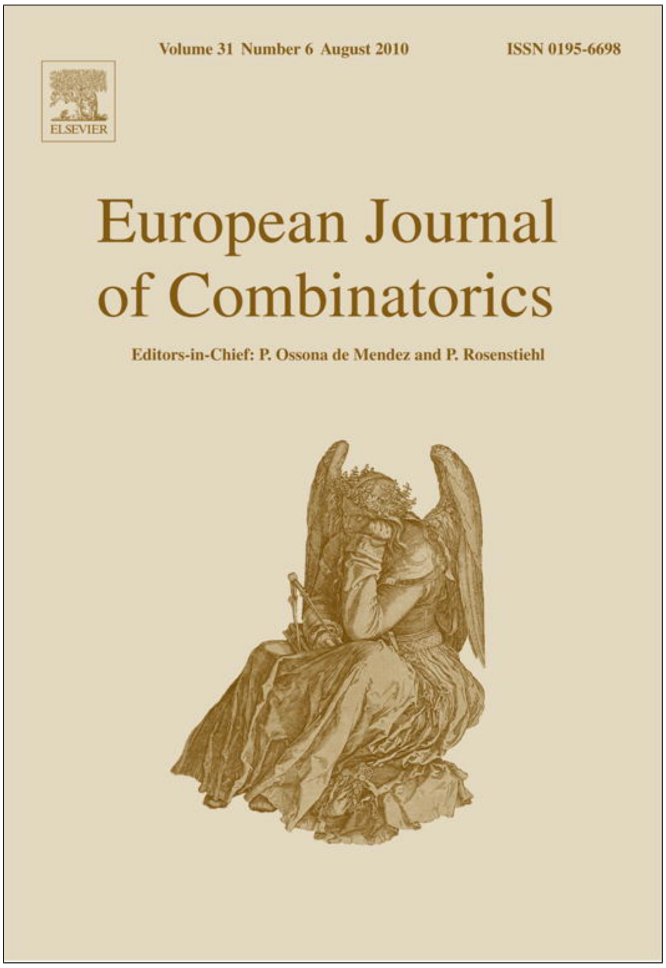
Volume 31 Number 6 August 2010
ISSN 0195-6698
ELSEVIER
European Journal of Combinatorics
Editors-in-Chief: P. Ossona de Mendez and P. Rosenstiehl

European Journal of Combinatorics 31 (2010) 1553–1559

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Strongly regular graphs with parameters $(4m^4, 2m^4 + m^2, m^4 + m^2, m^4 + m^2)$ exist for all $m > 1$

Willem H. Haemers [a], Qing Xiang [b]

[a] Department of Econometrics and O.R., Tilburg University, Tilburg, The Netherlands
[b] Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

ARTICLE INFO

Article history:
Available online 11 August 2009

ABSTRACT

Using results on Hadamard difference sets, we construct regular graphical Hadamard matrices of negative type of order $4m^4$ for every positive integer m. If $m > 1$, such a Hadamard matrix is equivalent to a strongly regular graph with parameters $(4m^4, 2m^4 + m^2, m^4 + m^2, m^4 + m^2)$. Strongly regular graphs with these parameters have been called *max energy graphs*, because they have maximal energy (as defined by Gutman) among all graphs on $4m^4$ vertices. For odd $m \geq 3$ the strongly regular graphs seem to be new.

1. Max energy graphs

A *strongly regular graph* (srg) with parameters (n, k, λ, μ) is a graph with n vertices that is regular of valency k $(1 \leq k \leq n - 2)$ and that has the following properties:

- For any two adjacent vertices x, y, there are exactly λ vertices adjacent to both x and y.
- For any two nonadjacent vertices x, y, there are exactly μ vertices adjacent to both x and y.

A disconnected srg is the disjoint union of cliques of the same size. The adjacency matrix of a connected srg with parameters (n, k, λ, μ) has three distinct eigenvalues k, r and s $(k > r \geq 0 > s)$, of multiplicity 1, f and g, respectively, where

$$r + s = \lambda - \mu, \qquad rs = \mu - k, \qquad f + g = n - 1, \qquad k + fr + gs = 0. \tag{1}$$

The *energy* $\mathcal{E}(\Gamma)$ of a graph Γ is the sum of the absolute values of the eigenvalues of its adjacency matrix. The concept of energy of a graph was introduced by Gutman in 1978 (see [5]), and it originated from theoretical chemistry. The recent talk by Stevanović [10] provides a good survey of research results on energy of graphs.

