&=q\eta(f)\big(\eta(s_1)\xi_p^{-\mathrm{tr}(\frac{u^2}{4v}-\frac{e^2u^2s_1}{4fv})}+\eta(s_2s_3)\xi_p^{-\mathrm{tr}(\frac{u^2s_2}{4v}-\frac{e^2u^2s_3}{4fv})}\big).\end{aligned}\tag{2.3}$$

Note that

$$\frac{u^2}{4v}-\frac{e^2u^2s_1}{4fv}=\frac{u^2}{4fv}\big(f-e^2s_1\big),$$
$$\frac{u^2s_2}{4v}-\frac{e^2u^2s_3}{4fv}=\frac{u^2}{4fv}\big(fs_2-e^2s_3\big).$$

By the definitions of s_1 and s_3, we have $(f-e^2s_1)=(fs_2-e^2s_3)$. Therefore, $\frac{u^2}{4v}-\frac{e^2u^2s_1}{4fv}=\frac{u^2s_2}{4v}-\frac{e^2u^2s_3}{4fv}$. Also, by Lemma 2.1, $\eta(s_1)=-\eta(s_2s_3)$. Combining these two facts, we see from (2.3) that $a\overline{b'}+c\overline{d'}=0$. Similarly, one can show that $b\overline{c'}+d\overline{a'}=0$. Therefore we have shown that $a\overline{b'}+b\overline{c'}+c\overline{d'}+d\overline{a'}=0$.

To finish the proof we will show that

$$a\overline{c''}+b\overline{d''}=0,$$
$$c\overline{a''}+d\overline{b''}=0.$$

We compute $a\overline{c''}+b\overline{d''}$ as follows.

$$\begin{aligned}a\overline{c''}+b\overline{d''}&=q\eta\big(f^2\big)\eta(s_2)\xi_p^{-\mathrm{tr}(\frac{u^2}{4v}+\frac{u^2s_2}{4v})}+q\eta\big(f^2\big)\eta(s_1s_3)\xi_p^{-\mathrm{tr}(\frac{u^2s_1}{4v}+\frac{u^2s_3}{4v})}\\&=q\big(\eta(s_2)\xi_p^{-\mathrm{tr}(\frac{u^2}{4v}+\frac{u^2s_2}{4v})}+\eta(s_1s_3)\xi_p^{-\mathrm{tr}(\frac{u^2s_1}{4v}+\frac{u^2s_3}{4v})}\big).\end{aligned}\tag{2.4}$$

By the definitions of s_1 and s_3, we have $s_2+1=s_1+s_3$. Hence $\frac{u^2}{4v}+\frac{u^2s_2}{4v}=\frac{u^2s_1}{4v}+\frac{u^2s_3}{4v}$. Also by Lemma 2.1, $\eta(s_2)=-\eta(s_1s_3)$. Combining these two facts, we see from (2.4) that $a\overline{c''}+b\overline{d''}=0$. Similarly, we can show that $c\overline{a''}+d\overline{b''}=0$. It follows that $a\overline{c''}+c\overline{a''}+b\overline{d''}+d\overline{b''}=0$. The proof is now complete. □

Remark. When $q=p$ is a prime, $p\equiv 1 \pmod 4$, $p>9$, we have constructed a $(4p,p,4p,4)$ RDS in groups G'_{13} $(e=-f)$, G_{14} $(e=1)$, G_{15} $(e=-1)$, G_{16} $(e=f)$ as listed in [13].

3. Non-existence of $(2p, p, 2p, 2)$ RDS in groups of order $2p^2$

Throughout this section p is an odd prime. We will show that there does not exist a $(2p, p, 2p, 2)$ RDS in any group of order $2p^2$.

Let G be a group of order $2p^2$. Then G has a unique Sylow p-subgroup H of order p^2. (This is an easy consequence of Sylow's theorems.) Hence H is a normal subgroup of G.

We first consider the case where H is cyclic. In this case, H has a unique subgroup N of order p. Hence N is a normal subgroup of G. Also $C_G(N) \geqslant H$. If R is a $(2p, p, 2p, 2)$ RDS in G relative to N, then by Lemma 1.3, we have $p^2 | 2p$, which is impossible. So from now on, we assume that H is not cyclic, say $H = \langle a, b\colon\ a^p = b^p = 1,\ [a, b] = 1\rangle$.

Let $c \in G$ be an element of order 2. Then G is a semidirect product of H and $\{1, c\}$. Since $\mathrm{Aut}(H) \cong GL_2(\mathbb{F}_p)$, and every element of order 2 in $GL_2(\mathbb{F}_p)$ is conjugate to a diagonal matrix with ± 1's on the diagonal, there are three isomorphism types of semidirect products of H and $\{1, c\}$. Below we list the three non-isomorphic groups of order $2p^2$ with non-cyclic Sylow p-subgroup H:

$$G_1 = \big\langle a, b, c\colon\ a^p = b^p = c^2 = 1,\ [a, b] = 1,\ a^c = a^{-1},\ b^c = b^{-1}\big\rangle;$$

$$G_2 = \big\langle a, b, c\colon\ a^p = b^p = c^2 = 1,\ [a, b] = 1,\ a^c = a^{-1},\ [b, c] = 1\big\rangle;$$

$$G_3 = \big\langle a, b, c\colon\ a^p = b^p = c^2 = 1,\ [a, b] = [a, c] = [b, c] = 1\big\rangle.$$

In each G_i, $i = 1, 2, 3$, we consider the orbits of subgroups of order p under the action of the full automorphism group $\mathrm{Aut}(G_i)$. There is only one orbit of subgroups order p in G_1 and G_3, and there are three such orbits in G_2. We list the orbit representatives as follows:

(1) $G = G_1$, $N = \langle a\rangle$;
(2) $G = G_3$, $N = \langle a\rangle$;
(3) $G = G_2$, $N = \langle a\rangle$;
(4) $G = G_2$, $N = \langle b\rangle$;
(5) $G = G_2$, $N = \langle ab\rangle$.

We remark that case (5) is the only case where N is not a normal subgroup of G.

The following lemma will play an important role in our non-existence proof.

Lemma 3.1. *Let p be an odd prime, and let $a_0, a_1, \ldots, a_{p-1}$ be non-negative integers such that $\sum_{i=0}^{p-1} a_i = p$. If $A = \sum_{i=0}^{p-1} a_i \xi_p^i$ has modulus $\sqrt{2p}$, then $p = 7$, $a_s = 4$, $a_{2^i t+s} = 1$, $0 \leqslant i \leqslant 2$, for some integers s, t, $0 \leqslant s \leqslant 6$, $1 \leqslant t \leqslant 6$, and $a_j = 0$ for the rest j's.*

Proof. Since $A\bar{A} = 2p$, we have

$$(A)(\bar{A}) = (2)(p) = (2)(1 - \xi_p)^{p-1},$$

as ideals in $\mathbb{Z}[\xi_p]$. Since the ideal $(1 - \xi_p)$ is fixed by $\xi_p \mapsto \xi_p^{-1}$, we have

$$(1 - \xi_p)^{(p-1)/2} \mid (A).$$

Recall that $\Delta\bar{\Delta} = p$, $\bar{\Delta} = (\frac{-1}{p})\Delta$, we have $(\Delta) = (1 - \xi_p)^{(p-1)/2}$. Hence $(\Delta)|(A)$, and we may write

$$A = f(\xi_p)\Delta, \tag{3.1}$$

where $f(\xi_p) = \sum_{i=0}^{p-1} b_i \xi_p^i$ and $f(\xi_p)\overline{f(\xi_p)} = 2$, $b_i \in \mathbb{Z}$.

