New bounds for partial spreads of $\mathsf{H}(2d − 1, q^2)$ and partial ovoids of the Ree–Tits octagon

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A B S T R A C T

Two results are obtained that give upper bounds on partial spreads and partial ovoids respectively.

The first result is that the size of a partial spread of the Hermitian polar space $\mathsf{H}(3, q^2)$ is at most $\left((2p^3 + p)/3\right)^t + 1$, where $q = p^t$, p is a prime. For fixed p this bound is in $o(q^3)$, which is asymptotically better than the previous best known bound of $(q^3 + q + 2)/2$. Similar bounds for partial spreads of $\mathsf{H}(2d − 1, q^2)$, d even, are given.

The second result is that the size of a partial ovoid of the Ree–Tits octagon $\mathsf{O}(2^t)$ is at most $26^t + 1$. This bound, in particular, shows that the Ree–Tits octagon $\mathsf{O}(2^t)$ does not have an ovoid.

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1. Introduction

Determining clique numbers of graphs is a traditional topic of combinatorics. Partial spreads of polar spaces are cliques of the oppositeness graph defined on the generators of the polar space. While bounds for partial spreads and partial ovoids are a long studied topic since Thas [11] popularized the problem, many questions remain open. The purpose of this paper is to address two of these open questions.

Our first result gives an improved bound on the size of partial spreads of certain Hermitian polar spaces. An *Hermitian polar space* $\mathsf{H}(2d-1, q^2)$, for $q = p^t$ a prime power, is the incidence geometry arising from a non-degenerate Hermitian form f of $\mathbb{F}_{q^2}^{2d}$. Here the flats of $\mathsf{H}(2d-1, q^2)$ consist of all nonzero totally isotropic subspaces of $\mathbb{F}_{q^2}^{2d}$ with respect to the form f; incidence is the inclusion relation of the flats of $\mathsf{H}(2d-1, q^2)$. The maximal totally isotropic subspaces of $\mathbb{F}_{q^2}^{2d}$ with respect to the form f are called *generators* of $\mathsf{H}(2d-1, q^2)$. A *partial spread* of $\mathsf{H}(2d-1, q^2)$ is a set of pairwise disjoint generators of $\mathsf{H}(2d-1, q^2)$. A simple double counting argument shows that a partial spread of $\mathsf{H}(2d-1, q^2)$ has size at most $q^{2d-1} + 1$. When d is odd, a better upper bound of $q^d + 1$ is known [14], and partial spreads of that size exist [1], [8]. So we are interested in the case when d is even.

Theorem 1.1. *Let $q = p^t$ with p prime and $t \geq 1$. Let Y be a partial spread of $\mathsf{H}(2d-1, q^2)$, where d is even.*

(a) *If $d = 2$, then $|Y| \leq \left(\frac{2p^3 + p}{3} \right)^t + 1$.*
(b) *If $d = 2$ and $p = 3$, then $|Y| \leq 19^t$.*
(c) *If $d > 2$, then $|Y| \leq \left(p^{2d-1} - p^{\frac{p^{2d-2}-1}{p+1}} \right)^t + 1$.*

The previous best known bound is in $\Theta(q^{2d-1})$ [6,7], while even for $d = 2$ the largest known examples only have size $\Theta(q^2)$ [4]. Here we provide the first upper bound which is in $o(q^{2d-1})$ for fixed p. For $d = 2$ the previous best known bound is $(q^3 + q + 2)/2$ [6]. An easy calculation shows that this old bound is better than the bound in part (a) of Theorem 1.1 if $p = 2$ and $t \leq 2$ or if $t = 1$. But for fixed p (and let $q = p^t$), the bound in part (a) of Theorem 1.1 is in $o(q^3)$, which is asymptotically better than the bound of $(q^3 + q + 2)/2$. For $d > 2$ the new bound improves all previous bounds if $t > 1$.

Our second result is a bound on the size of a partial ovoid in the *Ree–Tits octagons*. A *generalized n-gon* of order (s, r) is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathsf{I})$, where elements of $\mathcal{P}$ are called *points*, elements of $\mathcal{L}$ are called *lines*, and $\mathsf{I} \subseteq \mathcal{P} \times \mathcal{L}$ is an *incidence relation* between the points and lines, which satisfies the following axioms [13]:

(a) Each line is incident with $s + 1$ points.
(b) Each point is incident with $r + 1$ lines.
(c) The *incidence graph* has diameter n and girth $2n$.

