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SYMMETRIC BUSH-TYPE HADAMARD MATRICES OF ORDER $4m^4$ EXIST FOR ALL ODD m

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ABSTRACT. Using reversible Hadamard difference sets, we construct symmetric Bush-type Hadamard matrices of order $4m^4$ for all odd integers m.

1. INTRODUCTION

A *Hadamard matrix* of order n is an n by n matrix H with entries ± 1, such that

$$HH^\top = nI_n,$$

where I_n is the identity matrix of order n. It can be easily shown that if n is the order of a Hadamard matrix, then $n = 1$, 2 or $n \equiv 0 \pmod 4$. The famous Hadamard matrix conjecture states that for every positive integer n divisible by 4, there exists a Hadamard matrix of order n. This conjecture is far from being proved. We refer the reader to [13] for a recent construction of a Hadamard matrix of order 428 (the smallest order for which an example of a Hadamard matrix was not known for many years). In this note, we concentrate on a class of Hadamard matrices of highly specialized form, namely the Bush-type Hadamard matrices.

Let n be a positive integer and let J_{2n} denote the matrix of order $2n$ with all entries being ones. A Hadamard matrix $H = (H_{ij})$ of order $4n^2$, where H_{ij} are $2n \times 2n$ block matrices, is said to be of *Bush-type* if

$$H_{ii} = J_{2n} \text{ and } H_{ij}J_{2n} = J_{2n}H_{ij} = 0, \tag{1.1}$$

for $i \neq j$, $1 \leq i, j \leq 2n$. K. A. Bush [3] proved that the existence of a projective plane of order $2n$ implies the existence of a symmetric Bush-type Hadamard matrix of order $4n^2$. So if one can prove the nonexistence of symmetric Bush-type Hadamard matrices of order $4n^2$, where n is odd, then the nonexistence of a projective plane of order $2n$, where n is odd, will follow. This was Bush's original motivation for introducing Bush-type Hadamard matrices. Wallis [18] showed that $n-1$ mutually orthogonal Latin squares of order $2n$ lead to a symmetric Bush-type Hadamard matrix of order $4n^2$. Goldbach and Claasen [7] also proved that certain 3-class association schemes can give rise to symmetric Bush-type Hadamard

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matrices. More recently, Kharaghani and his coauthors [15, 9, 10, 11, 12] rekindled the interest in Bush-type Hadamard matrices by showing that these matrices are very useful for constructions of symmetric designs and strongly regular graphs. Kharaghani [15] conjectured that Bush-type Hadamard matrices of order $4n^2$ exist for all n. While it is relatively easy to construct Bush-type Hadamard matrices of order $4n^2$ for all even n for which a Hadamard matrix of order $2n$ exists (see [14]), it is not easy to decide whether such matrices of order $4n^2$ exist if $n > 1$ is an odd integer. In a recent survey [12], Jungnickel and Kharaghani wrote "Bush-type Hadamard matrices of order $4n^2$, where n is odd, seem pretty hard to construct. Examples are known for $n = 3$, $n = 5$, and $n = 9$ (see [9], [10], and [11] respectively); all other cases are open". In this note, we will show that symmetric Bush-type Hadamard matrices of order $4m^4$ exist for all odd m.

We first note a relation between symmetric Bush-type Hadamard matrices and strongly regular graphs with certain properties. The following lemma is well known. A weaker form of the lemma appeared in [18]. For convenience of the reader, we provide a proof.

Lemma 1.1. *There exists a symmetric Bush-type Hadamard matrix of order* $4n^2$ *if and only if there exists a strongly regular graph (SRG in short) with parameters*

$$v = 4n^2,\ k = 2n^2 - n,\ \lambda = \mu = n^2 - n,$$

and with the additional property that the vertex set can be partitioned into $2n$ *disjoint cocliques of size* $2n$.

Proof. If $H = (H_{ij})$, where H_{ij} are $2n \times 2n$ block matrices, is a symmetric Bush-type Hadamard matrix of order $4n^2$, then the matrix $A = \frac{1}{2}(J - H)$ is symmetric and satisfies