Multiplying both sides of (3.1) by $\bar{\Delta}$, we have

$$\left(\sum_{i=0}^{p-1} a_i \xi_p^i\right)\left(\sum_{i=0}^{p-1}\left(\frac{-i}{p}\right)\xi_p^i\right) = p\left(\sum_{i=0}^{p-1} b_i \xi_p^i\right). \tag{3.2}$$

Comparing the coefficients of ξ_p^k, $k=0,1,\dots,(p-1)$, on both sides of (3.2), we find that there exists some $c\in\mathbb{Z}$ such that

$$\sum_i a_{k-i}\left(\frac{-i}{p}\right)=pb_k-c,\quad \forall k=0,1,\dots,(p-1).$$

Summing these equations over k, we get $c=\sum_{k=0}^{p-1}b_k$. Since $(\sum_i b_i\xi_p^i)(\sum_i b_i\xi_p^{-i})=2$, we have

$$c^2=\left(\sum_{i=0}^{p-1}b_i\right)^2\equiv 2\pmod{(1-\xi_p)\cap\mathbb{Z}}.$$

That is, $c^2\equiv 2\pmod{p}$. Hence $\ell:=c\pmod{p}\neq 0$. Write $c=pc_1+\ell$. Note that for all $k=0,1,\dots,(p-1)$, on one hand we have $|\sum_i a_{k-i}(\frac{-i}{p})|\leqslant\sum_{i\neq 0}a_{k-i}=p-a_k\leqslant p$, and on the other hand $|\sum_i a_{k-i}(\frac{-i}{p})|=|pb_k-c|=|p(b_k-c_1)-\ell|$. So we must have $\delta_k:=b_k-c_1=1$ or 0, for all $k=0,1,\dots,(p-1)$. Also since

$$pc_1+\ell=\sum_{k=0}^{p-1}b_k=\sum_{k=0}^{p-1}(c_1+\delta_k),$$

we have $\sum_k\delta_k=\ell$. Hence exactly ℓ of the δ_k's are equal to 1. It follows that $\sum_k b_k\xi_p^k=\sum_{j=1}^{\ell}\xi_p^{i_j}$. Let $S=\{i_j\colon 1\leqslant j\leqslant\ell\}\subset\mathbb{Z}_p$. Define $S(x)=\sum_{j=1}^{\ell}x^{i_j}\in\mathbb{Z}[x]/(x^p-1)$. Then

$$S(x)S(x^{-1})=2+\lambda T(x),$$

where $T(x)=1+x+x^2+\dots+x^{p-1}$, and λ is some non-negative integer. It follows that $\lambda=\ell-2$ and $\ell^2=2+\lambda p$. We then have $\lambda^2+(4-p)\lambda+2=0$. Hence $\lambda=1$ or 2, and $p=7$.

If $\lambda=1$, then S is a $(7,3,1)$ difference set in $\mathbb{Z}_7$. Since 2 is a multiplier of S (see [4, p. 323]), we have $S=\{t+s,2t+s,4t+s\}$ for some integers s,t, where $1\leqslant t\leqslant 6$. Now using $\sum_{i=0}^{6}a_i\xi_7^i=(\sum_{i=0}^{6}(\frac{i}{7})\xi_7^i)(\xi_7^{t+s}+\xi_7^{2t+s}+\xi_7^{4t+s})$, we find that there are no solutions for the a_i's when $(\frac{t}{p})=1$; and there is a unique set of solutions: $a_s=4$, $a_{2^it+s}=1$, $0\leqslant i\leqslant 2$, and $a_j=0$ for the remaining j's when $(\frac{t}{p})=-1$.

In the case where $\lambda=2$, similarly, we find that there are no solutions for the a_i's when $(\frac{t}{p})=-1$; and there is a unique set of solutions: $a_s=4$, $a_{2^it+s}=1$, $0\leqslant i\leqslant 2$, and $a_j=0$ for the remaining j's, when $(\frac{t}{p})=1$. □

We are now ready to state the main theorem in this section.

Theorem 3.2. *Let p be an odd prime. Then there does not exist a $(2p,p,2p,2)$ RDS in any group of order $2p^2$.*

Proof. By the analysis preceding Lemma 3.1, we only need to consider the five cases listed before Lemma 3.1. We use the same notation as in the discussion at the beginning of this section. Suppose R is a putative $(2p,p,2p,2)$ RDS in G relative to N. Write $R=R_1+R_2c$, where $R_i\in\mathbb{Z}[H]$, $H=\langle a\rangle\times\langle b\rangle\cong\mathbb{Z}_p\times\mathbb{Z}_p$. Then $RR^{(-1)}=2p+2(G-N)$. Hence we have

$$R_1R_1^{(-1)}+R_2R_2^{(-1)}=2p+2(H-N),\qquad R_1R_2^{(-c)}+R_2R_1^{(-c)}=2H.$$

Applying the principal character of H to the above equations, we find that $|R_1|=|R_2|=p$.

We now consider the five cases one by one.

Case 1. $G=G_1$ and $N=\langle a\rangle$. In this case we have $R_1R_2^{(-c)}=R_1R_2$. Hence $R_1R_1^{(-1)}+R_2R_2^{(-1)}=2p+2(H-N)$ and $R_1R_2=H$. For any $\chi\in\hat{H}$ whose restriction on N is non-principal, we have

$$\chi(R_1)\chi(R_2)=0,$$
$$\chi(R_1)\overline{\chi(R_1)}+\chi(R_2)\overline{\chi(R_2)}=2p.$$

Hence $|\chi(R_1)|^2=2p$ or 0. Let $S_1=\{\chi\in\hat{H}\colon \chi$ is non-principal on N and $|\chi(R_1)|^2=2p\}$. It is clear that the coefficient of 1_H in $R_1R_1^{(-1)}$ is $|R_1|=p$. This coefficient can also be calculated by using the inversion formula. We therefore have

$$p=\frac{1}{p^2}\sum_{\chi\in\hat{H}}\chi\big(R_1R_1^{(-1)}\big)=\frac{1}{p^2}\big(p^2+2p|S_1|\big).$$

It follows that $|S_1|=\frac{p(p-1)}{2}$. Now note that $Gal(\mathbb{Q}(\xi_p)/\mathbb{Q})$ acts on $\hat{H}$, and S_1 is fixed (setwise) under this action. Therefore S_1 is partitioned into orbits under the aforementioned action, each having size $p-1$. So $|S_1|\equiv 0 \pmod{p-1}$. But this is impossible since $|S_1|=\frac{p(p-1)}{2}$. We have reached the desired contradiction.