E-mail addresses: haemers@uvt.nl (W.H. Haemers), xiang@math.udel.edu (Q. Xiang).

0195-6698/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2009.07.009

If Γ is an srg, $\mathcal{E}(\Gamma) = k + fr - gs = -2gs$. By the use of (1) it is an easy exercise to see that the srg's of the title have energy $2m^4(1 + 2m^2)$. This equals an upper bound on the energy by Koolen and Moulton [8], who proved the following result.

Theorem 1. *Let Γ be a graph on n vertices. Then*

$$\mathcal{E}(\Gamma) \le \frac{n(1+\sqrt{n})}{2},$$

with equality holding if and only if Γ is an srg with parameters

$$\left(n, \frac{n+\sqrt{n}}{2}, \frac{n+2\sqrt{n}}{4}, \frac{n+2\sqrt{n}}{4}\right). \tag{2}$$

(It is not standard to call the complete graph strongly regular, but here it is convenient to do so, in order to make the statement correct for $n = 4$.) An srg with parameters (2) will be called *a max energy graph* of order n. The complement of a max energy graph is an srg with parameters

$$\left(n, \frac{n-\sqrt{n}}{2} - 1, \frac{n-2\sqrt{n}}{4} - 2, \frac{n-2\sqrt{n}}{4}\right).$$

However, this srg does not have maximal energy. Therefore we choose the complementary parameters for the title of this note. In [8], a family of max energy graphs of order 4^k was given. In [6], it is conjectured that a max energy graph of order n exists for all even squares n. To support this conjecture, several max energy graphs were constructed, including the case $n = 4m^4$, with m even. In this note, we will show that a max energy graph of order $4m^4$ also exists for all odd integers m. Thus we have:

Theorem 2. *A max energy graph of order $4m^4$ exists for every positive integer m.*

2. Regular graphical Hadamard matrices

A *Hadamard matrix* of order n is an $n \times n$ matrix H with entries ± 1, such that

$$HH^\top = nI_n,$$

where I_n is the identity matrix of order n. We will see below that max energy graphs are essentially the same objects as certain special Hadamard matrices.

Definition 3. A Hadamard matrix is said to be *graphical* if it is symmetric and it has constant diagonal.

Note that if H is a graphical Hadamard matrix of order n with δ on the diagonal, and J is the $n \times n$ all-ones matrix, then $A = \frac{1}{2}(J - \delta H)$ is the adjacency matrix of a graph on n vertices.

Definition 4. A Hadamard matrix is said to be *regular* if all its row and column sums are constant.

Let H be a Hadamard matrix of order n. If H is regular, then there exists an integer ℓ such that $H\mathbf{1} = H^\top\mathbf{1} = \ell\mathbf{1}$. Since $HH^\top = nI$, we have $\ell^2\mathbf{1} = n\mathbf{1}$. Hence $\ell = \pm\sqrt{n}$.

Definition 5. Let H be a regular graphical Hadamard matrix with row sum ℓ and δ on the diagonal. We say that H is of positive type, or type $+1$ (respectively, negative type, or type -1) if $\delta\ell > 0$ (respectively, $\delta\ell < 0$).

It has been observed (see [4], or [6]) that if H is a regular graphical Hadamard matrix with δ on its diagonal, of type ε, then $A = \frac{1}{2}(J - \delta H)$ is the adjacency matrix of an srg with parameters

$$\left(n, \frac{n-\varepsilon\sqrt{n}}{2}, \frac{n-2\varepsilon\sqrt{n}}{4}, \frac{n-2\varepsilon\sqrt{n}}{4}\right), \tag{3}$$

Conversely, if A is the adjacency matrix of an srg with parameters (3) then $J - 2A$ is a regular graphical Hadamard matrix of type ε. Thus a max energy graph is essentially the same as a regular graphical Hadamard matrix of negative type.

In [9] symmetric Bush type Hadamard matrices of order $4m^4$ have been constructed for every odd integer m. These matrices are regular graphical of positive type. The next section shows that a modification of the construction in [9] gives regular graphical Hadamard matrices of negative type, and therefore max energy graphs of order $4m^4$, for all odd m. Since it is easy to construct (see [6]) a regular graphical Hadamard matrix of order $4n$ of positive as well as of negative type from one of order n we have the following result.