$$A^2 = n^2 I + (n^2 - n)J.$$

Moreover the $2n \times 2n$ block matrices on the main diagonal of A are all zero matrices. Hence A is the adjacency matrix of an SRG with parameters $v = 4n^2, k = 2n^2 - n, \lambda = \mu = n^2 - n$, and with the additional property that the vertex set can be partitioned into $2n$ disjoint cocliques of size $2n$. Conversely, if A is the adjacency matrix of such an SRG, then the matrix $H = J - 2A$ is symmetric and satisfies $H^2 = 4n^2 I$. Since the vertex set of the SRG can be partitioned into $2n$ cocliques, each of size $2n$, we may arrange the rows and columns of H so that we can partition H into $H = (H_{ij})$, where the H_{ij} are $2n \times 2n$ block matrices and $H_{ii} = J_{2n}$. It remains to show that $H_{ij} J_{2n} = J_{2n} H_{ij} = 0$ for $i \neq j$, $1 \leq i, j \leq 2n$. Noting that the SRG has the smallest eigenvalue $-n$, we see that the cocliques of size $2n$ of the SRG meet the Delsarte bound (sometimes called the Hoffman bound also). By Proposition 1.3.2 [2, p. 10], every vertex in the SRG outside a coclique is adjacent to exactly n vertices of the coclique. This proves that $H_{ij} J_{2n} = J_{2n} H_{ij} = 0$ for $i \neq j$, $1 \leq i, j \leq 2n$. The proof is complete. □

2. Symmetric Bush-type Hadamard matrices from reversible Hadamard difference sets

We start with a very brief introduction to difference sets. For a thorough treatment of difference sets, we refer the reader to [1, Chapter 6]. Let G be a finite group of order v. A k-element subset D of G is called a (v, k, λ) *difference set* in G if the list of "differences" $d_1d_2^{-1}$, $d_1, d_2 \in D$, $d_1 \neq d_2$, represents each non-identity element in G exactly λ times. Using multiplicative notation for the group operation, D is a (v, k, λ) difference set in G if and only if it satisfies the following equation in $\mathbb{Z}[G]$:

$$DD^{(-1)} = (k - \lambda)1_G + \lambda G, \tag{2.1}$$

where $D = \sum_{d\in D} d$, $D^{(-1)} = \sum_{d\in D} d^{-1}$, and 1_G is the identity element of G. A subset D of G is called *reversible* if $D^{(-1)} = D$. Note that if D is a reversible difference set, then

$$D^2 = (k - \lambda)1_G + \lambda G. \tag{2.2}$$

If furthermore we require that $1_G \notin D$, then from (2.2) we see that the Cayley graph $\mathbf{Cay}(G, D)$, with vertex set G and two vertices x and y being adjacent if and only if $xy^{-1} \in D$, is an SRG with parameters (v, k, λ, λ).

The difference sets considered in this note have parameters

$$(v, k, \lambda) = (4n^2, 2n^2 - n, n^2 - n).$$

These difference sets are called *Hadamard* difference sets (HDS), since their $(1, -1)$-incidence matrices are Hadamard matrices. Alternative names used by other authors are Menon difference sets and H-sets. We will show that reversible HDSs give rise to symmetric Bush-type Hadamard matrices.

Proposition 2.1. *Let D be a reversible HDS in a group G with parameters $(4n^2, 2n^2 - n, n^2 - n)$. If there exists a subgroup $H \leq G$ of order $2n$ such that $D \cap H = \emptyset$, then there exists a Bush-type symmetric Hadamard matrix of order $4n^2$.*

Proof. First note that the Cayley graph $\mathbf{Cay}(G, D)$ is strongly regular with parameters $v = 4n^2 - n, k = 2n^2 - n, \lambda = \mu = n^2 - n$. The cosets of H in G partition G. Let Hg be an arbitrary coset of H in G. Then any two elements $x, y \in Hg$ are not adjacent in $\mathbf{Cay}(G, D)$ since $xy^{-1} \in H$ and $D \cap H = \emptyset$. Therefore the vertex set of $\mathbf{Cay}(G, D)$ can be partitioned into $2n$ disjoint cocliques of size $2n$. By Lemma 1.1, the $(1, -1)$-adjacency matrix of $\mathbf{Cay}(G, D)$ is a symmetric Bush-type Hadamard matrix of order $4n^2$. □

Let $G = K \times W$ where $K = \{g_0 = 1, g_1, g_2, g_3\}$ is a Klein four group and W is a group of order n^2. Each subset D of G has a unique decomposition into a disjoint union $D = \bigcup_{i=0}^{3}(g_i, D_i)$ where $D_i \subset W$. Note that if D is a reversible HDS in G, then $(g_i, 1)D$ are also reversible HDSs for all i, $0 \leq i \leq 3$. This observation implies the following.