Case 2. $G=G_3$ and $N=\langle a\rangle$. In this case, the group G is abelian. For any $\chi\in\hat{H}$ whose restriction to N is non-principal, we have $|\chi(R_1\pm R_2)|^2=2p$. From the proof of Lemma 3.1, we have $\chi(R_1+R_2)=f_1\Delta$ and $\chi(R_1-R_2)=f_2\Delta$, where $f_i\in\mathbb{Z}[\xi_p]$ and $|f_i|^2=2$, for $i=1,2$. Since $(f_1-f_2)\Delta=2\chi(R_2)$, we have $2|(f_1-f_2)$ in $\mathbb{Z}[\xi_p]$. Let $f_2=f_1+2x$ for some $x\in\mathbb{Z}[\xi_p]$. Multiplying both sides of this equation by $\bar{f}_1$, we have

$$\bar{f}_1f_2=\bar{f}_1f_1+2x\bar{f}_1=2+2x\bar{f}_1.$$

So $2|\bar{f}_1f_2$. Let $\bar{f}_1f_2=2y$ for some $y\in\mathbb{Z}[\xi_p]$. Multiplying both sides of the equation by f_1, we obtain $f_2=f_1y$. Since both f_1 and f_2 have modulus $\sqrt{2}$, we have $f_1=\eta f_2$ for some root of unity $\eta\in\mathbb{Z}[\xi_p]$. Now $2\chi(R_1)=(f_1+f_2)\Delta=f_2(1+\eta)\Delta$. Multiplying this equation by its own complex conjugate, we find that $2|(1+\eta)\overline{(1+\eta)}$. Recall that η is a root of unity in $\mathbb{Z}[\xi_p]$ and $\gcd((2),(1-\xi_p))=1$, we see that $\eta=\pm1$. It follows that $|\chi(R_1)|^2=0$ or $2p$. Now the same arguments as those in the first case yield a contradiction.

Case 3. $G=G_2$ and $N=\langle a\rangle$. For any $(u,v)\in\mathbb{Z}_p^2$, we denote by $\chi_{u,v}$ the character of H defined by $\chi_{u,v}(a^{u'}b^{v'})=\xi_p^{uu'+vv'}$. Then $\chi_{u,v}((a^ib^j)^c)=\chi_{-u,v}(a^ib^j)$. So $\chi_{u,v}(R_i^{(-c)})=\chi_{u,-v}(R_i)$ for $i=1,2$. Let $\chi\in\hat{H}$ and $\chi|_N\neq1$. If χ is principal on $\langle b\rangle$, then from $R_1R_2^{(-c)}+R_2R_1^{(-c)}=2H$ we deduce that $\chi(R_1)\chi(R_2)=0$. Without loss of generality we assume that $\chi(R_1)=0$. Then $\chi(R_2)$ has modulus $\sqrt{2p}$. Since R_2 has size p, we have $p=7$ by Lemma 3.1. Noting that the characters $\chi_{u,0}$ with $u\in\mathbb{Z}_p^*$ form a single orbit of size $(p-1)$ under the action of $Gal(\mathbb{Q}(\xi_p)/\mathbb{Q})$, we have $\chi_{u,0}(R_1)=0$ for all $u\in\mathbb{Z}_p^*$.

From $R_1R_1^{(-1)}+R_2R_2^{(-1)}=2p+2(H-N)$, we have $R_1^{(c)}R_1^{(-c)}+R_2^{(c)}R_2^{(-c)}=2p+2(H-N)$. Now, apply a character χ which is non-principal on N to these group ring equations, we have

$$\big|\chi\big(R_1^{(c)}\big)\big|^2+\big|\chi\big(R_2^{(c)}\big)\big|^2=2p,\qquad \big|\chi(R_1)\big|^2+\big|\chi(R_2)\big|^2=2p,$$
$$\chi(R_1)\overline{\chi\big(R_2^{(c)}\big)}+\chi(R_2)\overline{\chi\big(R_1^{(c)}\big)}=\chi(R_1)\chi\big(R_2^{(-c)}\big)+\chi(R_2)\chi\big(R_1^{(-c)}\big)=0.$$

From the last equation, we have

$$\big|\chi(R_1)\big|^2\big|\chi\big(R_2^{(-c)}\big)\big|^2=\big|\chi(R_2)\big|^2\big|\chi\big(R_1^{(-c)}\big)\big|^2.$$

Substitute $|\chi(R_1)|^2$ by $2p-|\chi(R_2)|^2$, and $|\chi(R_1^{(-c)})|^2$ by $2p-|\chi(R_2^{(-c)})|^2$ in the above equation, we obtain

$$\big(2p-\big|\chi(R_2)\big|^2\big)\big|\chi\big(R_2^{(-c)}\big)\big|^2=\big|\chi(R_2)\big|^2\big(2p-\big|\chi\big(R_2^{(-c)}\big)\big|^2\big),$$

which simplifies to $|\chi(R_2)|^2=|\chi(R_2^{(-c)})|^2$. Similarly, we can show that $|\chi(R_1)|^2=|\chi(R_1^{(-c)})|^2$. Hence

$$\big|\chi_{u,v}(R_i)\big|=\big|\chi_{-u,-v}(R_i)\big|=\big|\chi_{-u,v}(R_i)\big|=\big|\chi_{u,-v}(R_i)\big|.$$

Thus the characters of H that are principal on neither N nor $\langle b\rangle$ are partitioned into subsets of size four of the form $\{\chi_{\epsilon_1 u,\epsilon_2 v}\colon\ \epsilon_1,\epsilon_2=\pm 1\}$, $u,v\in\mathbb{Z}_p^*$, where $|\chi_{\epsilon_1 u,\epsilon_2 v}(R_i)|=|\chi_{u,v}(R_i)|$. Now computing the coefficient of 1_H in $R_1R_1^{(-1)}$ by the inversion formula, we have

$$p=\frac{1}{p^2}\left(p^2+\sum_{u\in\mathbb{Z}_p^*}\left|\chi_{u,0}(R_1)\right|^2+\sum_{v\in\mathbb{Z}_p^*}\left|\chi_{0,v}(R_1)\right|^2+4x\right)=\frac{1}{p^2}\left(p^2+4x\right)$$

for some algebraic integer x. Hence $4|(p-1)$. But $p=7$: we have reached a contradiction.

Case 4. $G=G_2$ and $N=\langle b\rangle$. Let $\chi\in\hat{H}$ and $\chi|_N\neq 1$. If χ is principal on $\langle a\rangle$, then $\chi(R_i^{(-1)})=\chi(R_i^{(-c)})$ for $i=1,2$. By the same arguments as those in Case 2, we have $|\chi(R_1)|^2=2p$ or 0. In the former case, since $|R_1|=p$, we have $p=7$ by Lemma 3.1. In the latter case, we have $|\chi(R_2)|^2=2p$. Again since $|R_2|=p$, we have $p=7$ by Lemma 3.1. Now the same arguments as those in the third case yield a contradiction.