Theorem 6. *For every positive integer m there exists a regular graphical Hadamard matrix of order $4m^4$ of positive, as well as of negative type.*

In the Section 4, we show how in this case the negative type can be obtained from the positive type (and vice versa) by Seidel switching.

3. Hadamard difference sets

Let G be a finite group of order v. A k-element subset D of G is called a (v, k, λ) *difference set* in G if the list of "differences" $d_1 d_2^{-1}$, $d_1, d_2 \in D$, $d_1 \neq d_2$, represents each non-identity element in G exactly λ times. Using multiplicative notation for the group operation, D is a (v, k, λ) difference set in G if and only if it satisfies the following equation in $\mathbb{Z}[G]$:

$$DD^{(-1)} = (k - \lambda)1_G + \lambda G, \tag{4}$$

where $D = \sum_{d \in D} d$, $D^{(-1)} = \sum_{d \in D} d^{-1}$, and 1_G is the identity element of G. A subset D of G is called *reversible* if $D^{(-1)} = D$. The difference sets considered in this note have parameters

$$(v, k, \lambda) = (4n^2, 2n^2 \pm n, n^2 \pm n).$$

These difference sets are called *Hadamard* difference sets (HDS), since their (1,−1)-incidence matrices are Hadamard matrices. Alternative names used by other authors are Menon difference sets and H-sets.

Lemma 7. *Let t be a positive integer, and let D be a reversible $(4t^2, 2t^2 + t, t^2 + t)$ Hadamard difference set in a group G of order $4t^2$ such that $1_G \notin D$. Then there exists a $4t^2 \times 4t^2$ regular graphical Hadamard matrix of negative type.*

Proof. Let Cay(G, D) be the Cayley graph with vertex set G and "connection" set D. That is, the vertex set of Cay(G, D) is G, two vertices $x, y \in G$ are connected by an edge if and only if $xy^{-1} \in D$. Let A be the adjacency matrix of Cay(G, D). Since $1_G \notin D$, the diagonal entries of A are all zeros. Also A is symmetric because D is reversible. Since D is a Hadamard difference set, we have

$$A^2 = t^2 I + (t^2 + t)J. \tag{5}$$

Now let $H = J - 2A$. Then H is symmetric since A is. The diagonal entries of H are all ones (i.e., $\delta = 1$). The row sums of H are constant, and they are equal to $\ell := 4t^2 - 2(2t^2 + t) = -2t$. Hence $\delta\ell = -2t < 0$. Furthermore, from (5), we have

$$H^2 = 4t^2 I.$$

Therefore, H is a regular graphical Hadamard matrix of negative type. $\square$

We now aim at constructing a reversible $(4t^2, 2t^2 + t, t^2 + t)$ Hadamard difference set in a group G of order $4t^2$ with $1_G \notin D$ and $t = m^2$, m is an odd integer. We start by reviewing a construction in [12] of Hadamard difference sets in a group of order $4p^4$, where p is an odd prime.

Let p be an odd prime and PG$(3, p)$ be a three-dimensional projective space over GF(p). We will say that a set C of points in PG$(3, p)$ is of *type* Q if

$$|C| = \frac{(p^4 - 1)}{4(p - 1)},$$

and each plane of PG$(3, p)$ meets C in either $\frac{(p-1)^2}{4}$ points or $\frac{(p+1)^2}{4}$ points. For each set X of points in PG$(3, p)$ we denote by $\widetilde{X}$ the set of all non-zero vectors $v \in \mathrm{GF}(p)^4$ with the property that $\langle v \rangle \in X$, where $\langle v \rangle$ is the one-dimensional subspace of GF$(p)^4$ generated by v.

Let $S = \{L_1, L_2, \ldots, L_{p^2+1}\}$ be a spread of PG(3, p), and let C_0, C_1 be the two sets of type Q in PG(3, p) such that

$$\forall_{1\le i\le s}|C_0 \cap L_i| = \frac{p+1}{2} \quad \text{and} \quad \forall_{s+1\le i\le 2s}|C_1 \cap L_i| = \frac{p+1}{2}, \tag{6}$$

where $s = \frac{p^2+1}{2}$. (We note that if we take S to be the regular spread in PG(3, p), then examples of type Q sets C_0, C_1 in PG(3, p) satisfying (6) were first constructed in [13] when $p \equiv 3 \pmod 4$, in [3,12] when $p = 5, 13, 17$, and in [1] for all odd primes p.) As in [12] we set

$$C_2 := (L_1 \cup \cdots \cup L_s) \setminus C_0,$$
$$C_3 := (L_{s+1} \cup \cdots \cup L_{2s}) \setminus C_1.$$

Note that $C_0 \cup C_2 = L_1 \cup \cdots \cup L_s$ and $C_1 \cup C_3 = L_{s+1} \cup \cdots \cup L_{2s}$.