Here the incidence graph is the bipartite graph with $\mathcal{P} \cup \mathcal{L}$ as vertices, $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$ are adjacent if $(p, \ell) \in \mathsf{I}$.

The only known thick finite generalized octagons are the *Ree–Tits octagons* $\mathsf{O}(2^t)$, t odd, defined in [12], and their duals.

A *partial ovoid* of a generalized n-gon Γ is a set of points pairwise at distance n in the incidence graph. An easy counting argument shows that the size of a partial ovoid of a generalized octagon of order (s, r) is at most $(sr)^2 + 1$. A partial ovoid of a generalized octagon of order (s, r) is called an *ovoid* if it has the maximum possible size $(sr)^2 + 1$. The Ree–Tits octagon $\mathsf{O}(2^t)$ is a generalized octagon of order $(2^t, 4^t)$, so the size of an ovoid is $64^t + 1$. We shall establish the following bound.

Theorem 1.2. *The size of a partial ovoid of the Ree–Tits octagon* $\mathsf{O}(2^t)$*, t odd, is at most* $26^t + 1$.

Theorem 1.2 shows that the Ree–Tits octagon $\mathsf{O}(2^t)$ does not have an ovoid. This had been conjectured by Coolsaet and Van Maldeghem [5, p. 108]. For $\mathsf{O}(2)$, it was already shown [5] that a partial ovoid has at most 27 points, while for $t \geq 3$ no bounds better than the ovoid size were known. A computer search with the mixed-integer linear programming solver Gurobi showed that for $\mathsf{O}(2)$ the largest partial ovoid has size 24, so the bound of Theorem 1.2 is not tight. The nonexistence of ovoids further implies that the Ree–Tits octagon does not admit a line-domestic collineation [9, p. 582].

Both theorems are proved using the same approach based on the following simple and well known observation.

Lemma 1.3. *Let $(X, \sim)$ be a graph. Let A be the adjacency matrix of X. Let Y be a clique of X. Then*

$$|Y| \leq \begin{cases} rank_p(A) + 1, & \text{if } p \text{ divides } |Y| - 1, \\ rank_p(A), & \text{otherwise.} \end{cases}$$

Proof. Let J be the all-ones matrix of size $|Y| \times |Y|$. Let I be the identity matrix of size $|Y| \times |Y|$. As Y is a clique, the submatrix A' of A indexed by Y is $J - I$. Hence, the submatrix has p-rank $|Y| - 1$ if p divides $|Y| - 1$, and it has p-rank $|Y|$ if p does not divide $|Y| - 1$. As $\text{rank}_p(A') \leq \text{rank}_p(A)$, the assertion follows. $\square$

The feasibility of applying Lemma 1.3 depends, of course, on the ability to compute or to bound the p-ranks of the adjacency matrix in specific cases. In our cases, representation theory in characteristic p furnishes the necessary computations. The oppositeness graphs that we shall consider are defined by maximal distance in association schemes arising from the Hermitian dual polar spaces and generalized octagons. These maximal distance relations are specific examples of the general notion of oppositeness for flags in Tits buildings. It was shown in [10] that if G is a finite group of Lie type in characteristic p, then an oppositeness relation in the associated building defines a G-module

homomorphism (in characteristic p) whose image is a certain simple module for G and for a related semisimple algebraic group. Crucially, the calculation of the dimension of this module can be reduced to the prime field case (see Proposition 3.1). In the prime case computations have been made for many simple modules, including those needed for this paper ([2], [16]), as part of the representation theory of semisimple algebraic groups. In this way, we obtain the p-ranks of the oppositeness graphs for $\mathsf{O}(2^t)$ and $\mathsf{H}(3, q^2)$. In the other cases of $\mathsf{H}(2d-1, q^2)$ for $d > 2$, a bound on the p-rank is obtained by relating the oppositeness matrix with an idempotent matrix of the association scheme.

2. Two association schemes

A complete introduction to association schemes can be found in [3, Ch. 2].