Case 5. $G=G_2$ and $N=\langle ab\rangle$. Let χ_1 be the character of H which maps a to 1 and b to ξ_p. Then χ_1 is non-principal on N. Since $\chi_1|_{\langle a\rangle}=1$, we have $\chi_1(R_i^{(-1)})=\chi_1(R_i^{(-c)})$. Using the same arguments as those in Case 2, we have $|\chi_1(R_1)|^2=2p$ or $|\chi_1(R_2)|^2=2p$. Without loss of generality we assume that $|\chi_1(R_1)|^2=2p$. Since $|R_1\cap a^iN|=1$ for all $i=0,1,\ldots,(p-1)$, we can find a map $F_1:\mathbb{Z}_p\to\mathbb{Z}_p$ such that

$$R_1=\left\{a^{x+F_1(x)}b^{F_1(x)}\colon\ x\in\mathbb{Z}_p\right\}.$$

Let $a_i=|\{x\in\mathbb{Z}_p\colon\ F_1(x)=i\}|$. Then $\sum_{i=0}^{p-1}a_i=p$, $a_i\geqslant 0$, and $\chi_1(R_1)=\sum_{i=0}^{p-1}a_i\xi_p^i$. Since $|\chi_1(R_1)|^2=2p$, by Lemma 3.1, we have $p=7$, $a_s=4, a_{2^it+s}=1, 0\leqslant i\leqslant 2$, and $a_j=0$ for the remaining j, where s,t are two integers, $0\leqslant s\leqslant 6$ and $1\leqslant t\leqslant 6$. Assume that $F_1^{-1}(s)=\{i_1,i_2,i_3,i_4\}$, $F_1^{-1}(t+s)=\{i_5\}$, $F_1^{-1}(2t+s)=\{i_6\}$, $F_1^{-1}(4t+s)=\{i_7\}$. Now let χ_2 to be the character which maps a to ξ_p and b to 1. Then $\chi_2(R_i^{(-1)})=\chi_2(R_i)$. Combining this with $R_1R_2^{(-c)}+R_2R_1^{(-c)}=2H$, we deduce that $\chi_2(R_1)\chi_2(R_2)=0$. Hence $|\chi_2(R_1)|^2=0$ or 14. That is,

$$\chi_2(R_1)\xi_7^{-s}=\left(\sum_{j=1}^{4}\xi_7^{i_j}\right)+\xi_7^{i_5+t}+\xi_7^{i_6+2t}+\xi_7^{i_7+4t}$$

has modulus $\sqrt{14}$ or 0. We assume that t is a non-square of $\mathbb{Z}_7$. The case where t is a non-zero square in $\mathbb{Z}_7$ can be handled similarly.

We first consider the case where $|\chi_2(R_1)|^2=14$. Define

$$S(x):=\left(\sum_{j=1}^{4}x^{i_j}\right)+x^{i_5+t}+x^{i_6+2t}+x^{i_7+4t}\in\mathbb{Z}[x]/\left(x^7-1\right).$$

Then $S(x)S(x^{-1})=14+\lambda T(x)$, where $T(x)=1+x+x^2+\cdots+x^6$ and λ is a non-negative integer. It follows that $\lambda=\frac{1}{7}(7^2-14)=5$. Write $S(x)=\sum_{i=0}^{6}c_ix^i$. Since the i_j's are distinct, we have $0\leqslant c_i\leqslant 4$, for all i. Also $\sum_{i=0}^{6}c_i=7$ and $\sum_{i=0}^{6}c_i^2=19$. From these constraints, we find that there is only one possibility, namely $\{c_0,c_1,\ldots,c_6\}=\{4,1,1,1,0,0,0\}$. We may assume that $c_{i_1}=4$. It follows that $i_1=i_5+t=i_6+2t=i_7+4t$. After replacing R_1 by $a^{-i_1}b^{-s}R_1$ if necessary, we may assume that $i_1=0$ and $s=0$. In order for $(\sum_{j=2}^{4}\xi_7^{i_j})+4$ to have modulus $\sqrt{14}$, we must have $\{i_2,i_3,i_4\}=\{1,2,4\}$ or $\{3,5,6\}$ by Lemma 3.1. Since all i_j's are distinct and t is assumed to be a non-square modulo 7, we see that $\{i_2,i_3,i_4\}=\{3,5,6\}$. So F_1 maps all non-squares modulo 7 to 0, and maps each square modulo 7 to its additive inverse. Let χ_3 be the character that maps a to ξ_7 and b to ξ_7^u, and χ_4 be the one that maps a to ξ_7 and b to ξ_7^{-u}, where $u=2$ or 4. Then it is easy to see that $|\chi_3(R_1)|^2=7$, $|\chi_4(R_1)|^2=0$. But similar arguments to those in Case 3 show that we must have $|\chi_3(R_1)|=|\chi_4(R_1)|$: a contradiction.

Next we consider the case where $\chi_2(R_1) = 0$. We have

$$\sum_{j=1}^{4} \xi_7^{i_j} + \xi_7^{i_5+t} + \xi_7^{i_6+2t} + \xi_7^{i_7+4t} = 0.$$

Hence $\{i_1, i_2, i_3, i_4, i_5+t, i_6+2t, i_7+4t\} = \mathbb{Z}_7$. It follows that $\{i_5, i_6, i_7\} = \{i_5+t, i_6+2t, i_7+4t\}$. Since $t \neq 0$, we have either $(i_5, i_6, i_7) = (i_5, i_5-2t, i_5+t)$ or $(i_5, i_6, i_7) = (i_5, i_5+t, i_5+3t)$. By replacing R_1 with $a^{-i_5}b^{-s}R_1$ if necessary, we may assume that $s = 0$ and $i_5 = 0$. When $(i_5, i_6, i_7) = (0, -2t, t)$, apply the character χ_3' (respectively χ_4') that maps a to ξ_7 and b to ξ_7^u (respectively ξ_7^{-u}) to R_1, where $u = 3$, we find that $|\chi_3'(R_1)|^2 = 7$ and $|\chi_4'(R_1)|^2 = 0$. But again we should have $|\chi_3'(R_1)| = |\chi_4'(R_1)|$ as before: a contradiction. The case $(i_5, i_6, i_7) = (0, t, 3t)$ is similarly ruled out: take $u = 2$ (in the definition of χ_3' and χ_4'); then $|\chi_3'(R_1)|^2 = 14$ and $\chi_4'(R_1) = 0$, again contradicting $|\chi_3'(R_1)| = |\chi_4'(R_1)|$.

The proof of the theorem is now complete. □

4. Non-existence of $(4p, p, 4p, 4)$ RDS in abelian groups of order $4p^2$

Throughout this section we let G be an abelian group of order $4p^2$, p an odd prime. If G contains a $(4p, p, 4p, 4)$ RDS relative to a subgroup N of order p, then by Lemma 1.3 the Sylow p-subgroup of G is non-cyclic. Therefore in the rest of this section we always assume that the Sylow p-subgroup of G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

In this section we will first show that if $p \neq 3$ is an odd prime, then $G = \mathbb{Z}_2^2 \times \mathbb{Z}_p^2$ does not contain a $(4p, p, 4p, 4)$ RDS. We remark that $G = \mathbb{Z}_2^2 \times \mathbb{Z}_3^2$ indeed contains a $(12, 3, 12, 4)$ RDS, see [6] and [16].