Let A (resp. B) be a union of $(s-1)/2$ lines from $\{L_{s+1}, \ldots, L_{2s}\}$ (resp. $\{L_1, \ldots, L_s\}$). Let $K = \{1, a, b, ab\}$ and $W = (\mathrm{GF}(p)^4, +)$. Define

$$D_0 := \widetilde{C_0} \cup \widetilde{A},$$
$$D_2 := \widetilde{C_2} \cup \widetilde{A},$$
$$D_1 := \widetilde{C_1} \cup \widetilde{B},$$
$$D_3 := W \setminus (\widetilde{C_3} \cup \widetilde{B}).$$

Then

$$|D_0| = |D_1| = |D_2| = \frac{p^4 - p^2}{2}, \qquad |D_3| = \frac{p^4 + p^2}{2}.$$

By Theorem 2.2 [12] the set

$$D := (1, D_0) \cup (a, D_1) \cup (b, D_2) \cup (ab, D_3),$$

is a reversible $(4p^4, 2p^4 - p^2, p^4 - p^2)$ difference set in the group $K \times W$. We note that $0 \notin D_i$, for $i = 0, 1, 2$, but $0 \in D_3$ since $0 \notin \widetilde{C_3} \cup \widetilde{B}$.

Next we recall Turyn's composition theorem. We will use the version as stated in Theorem 6.5 [2, p. 45]. For convenience we introduce the following notation. Let W_1, W_2 be two groups. For $A, B \subseteq W_1$ and $C, D \subseteq W_2$, we define the following subset of $W_1 \times W_2$.

$$\nabla(A, B; C, D) := \big((A \cap B) \times C'\big) \cup \big((A' \cap B') \times C\big) \cup \big((A \cap B') \times D'\big) \cup \big((A' \cap B) \times D\big),$$

where $A' = W_1 \setminus A$, $B' = W_1 \setminus B$, $C' = W_2 \setminus C$, and $D' = W_2 \setminus D$.

Theorem 8 (*Turyn [11]*)**.** *Let $K = \{1, a, b, ab\}$ be a Klein four group. Let*

$$E_1 = (1, A_0) \cup (a, A_1) \cup (b, A_2) \cup (ab, A_3)$$

and

$$E_2 = (1, B_0) \cup (a, B_1) \cup (b, B_2) \cup (ab, B_3)$$

be reversible Hadamard difference sets in groups $K \times W_1$ and $K \times W_2$, respectively, where $|W_1| = w_1^2$ and $|W_2| = w_2^2$, w_1 and w_2 are odd, $A_i \subseteq W_1$ and $B_i \subseteq W_2$, and

$$|A_0| = |A_1| = |A_2| = \frac{w_1^2 - w_1}{2}, \qquad |A_3| = \frac{w_1^2 + w_1}{2},$$
$$|B_0| = |B_1| = |B_2| = \frac{w_2^2 - w_2}{2}, \qquad |B_3| = \frac{w_2^2 + w_2}{2}.$$

Let

$$E = (1, \nabla(A_0, A_1; B_0, B_1)) \cup (a, \nabla(A_0, A_1; B_2, B_3))$$
$$\cup\, (b, \nabla(A_2, A_3; B_0, B_1)) \cup (ab, \nabla(A_2, A_3; B_2, B_3)).$$

Then

$$|\nabla(A_0, A_1; B_0, B_1)| = |\nabla(A_0, A_1; B_2, B_3)| = |\nabla(A_2, A_3; B_0, B_1)| = \frac{w_1^2 w_2^2 - w_1 w_2}{2},$$

$$|\nabla(A_2, A_3; B_2, B_3)| = \frac{w_1^2 w_2^2 + w_1 w_2}{2},$$

and E is a reversible $(4w_1^2w_2^2, 2w_1^2w_2^2 - w_1w_2, w_1^2w_2^2 - w_1w_2)$ Hadamard difference set in the group $K \times W_1 \times W_2$.