Definition 2.1. Let X be a finite set of size n. An *association scheme* with $d + 1$ classes is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a set of symmetric binary relations on X with the following properties:

(a) $\mathcal{R}$ is a partition of $X \times X$.
(b) R_0 is the identity relation.
(c) There are numbers p_{ij}^k such that for $x, y \in X$ with $x R_k y$ there are exactly p_{ij}^k elements z with $x R_i z$ and $z R_j y$.

The relations R_i are described by their *adjacency matrices* $A_i \in \mathbb{C}^{n,n}$ defined by

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } x R_i y, \\ 0, & \text{otherwise.} \end{cases}$$

In this paper the matrix A_d is referred to as the *oppositeness matrix*. Denote the all-ones matrix by J. There exist (see [3, p. 45]) idempotent Hermitian matrices $E_j \in \mathbb{C}^{n,n}$ with the properties

$$\sum_{j=0}^{d} E_j = I, \qquad\qquad E_0 = n^{-1} J,$$

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \qquad\qquad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i,$$

where $P = (P_{ij}) \in \mathbb{C}^{d+1,d+1}$ and $Q = (Q_{ij}) \in \mathbb{C}^{d+1,d+1}$ are the so-called eigenmatrices of the association scheme. The P_{ij} are the eigenvalues of A_j. The multiplicity f_i of P_{ij} satisfies

$$f_i = \operatorname{rank}(E_i) = \operatorname{tr}(E_i) = Q_{0i}.$$

In this paper we consider the association schemes corresponding to the dual polar graph of $\mathsf{H}(2d-1, q^2)$ and the Ree–Tits octagon. For the dual polar graph of $\mathsf{H}(2d-1, q^2)$

we have the following situation. Here X is the set of generators of $\mathsf{H}(2d-1, q^2)$ and two generators a, b are in relation R_i, where $0 \leq i \leq d$, if and only if a and b intersect in codimension i. The dual polar graph of $\mathsf{H}(2d-1, q^2)$ has diameter d and is distance regular.

As in [7], for the dual polar graph of $\mathsf{H}(2d-1, q^2)$ we obtain

$$f_d = q^{2d} \frac{q^{1-2d}+1}{q+1} = q^{2d-1} - q\frac{q^{2d-2}-1}{q+1},$$

and

$$Q_{id} = \frac{P_{di}}{P_{0i}} Q_{0d} = f_d(-q)^{-i}.$$

For the Ree–Tits octagon $\mathsf{O}(2^t)$ we consider the following association scheme: the set X is the set of points of the octagon, and two points a and b are in relation R_i, where $0 \leq i \leq 4$, if their distance in the incidence graph of $\mathsf{O}(2^t)$ is $2i$. In this case A_4 is the oppositeness matrix.

3. p-ranks of the oppositeness matrices

The following result reduces the computation of p-ranks of oppositeness matrices to the case of the prime field. It is an application of Steinberg's Tensor Product Theorem.

Proposition 3.1 *(Sin [10, Prop. 5.2]). Let $G(q)$, $q = p^t$ a prime power, be a finite group of Lie type and let $A(q)$ denote the oppositeness matrix for objects of a fixed self-opposite type in the building of $G(q)$. Then*

$$rank_p(A(q)) = rank_p(A(p))^t.$$

Remark 3.2. The meaning of q in Proposition 3.1 needs to be explained. For (untwisted) Chevalley groups q is simply the cardinality of the field used to define the group. In the case of Ree groups of type F_4, there is a Steinberg endomorphism τ of the algebraic group F_4 (over an algebraic closure of $\mathbb{F}_2$) such that the Ree group that we denote by $G(q)$ is the subgroup of fixed points of τ, and the subgroup of fixed points of τ^2 is the Chevalley group $\mathsf{F}_4(q)$, where q is an odd power of 2. In general, the relation between $G(q)$ and q is that the highest weight of the simple module in [10, Theorem 4.1] is of the form $(q-1)\tilde{\omega}$, where $\tilde{\omega}$ is a sum of fundamental weights. We refer to §5 of [10] for a detailed description.

3.1. Proof of Theorem 1.1

Lemma 3.3.