Theorem 4.1. *Let $p \geqslant 5$ be an odd prime. Then there does not exist a $(4p, p, 4p, 4)$ relative difference set in $G = \mathbb{Z}_2^2 \times \mathbb{Z}_p^2$.*

Proof. We write $G = \langle \alpha_1\colon\ \alpha_1^2 = 1\rangle \times \langle \alpha_2\colon\ \alpha_2^2 = 1\rangle \times \mathbb{Z}_p^2$ and $H := \mathbb{Z}_p^2 < G$. Assume that R is a $(4p, p, 4p, 4)$ RDS in G relative to a subgroup N of order p. Since the subgroups of order p of G form a single orbit under the action of $\mathrm{Aut}(G)$, we may choose N to be $\{0\} \times \mathbb{Z}_p < H$. By the definition of an RDS, we have

$$RR^{(-1)} = 4p + 4(G - N) \quad \text{in } \mathbb{Z}[G]. \tag{4.1}$$

On one hand, if $\theta \in \hat{G}$ and $\theta|_N = 1$, then by applying θ to both sides of (4.1) we obtain that $\theta(R) = 0$. On the other hand, if $\theta \in \hat{G}$ and $\theta|_N \neq 1$, then by applying θ to both sides of (4.1) we obtain that $\theta(R)\overline{\theta(R)} = 4p$; by the same arguments as those at the beginning of the proof of Lemma 3.1, we find that $\theta(R) = f_0(\xi_p)\Delta$, where $|f_0(\xi_p)|^2 = 4$ and $f_0(x) \in \mathbb{Z}[x]$. Write

$$R = R_1 + R_2\alpha_1 + R_3\alpha_2 + R_4\alpha_1\alpha_2, \tag{4.2}$$

where $R_i \subset H$ for all $1 \leqslant i \leqslant 4$. By applying the characters of G whose restrictions to H are trivial to both sides of (4.2), we have

$$\begin{aligned} &|R_1| + |R_2| + |R_3| + |R_4| = 4p,\\ &|R_1| - |R_2| + |R_3| - |R_4| = 0,\\ &|R_1| - |R_2| - |R_3| + |R_4| = 0,\\ &|R_1| + |R_2| - |R_3| - |R_4| = 0. \end{aligned} \tag{4.3}$$

From these equations, we find that $|R_1| = |R_2| = |R_3| = |R_4| = p$.

The characters of H are of the form $\chi_{u,v}(u', v') = \xi_p^{uu'+vv'}$, $\forall (u', v') \in H$. For any character χ of H that is non-principal on N, write $(a, b, c, d) = (\chi(R_1), \chi(R_2), \chi(R_3), \chi(R_4))$. By applying the characters of G whose restrictions to H equal χ to both sides of (4.2), we have

$$\begin{aligned} &a+b+c+d=f_1(\xi_p)\Delta, \qquad a-b+c-d=f_2(\xi_p)\Delta, \\ &a-b-c+d=f_3(\xi_p)\Delta, \qquad a+b-c-d=f_4(\xi_p)\Delta, \end{aligned} \tag{4.4}$$

where $|f_i(\xi_p)|^2=4$ and $f_i(x)\in\mathbb{Z}[x]$, for $i=1,2,3,4$. To simplify notation, we will usually write $f_i(\xi_p)$ as f_i. Solving for a,b,c,d, we obtain,

$$a=\frac{1}{4}(f_1+f_2+f_3+f_4)\Delta, \qquad b=\frac{1}{4}(f_1-f_2-f_3+f_4)\Delta,$$
$$c=\frac{1}{4}(f_1+f_2-f_3-f_4)\Delta, \qquad d=\frac{1}{4}(f_1-f_2+f_3-f_4)\Delta.$$

Note that a,b,c,d are all algebraic integers. We consider two cases.

Case 1. $\mathrm{ord}_p(2)$ is odd. Let $(2)=Q_1\cdots Q_g\bar{Q}_1\cdots\bar{Q}_g$ be the prime ideal decomposition of (2) in $\mathbb{Z}[\xi_p]$. (Note that since $\mathrm{ord}_p(2)$ is odd, the decomposition group of Q_ℓ does not contain the complex conjugation.) For each i, $1\leqslant i\leqslant 4$, let

$$(f_i)=Q_1^{r_{i1}}\cdots Q_g^{r_{ig}}\bar{Q}_1^{s_{i1}}\cdots\bar{Q}_g^{s_{ig}},$$

where $r_{i\ell},s_{i\ell}\geqslant 0$. Then from $f_i\bar{f}_i=4$ we obtain that $r_{i\ell}+s_{i\ell}=2$, $\forall\ell=1,2,\dots,g$.

We claim that $(f_i)=(f_j)$, $\forall 1\leqslant i,j\leqslant 4$. The proof of the claim goes as follows. First note that by subtracting the two equations in (4.4) that involve f_i and f_j, we find that $2|(f_i-f_j)$. Hence $Q_\ell|(f_i-f_j)$ as well as $\overline{Q_\ell}|(f_i-f_j)$ for each ℓ. If $r_{i\ell}=0$ for some ℓ, then Q_ℓ does not divide f_j since otherwise from $f_j\in Q_\ell$ and $f_i-f_j\in Q_\ell$ we obtain $f_i\in Q_\ell$, i.e. $Q_\ell|(f_i)$. So we must have $r_{j\ell}=0$. Hence $s_{i\ell}=s_{j\ell}=2$. Similarly, if $s_{i\ell}=0$ for some ℓ, then $s_{j\ell}=0$ and $r_{i\ell}=r_{j\ell}=2$. If $r_{i\ell}=s_{i\ell}=1$ for some ℓ, then neither $r_{j\ell}$ nor $s_{j\ell}$ can be zero for otherwise $r_{i\ell}=0$ or 2 from the above analysis. It follows that $r_{j\ell}=s_{j\ell}=1$. We have thus proved that (f_i) and (f_j) has the same prime ideal decomposition. Hence $(f_i)=(f_j)$. It follows that $f_i=f_1\mu_i$, where $\mu_1=1$ and μ_i, $2\leqslant i\leqslant 4$, are $2p$th roots of unity. Furthermore, if $f_i\neq\pm f_j$ for some i,j, then since $(\mu_i-\mu_j)$ and (2) have no common prime ideal divisor, and $2|(f_i-f_j)$, we have $(f_i)=(f_j)=(2)$. There are two possibilities to consider.

(i) $\mu_i=\pm1$, $\forall i\in\{1,2,3,4\}$. In this case, noting that $\mu_1=1$, we see that $\sum_{i=1}^4\mu_i$ can only take one of the values $0,4,\pm2$. If $\sum_{i=1}^4\mu_i=0$ or 4, then $(\mu_1,\mu_2,\mu_3,\mu_4)$ must be one of

$$(1,1,1,1),\ (1,-1,1,-1),\ (1,1,-1,-1),\ (1,-1,-1,1).$$

In each case, exactly one of a,b,c,d has modulus $\sqrt{4p}$ and the others are 0. If $\sum_{i=1}^4\mu_i=\pm2$, then three of μ_i, $i=1,2,3,4$, are equal. We must have $(f_1)=(2)$ since a is an algebraic integer. It follows that $\{a,b,c,d\}=\Delta\cdot\{\eta,\eta,\eta,-\eta\}$ for some root of unity η.