Proposition 9. *Let $m > 1$ be an integer, and let $m = p_1p_2 \cdots p_t$, where p_i, $i = 1, 2, \ldots, t$, are (not necessarily distinct) odd primes. Let $K = \{1, a, b, ab\}$ be a Klein four group and $W = \mathbb{Z}_{p_1}^4 \times \cdots \times \mathbb{Z}_{p_t}^4$. Then there exists a reversible Hadamard difference set E in $G = K \times W$ such that*

$$E = (1, E_0) \cup (a, E_1) \cup (b, E_2) \cup (ab, E_3),$$

where $E_i \subset W$, $|E_0| = |E_1| = |E_2| = \frac{m^4-m^2}{2}$, $|E_3| = \frac{m^4+m^2}{2}$, and $1_W \notin E_i$ for $i = 0, 1, 2$, but $1_W \in E_3$.

Proof. We use induction on t. If $t = 1$, then the construction following the proof of Lemma 7 and preceding Turyn's composition theorem guarantees the existence of the required difference set. Assume that the proposition is true when m is a product of $(t - 1)$ primes. We will prove that the proposition is true when $m = p_1p_2 \cdots p_t$, where p_i, $i = 1, 2, \ldots, t$, are odd primes.

Let $w_1 = p_1^2p_2^2 \cdots p_{t-1}^2$. Then by induction hypothesis, there exists a reversible difference set

$$E_1 = (1, A_0) \cup (a, A_1) \cup (b, A_2) \cup (ab, A_3)$$

in $K \times W_1$, where $W_1 = \mathbb{Z}_{p_1}^4 \times \cdots \times \mathbb{Z}_{p_{t-1}}^4$, $A_i \subset W_1$, for all $i = 0, 1, 2, 3$,

$$|A_0| = |A_1| = |A_2| = \frac{w_1^2 - w_1}{2}, \qquad |A_3| = \frac{w_1^2 + w_1}{2},$$

and $1_{W_1} \notin A_i$ for $i = 0, 1, 2$, but $1_{W_1} \in A_3$.

Let $w_2 = p_t^2$. Again by the construction following the proof of Lemma 7, there exists a reversible difference set

$$E_2 = (1, B_0) \cup (a, B_1) \cup (b, B_2) \cup (ab, B_3)$$

in $K \times W_2$, where $W_2 = \mathbb{Z}_{p_t}^4$, $B_i \subset W_2$, for all $i = 0, 1, 2, 3$,

$$|B_0| = |B_1| = |B_2| = \frac{w_2^2 - w_2}{2}, \qquad |B_3| = \frac{w_2^2 + w_2}{2},$$

and $1_{W_2} \notin B_i$ for $i = 0, 1, 2$, but $1_{W_2} \in B_3$. By Theorem 8, we know that

$$\begin{aligned} E &= (1, \nabla(A_0, A_1; B_0, B_1)) \cup (a, \nabla(A_0, A_1; B_2, B_3)) \\ &\quad \cup (b, \nabla(A_2, A_3; B_0, B_1)) \cup (ab, \nabla(A_2, A_3; B_2, B_3)) \end{aligned}$$

is a reversible Hadamard difference set in $K \times W_1 \times W_2 = K \times W$, and

$$|\nabla(A_0, A_1; B_0, B_1)| = |\nabla(A_0, A_1; B_2, B_3)| = |\nabla(A_2, A_3; B_0, B_1)| = \frac{m^4 - m^2}{2},$$

$$|\nabla(A_2, A_3; B_2, B_3)| = \frac{m^4 + m^2}{2}.$$

Next it is straightforward to check that $1_W \notin \nabla(A_0, A_1; B_0, B_1)$, $1_W \notin \nabla(A_0, A_1; B_2, B_3)$, $1_W \notin \nabla(A_2, A_3; B_0, B_1)$, but $1_W \in \nabla(A_2, A_3; B_2, B_3)$. The proof of the proposition is complete. □

Theorem 10. *Let m be a positive odd integer. Then there exists a $4m^4 \times 4m^4$ regular graphical Hadamard matrix of negative type.*

Proof. When $m = 1$, one can easily demonstrate a 4×4 regular graphical Hadamard matrix of negative type. Therefore we will assume that m is an odd integer greater than 1.