(a) *The p-rank of the oppositeness matrix of lines of $\mathsf{H}(3, p^2)$ is $\frac{2p^3+p}{3}$.*
(b) *The p-rank of the oppositeness matrix of generators of $\mathsf{H}(2d-1, p^2)$, d even, is at most $p^{2d-1} - p\frac{p^{2d-2}-1}{p+1}$.*

Proof. Notice that $\mathsf{H}(3, p^2)$ is dual to $\mathsf{Q}^-(5, p)$, so the oppositeness matrix of lines of $\mathsf{H}(3, p^2)$ is the oppositeness matrix of points in $\mathsf{Q}^-(5, p)$. By [10, Example 6.2], the p-rank of this oppositeness matrix is equal to the dimension of the simple module for a simply connected algebraic group of type D_3 with highest weight equal to $(p - 1)\omega_1$, under the standard parametrization of simple modules, where ω_1 is the first fundamental weight. This dimension was calculated in [2, Theorem 1.2]. Applying Theorem 1.2 of [2] with $\ell = 3$ and $r = p - 1$, for the oppositeness matrix A_2 we obtain

$$\mathrm{rank}_p(A_2) = \frac{2p^3 + p}{3}.$$

Part (a) of the lemma follows.

For (b) notice that the matrix E_d of the dual polar graph of $\mathsf{H}(2d - 1, p^2)$ has rank

$$f_d = p^{2d-1} - p\frac{p^{2d-2} - 1}{p + 1}.$$

When d even the matrix $np^{d-1}E_d$ has only integer entries and we have $A_d \equiv np^{d-1}E_d$ mod p. Hence, $\mathrm{rank}_p(A_d) = \mathrm{rank}_p(np^{d-1}E_d) \le \mathrm{rank}(np^{d-1}E_d) = \mathrm{rank}(E_d) = p^{2d-1} - p\frac{p^{2d-2}-1}{p+1}$. $\square$

Theorem 1.1 now follows by combining Lemma 3.3, Proposition 3.1 and Lemma 1.3. Notice here that $\frac{2p^3+p}{3}$ is divisible by p unless $p = 3$. $\square$

3.2. Proof of Theorem 1.2

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a set of fundamental roots for a root system of type F_4, where α_1 is a long root at one end of the Dynkin diagram and α_4 is the short root at the other end. According to [10, p. 282, para. 3], the 2-rank of the oppositeness matrix of $\mathsf{O}(2)$ is equal to the dimension of the simple module of the group F_4 whose highest weight is the fundamental weight corresponding to α_4. The dimension can be found in several ways. First, Veldkamp calculated all simple modules of the group F_4 in characteristic 2. (See [16, Table 2]; the highest weight in question is denoted $d_4 = 0001$.) Secondly, this module happens to be the modulo 2 reduction of the corresponding Weyl module in characteristic zero, so its dimension is also given by Weyl's Dimension Formula. Finally, it is also easy to compute the oppositeness matrix of $\mathsf{O}(2)$ and its 2-rank by computer. Hence, we obtain the following result.

Lemma 3.4. *The* 2-*rank of the oppositeness matrix of* $\mathsf{O}(2)$ *is* 26.

Theorem 1.2 now follows by combining Proposition 3.1, Lemma 3.4 and Lemma 1.3. $\square$

4. Final remarks

The association scheme technique used also work for partial ovoids of the twisted triality hexagon of order (q, q^3), where we obtain the p-rank $\left((4p^5 + p)/5\right)^t$, but there a better bound of order $q^3 + 1$ is known [15, Theorem 6.4.19]. For all other known generalized polygons p-ranks do not improve any known bounds for partial ovoids.

The bounds on partial spreads $\mathsf{H}(2d - 1, q^2)$, $d > 2$ even, in Theorem 1.1 could be improved by calculating exact p-ranks. We would expect the bounds thus obtained to be much better than the ones given in Theorem 1.1, were we to extrapolate from the case of $\mathsf{H}(5, q^2)$. There, the corresponding p-rank is $\left((11p^5 + 5p^3 + 4p)/20\right)^t$, while an argument as in Lemma 3.3 (b) only yields $\left(p^5 - p^4 + p^3 - p^2 + p\right)^t$ as an upper bound. Unfortunately, calculating the exact p-rank seems to be too complicated. Notice however that this is a finite problem for fixed p and fixed d as one only has to calculate the p-rank for $t = 1$ by Proposition 3.1.

Acknowledgments

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