(ii) Some μ_i is not equal to ±1. In this case, we have $(f_i)=(2)$, $\forall i\in\{1,2,3,4\}$, by the analysis immediately preceding (i). So we write $f_i=2\omega_i$ with ω_i a root of unity, for each i. It is clear that any subset of size $p-1$ of $X:=\{1,\xi_p,\dots,\xi_p^{p-1}\}$ forms an integral basis of $\mathbb{Z}[\xi_p]$. So any k-subset of X can be completed to an integral basis of $\mathbb{Z}[\xi_p]$ when $k<p-1$. Write $\omega_i=\epsilon_i\xi_p^{\ell_i}$ with $\ell_i\in\mathbb{Z}_p$ and $\epsilon_i=\pm1$ for each i. Then at least two of the ℓ_i's are distinct, and the distinct elements among the four $\xi_p^{\ell_i}$'s can be completed to an integral basis of $\mathbb{Z}[\xi_p]$ as we remarked. We only consider the case where $\ell_1\neq\ell_2$. The remaining cases are similar. From $a=\frac{\sum_{i=1}^4\omega_i}{2}\Delta$, we see that $\frac{\sum_{i=1}^4\omega_i}{2}$ is an algebraic integer. Hence the sum of coefficients of $\xi_p^{\ell_1}$ (respectively $\xi_p^{\ell_2}$) is even in $\sum_{i=1}^4\omega_i$. Therefore we must have $\{\ell_3,\ell_4\}=\{\ell_1,\ell_2\}$, which in turn implies that $\{\omega_3,\omega_4\}$ is one of $\pm\{\omega_1,\omega_2\}$, $\pm\{\omega_1,-\omega_2\}$. Case-by-case examinations show that we must have either $\{a,b,c,d\}=\Delta\cdot\{\eta_1,\eta_1,\eta_2,-\eta_2\}$ or $\{a,b,c,d\}=\Delta\cdot\{\eta_1+\eta_2,\eta_1-\eta_2,0,0\}$, where both η_1 and η_2 are roots of unity and $\eta_1\neq\pm\eta_2$.

To summarize, we have the following three possibilities for (a,b,c,d):

(1A) exactly one has modulus $\sqrt{4p}$, and the others are 0;
(2A) $\{a,b,c,d\}=\Delta\cdot\{\eta_1,\eta_1,\eta_2,-\eta_2\}$;
(2B) $\{a,b,c,d\}=\Delta\cdot\{\eta_1+\eta_2,\eta_1-\eta_2,0,0\}$, with $\eta_1\neq\pm\eta_2$,

where η_1,η_2 are roots of unity in $\mathbb{Z}[\xi_p]$.

Case 2. $\mathrm{ord}_p(2)$ is even. In this case, each prime ideal divisor of (2) in $\mathbb{Z}[\xi_p]$ is fixed by the complex conjugation. So $f_i = 2\mu_i$ for some root of unity μ_i for each i. The same arguments as those in the above case work for this case; and there are also three possibilities as listed above. In particular, a, b, c, d are multiples of 2Δ in case (1A) this time. In the following, we will consider the $\mathrm{ord}_p(2)$ even case and the $\mathrm{ord}_p(2)$ odd case together.

First we prove that case (2B) does not occur. Assume to the contrary that $\chi := \chi_{u,u'}$ with $u' \neq 0$ is a character of H such that case (2B) occurs. Then $\chi(R_i) = (1 - \xi_p^{\ell})\xi_p^{\ell'}\epsilon\Delta$ for some i, where $\ell \in \mathbb{Z}_p^*$ and $\epsilon = \pm 1$. Since R_i meets each coset of N in H in a unique element we may write $R_i = \{(x, f(x)) \colon x \in \mathbb{Z}_p\}$, where $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a function. Define $F(x) := ux + u'f(x) - \ell'$ and $a_j := |\{x \in \mathbb{Z}_p \colon F(x) = j\}|$, $\forall j \in \mathbb{Z}_p$. Then

$$\chi(R_i)\xi_p^{-\ell'} = \sum_{x\in\mathbb{Z}_p} \xi_p^{F(x)} = \sum_{j=0}^{p-1} a_j \xi_p^j = \left(1 - \xi_p^{\ell}\right)\epsilon\Delta = \sum_{j=0}^{p-1}\left[\left(\frac{j}{p}\right) - \left(\frac{j-\ell}{p}\right)\right]\epsilon\xi_p^j.$$

Comparing the coefficients of ξ^j on the two sides of the above equation, we find that $a_j - a_0 = [(\frac{j}{p}) - (\frac{j-\ell}{p}) + (\frac{-\ell}{p})]\epsilon$. Together with $\sum_{j=0}^{p-1} a_j = p$, we deduce that $a_j = 1 + [(\frac{j}{p}) - (\frac{j-\ell}{p})]\epsilon$. We now show that there exists j, $1 \leqslant j \leqslant p-1$, such that a_j is negative. Let (RN) (respectively (NR)) be the number of pairs $(x, x-\ell)$ in the set $1, 2, \dots, p-1$ such that x (respectively $x - \ell$) is a non-zero square modulo p and $x - \ell$ (respectively x) is a non-square modulo p. Then by elementary number theory (see, e.g. [15, p. 64]), we find that

$$(RN) = \frac{p-1}{4} + \frac{1}{2}\big(\delta(-\ell \in Q) - \delta(\ell \in Q)\big),$$
$$(NR) = \frac{p-1}{4} - \frac{1}{2}\big(\delta(-\ell \in Q) - \delta(\ell \in Q)\big),$$

where δ is the Kronecker delta function and Q is the set of non-zero squares modulo p. Since $p \geqslant 5$, both (RN) and (NR) are positive. Hence there exists $j \in \mathbb{Z}_p^*$ such that $-(\frac{j}{p}) = (\frac{j-\ell}{p}) = \epsilon$. It follows that $a_j = -1 < 0$: a contradiction. Therefore case (2B) cannot occur.

Next we show that case (1A) does not occur. Assume to the contrary that $\chi := \chi_{u,u'}$ with $u' \neq 0$ is a character of H such that case (1A) occurs. Then $\chi(R_i) = (\sum_j b_j \xi_p^j)\Delta$ for some i, where $(\sum_j b_j \xi_p^j)(\sum_j b_j \xi_p^{-j}) = 4$, $b_j \in \mathbb{Z}$. Since R_i meets each coset of N in H in a unique element we may write $R_i = \{(x, f(x)) \colon x \in \mathbb{Z}_p\}$, where $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a function. Define $F(x) := ux + u'f(x)$ and $a_j := |\{x \in \mathbb{Z}_p \colon F(x) = j\}|$. Then $\chi(R_i) = \sum_j a_j \xi_p^j$. Multiplying both sides of the following equation

$$\sum_j a_j \xi_p^j = \left(\sum_j \left(\frac{j}{p}\right)\xi_p^j\right)\left(\sum_j b_j \xi_p^j\right)$$

by $\overline{\Delta}$, we get

$$\left(\sum_j a_j \xi_p^j\right)\left(\sum_j \left(\frac{-j}{p}\right)\xi_p^j\right) = p\left(\sum_j b_j \xi_p^j\right).$$

The following arguments are similar to those in the proof of Lemma 3.1. By comparing coefficients of ξ_p^k, we get

$$\sum_j a_{k-j}\left(\frac{-j}{p}\right) = pb_k - c, \quad \forall k \in \mathbb{Z}_p,$$

for some integer c. Summing the above equations over k, we get $c = \sum_j b_j$. Since $(\sum_j b_j \xi_p^j) \times (\sum_j b_j \xi_p^{-j}) = 4$, we have $c^2 \equiv 4 \bmod ((1 - \xi_p) \cap \mathbb{Z})$, i.e., $c^2 \equiv 4 \pmod{p}$. Hence $c \equiv \pm 2 \pmod{p}$. Write $c = pc_1 + 2\epsilon$ with $\epsilon = \pm 1$. Note that

$$\left|p(b_k-c_1)-2\epsilon\right| = |pb_k-c| = \left|\sum_j a_{k-j}\left(\frac{-j}{p}\right)\right| \leqslant p-a_k \leqslant p.$$

So if $\epsilon=1$, then $\delta_k := b_k-c_1 = 1$ or 0. Since $pc_1+2=\sum_j b_j=\sum_j(c_1+\delta_j)$, we have $\sum_j \delta_j = 2$. Hence only two of the δ_j's are equal to 1. It follows that $\sum_j b_j\xi_p^j = \xi_p^{i_1}+\xi_p^{i_2}$ with $i_1 \neq i_2 \in \mathbb{Z}_p$. Now $\xi_p^{i_1}+\xi_p^{i_2}$ clearly cannot have modulus 2: a contradiction. The case where $\epsilon=-1$ is similarly ruled out.