Proposition 2.2. *Let $K = \{g_0 = 1, g_1, g_2, g_3\}$ be a Klein four group. Let $D = \bigcup_{\ell=0}^{3}(g_\ell, D_\ell)$ be a reversible Hadamard difference set in the group $G = K \times W$, where $D_\ell \subseteq W$ and $|W| = n^2$. If there exists a subgroup $P \leq W$ of order n such that $P \cap D_i = P \cap D_j = \emptyset$ for some $i \neq j$, $0 \leq i, j \leq 3$, then there exists a symmetric Bush-type Hadamard matrix of order $4n^2$.*

Proof. Let $E = (g_i, 1)D$. By the above observation, E is a reversible HDS in G. Let $H = (g_0, 1)P \cup (g_ig_j, 1)P$. Then H is a subgroup of G of order $2n$ and $H \cap E = \emptyset$. By Proposition 2.1, E gives rise to a symmetric Bush-type Hadamard matrix of order $4n^2$. □

3. Construction of symmetric Bush-type Hadamard matrices of order $4m^4$ for all odd m

A symmetric Bush-type Hadamard matrix H of order 4 is exhibited below:

$$H = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

So we will only be concerned with constructions of symmetric Bush-type Hadamard matrices of order $4m^4$ for odd $m > 1$. We will first construct symmetric Bush-type Hadamard matrices of order $4p^4$, where p is an odd prime, from certain $(4p^4, 2p^4 - p^2, p^4 - p^2)$ HDS. To this end, we need to recall a construction of an HDS with these parameters from [19]. Let p be an odd prime and $\mathrm{PG}(3,p)$ be a three-dimensional projective space over $\mathrm{GF}(p)$. We will say that a set C of points in $\mathrm{PG}(3,p)$ is of *type* Q if

$$|C| = \frac{(p^4-1)}{4(p-1)}$$

and each plane of $\mathrm{PG}(3,p)$ meets C in either $\frac{(p-1)^2}{4}$ points or $\frac{(p+1)^2}{4}$ points. For each set X of points in $\mathrm{PG}(3,p)$ we denote by $\widetilde{X}$ the set of all non-zero vectors $v \in \mathrm{GF}(p)^4$ with the property that $\langle v \rangle \in X$, where $\langle v \rangle$ is the 1-dimensional subspace of $\mathrm{GF}(p)^4$ generated by v.

Let $S = \{L_1, L_2, \ldots, L_{p^2+1}\}$ be a spread of $\mathrm{PG}(3,p)$ and let C_0, C_1 be two sets of type Q in $\mathrm{PG}(3,p)$ such that

$$\forall_{1 \le i \le s}\ |C_0 \cap L_i| = \frac{p+1}{2} \text{ and } \forall_{s+1 \le i \le 2s}\ \ |C_1 \cap L_i| = \frac{p+1}{2}, \tag{3.1}$$

where $s := \frac{p^2+1}{2}$. (We note that if we take S to be the regular spread in $\mathrm{PG}(3,p)$, then examples of type Q sets C_0, C_1 in $\mathrm{PG}(3,p)$ satisfying (3.1) were first constructed in [20] when $p \equiv 3 \pmod 4$, in [6, 19] when $p = 5, 13, 17$, and in [4] for all odd primes p.) As in [19] we set

$$\begin{aligned} C_2 &:= \quad (L_1 \cup \ldots \cup L_s) \setminus C_0, \\ C_3 &:= \quad (L_{s+1} \cup \ldots \cup L_{2s}) \setminus C_1. \end{aligned}$$

Note that $C_0 \cup C_2 = L_1 \cup \ldots \cup L_s$ and $C_1 \cup C_3 = L_{s+1} \cup \ldots \cup L_{2s}$.

Let A (respectively B) be a union of $(s-1)/2$ lines from $\{L_{s+1}, \ldots, L_{2s}\}$ (respectively $\{L_1, \ldots, L_s\}$). Let $K = \{g_0 = 1, g_1, g_2, g_3\}$ and $W = (\mathrm{GF}(p)^4, +)$. Denote

$$\begin{aligned} D_0 &:= \widetilde{C_0} \cup \widetilde{A}, \\ D_2 &:= \widetilde{C_2} \cup \widetilde{A}, \\ D_1 &:= \widetilde{C_1} \cup \widetilde{B}, \\ D_3 &:= W \setminus (\widetilde{C_3} \cup \widetilde{B}). \end{aligned}$$

Then

$$|D_0| = |D_1| = |D_2| = \frac{p^4 - p^2}{2},\ |D_3| = \frac{p^4 + p^2}{2}.$$

By Theorem 2.2 [19] the set

$$D := (g_0, D_0) \cup (g_1, D_1) \cup (g_2, D_2) \cup (g_3, D_3)$$

is a reversible $(4p^4, 2p^4 - p^2, p^4 - p^2)$ difference set in the group $K \times W$.