So we have proved that for each character χ of H that is non-principal on N only case (2A) can possibly occur. Write $R_i := \{(x,h_i(x))\colon x \in \mathbb{Z}_p\} \subset H$ for all $i=1,2,3,4$, where $h_i : \mathbb{Z}_p \to \mathbb{Z}_p$. For any $\chi \in \hat{H}$ and $\chi|_N \neq 1$, we have $|\chi(R_i)| = \sqrt{p}$ for each i. This implies that each h_i is a p-ary bent function from $\mathbb{Z}_p$ to itself. By Theorem 1.4, we have $h_i(x)=a_ix^2+b_ix+c_i$, $a_i \neq 0$, $a_i,b_i,c_i \in \mathbb{Z}_p$ for each i. For any $u \in \mathbb{Z}_p$, we write $\chi_u := \chi_{u,1}$, which is a character of H and whose restriction to N is non-principal. Define for each $u \in \mathbb{Z}_p$ the following 4-tuple

$$(A_{1u},A_{2u},A_{3u},A_{4u}) = \bigl(\chi_u(R_1),\chi_u(R_2),\chi_u(R_3),\chi_u(R_4)\bigr).$$

We have $A_{iu} = \Delta\xi_p^{\frac{-(b_i+u)^2}{4a_i}+c_i}(\frac{a_i}{p})$ by direct computations. Hence to meet the conditions in case (2A), we must have three of $(\frac{a_i}{p})$ being equal and the fourth being distinct from them. Without loss of generality we assume that

$$\left(\frac{a_1}{p}\right)=\left(\frac{a_2}{p}\right)=\left(\frac{a_3}{p}\right)=-\left(\frac{a_4}{p}\right).$$

For each $u \in \mathbb{Z}_p$, one of the following should occur:

(i) $A_{1u}=A_{2u}$, $A_{3u}=-A_{4u}$;
(ii) $A_{1u}=A_{3u}$, $A_{2u}=-A_{4u}$;
(iii) $A_{2u}=A_{3u}$, $A_{1u}=-A_{4u}$.

If we are in case (i), then $a_3 \neq a_4$ since $(\frac{a_3}{p})=-(\frac{a_4}{p})$, and $-\frac{(b_3+u)^2}{4a_3}+c_3=-\frac{(b_4+u)^2}{4a_4}+c_4$. The last equation is quadratic in u (the coefficient of u^2 is $\frac{a_3-a_4}{4a_3a_4} \neq 0$). Therefore there are at most two u's satisfying that equation. In other words, case (i) occurs for at most two values of u. The same is true for the other two cases. Now note that for any $u \in \mathbb{Z}_p$, one of the above three cases must occur. It follows that $p \leqslant 6$. Hence $p=5$ (since p is assumed to be greater than or equal to 5). It will be convenient to define

$$U_1=\{u \in \mathbb{Z}_5 \mid A_{1u}=A_{2u},\ A_{3u}=-A_{4u}\},$$
$$U_2=\{u \in \mathbb{Z}_5 \mid A_{1u}=A_{3u},\ A_{2u}=-A_{4u}\},$$
$$U_3=\{u \in \mathbb{Z}_5 \mid A_{2u}=A_{3u},\ A_{1u}=-A_{4u}\}.$$

By the above analysis, we have $U_1 \cup U_2 \cup U_3 = \mathbb{Z}_5$, $1 \leqslant |U_i| \leqslant 2$ for all i, and $U_i \neq U_j$ for $1 \leqslant i \neq j \leqslant 5$.

We first claim that it is impossible to have $a_1=a_2=a_3$. If $a_1=a_2$, then $b_1 \neq b_2$ since otherwise $(a_1,b_1,c_1)=(a_2,b_2,c_2)$, which implies $U_2=U_3$, a contradiction. Therefore, if $a_1=a_2$, then $A_{1u}=A_{2u}$ becomes a degree one equation in u, which has at most one solution; hence $|U_1|=1$. By the same reasoning we see that if $a_1=a_2=a_3$, then $|U_1|=|U_2|=|U_3|=1$, which is clearly impossible.

Now recall that $(\frac{a_1}{5})=(\frac{a_2}{5})=(\frac{a_3}{5})$. Since there are two non-zero squares and two non-squares in $\mathbb{Z}_5$, we must have two of a_1,a_2,a_3 being equal. Without loss of generality assume that $a_1=a_2=-a_3$. After replacing R by $R^\sigma g$ for some $g \in G$ and $\sigma \in \mathrm{Aut}(G)$ which fixes elements of $\langle\alpha_1\rangle \times \langle\alpha_2\rangle$, we may assume that $h_1(x)=x^2$ (hence $a_1=1$, $b_1=c_1=0$). In the following we study the case where $a_4=2$. The case where $a_4=-2$ can be handled similarly.

Now that we assumed $a_1=a_2$, by the above reasoning we must have $b_1 \neq b_2$, that is $b_2 \neq 0$ since b_1 is now assumed to be 0. We must have $|U_1|=1$, $|U_2|=|U_3|=2$, and U_1, U_2 and U_3 are mutually disjoint.

Solving $A_{1u} = A_{2u}$, we see that the unique element of U_1 is $u = 2b_2 - \frac{c_2}{2b_2}$, which must also satisfy

$$u^2 + (-b_4 - 2b_3)u + 2b_4^2 - c_4 + c_3 - b_3^2 = 0. \tag{4.5}$$

This last equation comes from $A_{3u} = -A_{4u}$.

Any element $u \in U_2$ must satisfy

$$2u^2 + 2b_3 u + b_3^2 - c_3 = 0, \tag{4.6}$$

$$3u^2 + (-b_4 + 2b_2)u + 2b_4^2 - c_4 + c_2 + b_2^2 = 0. \tag{4.7}$$

Since $|U_2| = 2$, the two equations above should have two distinct common solutions. So by comparing coefficients we have $b_4 = 2b_2 + 2b_3$ and $2b_4^2 - c_4 + b_3^2 - c_3 + b_2^2 + c_2 = 0$.

Now, $u = 2b_2 - \frac{c_2}{2b_2} \in U_1$ cannot be a solution to (4.6). But adding twice of (4.6) to (4.5) gives $u = 2b_2 - \frac{c_2}{2b_2}$: a contradiction.