Pick an arbitrary line, say L_a, from the set $\{L_{s+1}, ..., L_{2s}\}$ such that $L_a \cap A = \emptyset$. Then $P := \widetilde{L_a} \cup \{0\}$ is a subgroup of W of order p^2 such that $P \cap D_0 = P \cap D_2 = \emptyset$. Now Proposition 2.2 implies that there exists a symmetric Bush-type Hadamard matrix of order $4p^4$. Therefore we have proved

Theorem 3.1. *There exists a symmetric Bush-type Hadamard matrix of order* $4p^4$ *for every odd prime* p*.*

In order to build a symmetric Bush-type Hadamard matrix of order $4m^4$ for arbitrary odd $m > 1$ we need to use Turyn's composition theorem [17]. We also need the following simple

Proposition 3.2. *There exists a subgroup* $Q \leq W$ *of order* p^2 *such that* $Q \subseteq D_3$ *and* $Q \cap D_1 = \emptyset$*.*

Proof. Pick an arbitrary line L_b from $\{L_1, ..., L_s\}$ such that $L_b \cap B = \emptyset$ and set $Q := \{0\} \cup \widetilde{L_b}$. The conclusion of the proposition follows. □

Next we recall Turyn's composition theorem. We will use the version as stated in Theorem 6.5 [5, p. 45]. For convenience we introduce the following notation. Let W_1, W_2 be two groups. For $A, B \subseteq W_1$ and $C, D \subseteq W_2$, we define the following subset of $W_1 \times W_2$:

$$\nabla(A, B; C, D) := \big((A \cap B) \times C'\big) \cup \big((A' \cap B') \times C\big) \cup \big((A \cap B') \times D'\big) \cup \big((A' \cap B) \times D\big),$$

where $A' = W_1 \setminus A$, $B' = W_1 \setminus B$, $C' = W_2 \setminus C$, and $D' = W_2 \setminus D$.

Theorem 3.3 (Turyn [17])**.** *Let* $K = \{g_0, g_1, g_2, g_3\}$ *be a Klein four group. Let* $E_1 = \bigcup_{i=0}^{3} (g_i, A_i)$ *and* $E_2 = \bigcup_{i=0}^{3} (g_i, B_i)$ *be reversible Hadamard difference sets in groups* $K \times W_1$ *and* $K \times W_2$*, respectively, where* $|W_1| = w_1^2$ *and* $|W_2| = w_2^2$*,* w_1 *and* w_2 *are odd,* $A_i \subseteq W_1$ *and* $B_i \subseteq W_2$*, and*

$$|A_0| = |A_1| = |A_2| = \frac{w_1^2 - w_1}{2},\ |A_3| = \frac{w_1^2 + w_1}{2},$$
$$|B_0| = \frac{w_2^2 + w_2}{2},\ |B_1| = |B_2| = |B_3| = \frac{w_2^2 - w_2}{2}.$$

Define

$$\begin{aligned} E &:= (g_0, \nabla(A_0, A_1; B_0, B_1)) \cup (g_1, \nabla(A_0, A_1; B_2, B_3)) \\ &\cup \ (g_2, \nabla(A_2, A_3; B_0, B_1)) \cup (g_3, \nabla(A_2, A_3; B_2, B_3)). \end{aligned}$$

Then

$$|\nabla(A_0, A_1; B_0, B_1)| = \frac{w_1^2 w_2^2 + w_1 w_2}{2},$$
$$|\nabla(A_0, A_1; B_2, B_3)| = |\nabla(A_2, A_3; B_0, B_1)| = |\nabla(A_2, A_3; B_2, B_3)| = \frac{w_1^2 w_2^2 - w_1 w_2}{2},$$

and E *is a reversible* $(4w_1^2 w_2^2, 2w_1^2 w_2^2 - w_1 w_2, w_1^2 w_2^2 - w_1 w_2)$ *Hadamard difference set in the group* $K \times W_1 \times W_2$*.*

Proposition 3.4. *With the assumptions as in Theorem* 3.3*, let* $Q \leq W_1$ *and* $P \leq W_2$ *be such that* $Q \cap A_2 = \emptyset, Q \subseteq A_3$ *and* $P \cap B_1 = \emptyset, P \cap B_3 = \emptyset$*. Then* $(Q \times P) \cap \nabla(A_2, A_3; B_0, B_1) = \emptyset$ *and* $(Q \times P) \cap \nabla(A_2, A_3; B_2, B_3) = \emptyset$*.*

Proof. It follows from the intersections

$$\begin{aligned} Q\cap(A_2\cap A_3) &= \emptyset, \\ Q\cap(A_2'\cap A_3') &= \emptyset, \\ Q\cap(A_2\cap A_3') &= \emptyset, \\ Q\cap(A_2'\cap A_3) &= Q \end{aligned}$$

that $\nabla(A_2, A_3; B_0, B_1)\cap(Q\times P) = Q\times(B_1\cap P) = \emptyset$ and $\nabla(A_2, A_3; B_2, B_3)\cap(Q\times P) = Q\times(B_3\cap P) = \emptyset$. □

Theorem 3.5. *There exists a symmetric Bush-type Hadamard matrix of order* $4m^4$ *for all odd* m.