We have shown that for any character χ of H that is non-principal on N, none of the cases (1A), (2A), (2B) can occur. Therefore for an odd prime $p \geqslant 5$, a $(4p, p, 4p, 4)$ RDS in G cannot exist. The proof is complete. □

Theorem 4.2. *Let p be an odd prime. Then there does not exist a $(4p, p, 4p, 4)$ relative difference set in $G = \mathbb{Z}_4 \times \mathbb{Z}_p^2$.*

Proof. We write $G = \langle \alpha\colon\ \alpha^4 = 1\rangle \times \mathbb{Z}_p^2$ and $H := \mathbb{Z}_p^2 < G$. Assume that R is a $(4p, p, 4p, 4)$ RDS in G relative to a subgroup N of order p. Since the subgroups of order p of G form a single orbit under the action of $\mathrm{Aut}(G)$, we may choose N to be $\{0\} \times \mathbb{Z}_p < H$. By the definition of an RDS, we have

$$RR^{(-1)} = 4p + 4(G - N) \quad \text{in } \mathbb{Z}[G]. \tag{4.8}$$

On one hand, if $\theta \in \hat{G}$ and $\theta|_N = 1$, then by applying θ to both sides of (4.8) we obtain that $\theta(R) = 0$. On the other hand, if $\theta \in \hat{G}$ and $\theta|_N \neq 1$, then by applying θ to both sides of (4.8) we obtain that $\theta(R)\overline{\theta(R)} = 4p$; by the same arguments as those at the beginning of the proof of Lemma 3.1, we find that $\theta(R) = f(\xi_p)\Delta$, where $|f(\xi_p)|^2 = 4$ and $f(x) \in \mathbb{Z}[x]$. Write

$$R = R_0 + R_1\alpha_1 + R_2\alpha^2 + R_3\alpha^3, \tag{4.9}$$

where $R_j \subset H$ for $j = 0, 1, 2$ and 3. Applying the characters of G whose restrictions to H are trivial to both sides of (4.9), we have

$$\begin{aligned} &|R_0| + |R_1| + |R_2| + |R_3| = 4p, \\ &|R_0| - |R_1| + |R_2| - |R_3| = 0, \\ &|R_0| + i|R_1| - |R_2| - i|R_3| = 0, \\ &|R_0| - i|R_1| - |R_2| + i|R_3| = 0, \end{aligned} \tag{4.10}$$

where $i^2 = -1$. From these equations, we find that $|R_0| = |R_1| = |R_2| = |R_3| = p$.

For any character $\chi \in \hat{H}$ that is non-principal on N, write $(a, b, c, d) = (\chi(R_0), \chi(R_1), \chi(R_2), \chi(R_3))$. By applying the characters of G whose restrictions to H are χ we obtain

$$\begin{aligned} &a + b + c + d = f_1(\xi_p)\Delta, \\ &a - b + c - d = f_2(\xi_p)\Delta, \\ &\left|(a - c) + (b - d)i\right|^2 = 4p, \end{aligned} \tag{4.11}$$

where $|f_j(\xi_p)|^2 = 4$, $j = 1, 2$, with $f_j(x) \in \mathbb{Z}[x]$. From the first two equations in (4.11), we find that $2(b+d) = \Delta(f_1 - f_2)$. Hence $2|(f_1 - f_2)$. By the same arguments as those in the proof of Theorem 4.1, we deduce that $f_2 = f_1\eta$ for some $2p$th root of unity $\eta \in \mathbb{Z}[\xi_p]$. We show that η has to be ± 1.

Assume to the contrary that $\eta \neq \pm 1$. Then from $\Delta f_1(1-\eta) = 2(b+d)$ and $\Delta f_1(1+\eta) = 2(a+c)$ we find that $2|f_1$. It follows that $2|f_2$. We thus have $f_1 = 2\eta_1$ and $f_2 = 2\eta_2$ for some roots of unity $\eta_1, \eta_2 \in \mathbb{Z}[\xi_p]$. Denote $a+c = (\eta_1+\eta_2)\Delta$ by x and $b+d = (\eta_1-\eta_2)\Delta$ by y. Expanding $|(a-c)+(b-d)i|^2 = |(x-2c)+(y-2d)i|^2 = 4p$ and noting that 1, i are linearly independent over $\mathbb{Z}[\xi_p]$, we get

$$x\bar{c} + \bar{x}c + y\bar{d} + \bar{y}d = 2c\bar{c} + 2d\bar{d},$$

$$x\bar{d} + \bar{y}c - y\bar{c} - \bar{x}d - 2c\bar{d} + 2d\bar{c} = p(\eta - \bar{\eta}).$$

Here we have used the facts that $x\bar{x} + y\bar{y} = 4p$ and $x\bar{y} - \bar{x}y = 2p(\eta - \bar{\eta})$. In $\mathbb{Z}[\xi_p]$, we have $x \equiv y \pmod 2$, $\bar{x} \equiv \bar{y} \pmod 2$. So from the above two equations we have $p(\eta - \bar{\eta}) \equiv 0 \pmod 2$: a contradiction. Therefore we have proved that $\eta = \pm 1$. It follows that for an arbitrary character χ of H that is non-principal on N we have

$$\chi(R_0+R_2) = 0, \qquad \left|\chi(R_1+R_3)\right| = \sqrt{4p},$$

or

$$\chi(R_1+R_3) = 0, \qquad \left|\chi(R_0+R_2)\right| = \sqrt{4p}.$$

We also note that for a non-trivial character χ of H that is principal on N we have $\chi(R_0+R_2) = \chi(R_1+R_3) = 0$ (the argument is similar to the one we used to find $|R_j|$). By the inversion formula, the coefficient of the identity in $(R_0+R_2)(R_0+R_2)^{(-1)}$ is

$$\frac{1}{p^2}\left(4p^2 + 4pz\right) = 4 + \frac{4z}{p},$$

where

$$z = \left|\left\{\chi \in \hat{H}\colon\ \chi|_N \neq 1,\ \left|\chi(R_0+R_2)\right| = \sqrt{4p}\right\}\right|.$$

Hence we have $p|z$. Noting that the above set of characters is stable under the action of $Gal(\mathbb{Q}(\xi_p)/\mathbb{Q})$ on $\hat{H}$, we see that its elements are partitioned into orbits, each of size $p-1$. Hence $(p-1)|z$. So $z = 0$ or $z = (p-1)p$. If $z = (p-1)p$, then $\chi(R_1+R_3) = 0$ for all non-principal character χ of H. It follows that $R_1 + R_3 = \lambda H$ for some positive integer λ. This is clearly impossible since $|R_1| = |R_3| = p$. The case $z = 0$ is similarly ruled out. The proof is complete. □

By the analysis at the very beginning of this section, and combining Theorems 4.1 and 4.2 with the known example of a $(12, 3, 12, 4)$ RDS in $\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$ in [6] we have

Theorem 4.3. *Let p be an odd prime. An abelian group G of order $4p^2$ contains a $(4p, p, 4p, 4)$ relative difference set if and only if $G = \mathbb{Z}_2^2 \times \mathbb{Z}_3^2$.*

5. Conclusion

A (v, k, λ) difference set D in a non-abelian group of order v is said to be *genuinely non-abelian* if none of the abelian groups of the same order contains a difference set with these parameters. The first genuinely non-abelian difference set was constructed by K. Smith in [21], and its parameters are $(100, 45, 20)$.

We define a genuinely non-abelian relative difference set in the analogous way. Combining the construction in Section 2 and the non-existence results in Section 4, we therefore have constructed an infinite family of genuinely non-abelian semi-regular relative difference sets with parameters $(4p, p, 4p, 4)$, where $p \equiv 1 \pmod 4$ is a prime and $p > 9$. As far as we know, this is the first infinite family of genuinely non-abelian $(m, n, m, m/n)$ relative difference sets with $n > 2$.

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