Proof. We only need to prove the theorem for odd $m > 1$. Let $K = \{g_0, g_1, g_2, g_3\}$ be a Klein four group. Let p and q be two odd primes, not necessarily distinct, and let $W_1 = (\mathrm{GF}(p)^4, +)$ and $W_2 = (\mathrm{GF}(q)^4, +)$. By the construction before the statement of Theorem 3.1, we can construct a reversible HDS

$$E_1 = (g_0, A_0)\cup(g_1, A_1)\cup(g_2, A_2)\cup(g_3, A_3)$$

in $K\times W_1$ such that

$$|A_0| = |A_1| = |A_2| = \frac{p^4-p^2}{2},\ |A_3| = \frac{p^4+p^2}{2},$$

and there exists a subgroup $Q\le W_1$ of order p^2 with the property that $Q\cap A_2 = \emptyset, Q\subset A_3$. (See Proposition 3.2. Note that here the A_i are a renumbering of the D_i; any renumbering of the D_i still yields a reversible difference set.) Also we can construct a reversible HDS

$$E_2 = (g_0, B_0)\cup(g_1, B_1)\cup(g_2, B_2)\cup(g_3, B_3)$$

in $K\times W_2$ such that

$$|B_0| = \frac{q^4+q^2}{2},\ |B_1| = |B_2| = |B_3| = \frac{q^4-q^2}{2},$$

and there exists a subgroup $P\le W_2$ of order q^2 with the property that $P\cap B_1 = \emptyset, P\cap B_3 = \emptyset$ (see the paragraph before the statement of Theorem 3.1). Now we apply Theorem 3.3 to E_1 and E_2 to obtain a reversible HDS

$$\begin{aligned} E &= (g_0, \nabla(A_0, A_1; B_0, B_1))\cup(g_1, \nabla(A_0, A_1; B_2, B_3)) \\ &\cup (g_2, \nabla(A_2, A_3; B_0, B_1))\cup(g_3, \nabla(A_2, A_3; B_2, B_3)) \end{aligned}$$

of size $2p^4q^4 - p^2q^2$ in $K\times W_1\times W_2$. By Proposition 3.4, we have

$$(Q\times P)\cap\nabla(A_2, A_3; B_0, B_1) = \emptyset,\ (Q\times P)\cap\nabla(A_2, A_3; B_2, B_3) = \emptyset. \tag{3.2}$$

By Proposition 2.2, there exists a symmetric Bush-type Hadamard matrix of order $4(pq)^4$. Now note that $|Q\times P| = p^2q^2$, $|\nabla(A_2, A_3; B_0, B_1)| = |\nabla(A_2, A_3; B_2, B_3)| = \frac{p^4q^4-p^2q^2}{2}$, and E satisfies property (3.2). We can repeatedly use the above process to produce a reversible HDS satisfying the condition of Proposition 2.2; hence there exists a symmetric Bush-type Hadamard matrix of order $4m^4$ for all odd $m > 1$. The proof is complete. □

Kharaghani [15, 16] showed how to use Bush-type Hadamard matrices to simplify Ionin's method [8] for constructing symmetric designs. Based on his constructions in [15, 16], we draw the following consequences of Theorem 3.5.

Theorem 3.6. *Let m be an odd integer. If $q = (2m^2 - 1)^2$ is a prime power, then there exist twin symmetric designs with parameters*

$$v = 4m^4 \frac{(q^{\ell+1} - 1)}{q - 1},\ k = q^\ell(2m^4 - m^2),\ \lambda = q^\ell(m^4 - m^2),$$

for every positive integer ℓ.

Theorem 3.7. *Let m be an odd integer. If $q = (2m^2 + 1)^2$ is a prime power, then there exist Siamese twin symmetric designs with parameters*

$$v = 4m^4 \frac{(q^{\ell+1} - 1)}{q - 1},\ k = q^\ell(2m^4 + m^2),\ \lambda = q^\ell(m^4 + m^2),$$

for every positive integer ℓ.

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