By Proposition 9, there exists a reversible Hadamard difference set $E = (1, E_0) \cup (a, E_1) \cup (b, E_2) \cup (ab, E_3)$ in a group $G = K \times H$, where $K = \{1, a, b, ab\}$ is a Klein four group and H is an abelian group of order m^4, such that $|E_0| = |E_1| = |E_2| = \frac{m^4-m^2}{2}$, $|E_3| = \frac{m^4+m^2}{2}$, and $1_H \notin E_i$ for $i = 0, 1, 2$, but $1_H \in E_3$.

Let $E' = (K \times H) \setminus E$. That is, E' is the complement of E. And let $D = (ab, 1_H)E'$. Then D is a reversible $(4m^4, 2m^4 + m^2, m^4 + m^2)$ Hadamard difference set in $K \times H$. Since

$$D = (ab, E_0') \cup (b, E_1') \cup (a, E_2') \cup (1, E_3'),$$

and $1_H \notin E_3'$, we see that $1_G \notin D$. Applying Lemma 7 (with $t = m^2$), we conclude that there exists a $4m^4 \times 4m^4$ regular graphical Hadamard matrix of negative type. □

4. Seidel switching

Consider a graphical Hadamard matrix H of order n. Let X be a subset of $\{1, \ldots, n\}$. If we multiply rows and columns indexed by X by -1, we again obtain a graphical Hadamard matrix. The operation on the corresponding graph is called *Seidel switching*. In some cases it is possible to switch a graphical Hadamard matrix of positive type into one of negative type (and vice versa). Here we will show that this is indeed the case for the graphical Hadamard matrices constructed in [9], which leads to graphical Hadamard matrices of negative type constructed in the previous section.

Lemma 11. *Suppose*

$$H = \begin{bmatrix} H_1 & H_{12} \\ H_{12}^\top & H_2 \end{bmatrix}$$

is a regular graphical Hadamard matrix of order n. Furthermore assume that H_1 and H_2 have row sum 0. *Then there exist regular graphical Hadamard matrix of order n of positive type, as well as one of negative type.*

Proof. Consider

$$H' = \begin{bmatrix} H_1 & -H_{12} \\ -H_{12}^\top & H_2 \end{bmatrix}.$$

Then H' clearly is again a graphical Hadamard matrix with the same diagonal as H. Let ℓ be the row sum of H. Then, since H_1 and H_2 have row sum 0, H_{12} has row and column sum ℓ. This implies that H' is regular with row and column sum $-\ell$. So the type of H' is opposite to the type of H. □

Note that H_1, H_2 and H_{12} all have size $n/2 \times n/2$. Next we need to see that the construction in [9], admits the required partition. The construction uses a reversible Hadamard difference set of the form

$$D = (1, E_0) \cup (a, E_1) \cup (b, E_2) \cup (ab, E_3),$$

in the group $\{1, a, b, ab\} \times W$, with $|E_0| = \frac{m^4+m^2}{2}$ and $|E_1| = |E_2| = |E_3| = \frac{m^4-m^2}{2}$. The Hadamard matrix H is symmetric of Bush type, and therefore regular graphical of positive type. Consider the partition of H corresponding to the partition of $\{1, a, b, ab\} \times W$ into the cosets of the index two subgroup $\{1, a\} \times W$. This partitions D into $D_1 = (1, E_0) \cup (a, E_1)$ and $D_2 = (b, E_2) \cup (ab, E_3)$. We have that $|E_0| + |E_1| = m^4$, therefore H satisfies the condition of Lemma 11, and Seidel switching with respect to this partition gives a regular graphical Hadamard matrix H' of negative type.

And, of course, also the construction presented in Proposition 9 admits the structure of Lemma 11. Indeed, take the partition of the group $\{1, a, b, ab\} \times W$ into the cosets of $\{1, ab\} \times W$.

5. Smallest open cases

The last section of [6] contains a discussion on the status of the conjecture that there exist a max energy graph and a regular graphical Hadamard matrix of negative type of order $n = 4t^2$ for all t. It was stated that the first open case is $t = 7, n = 196$. A careful search of the literature reveals that a regular graphical Hadamard matrix of order 196 of positive type as well as one of negative type does exist, see [7, p. 258].

The case $t = 9, n = 324$ is constructed in this paper. Also existence has been established of regular graphical Hadamard matrices of positive and negative type for $t = 8$ and $t = 10$ (see [6]). Therefore now the first open case is $t = 11, n = 484$. We know of no graphical Hadamard matrix of this order.

Acknowledgement

The research of Qing Xiang was supported in part by NSF Grant DMS 0701049.

References

[1] Y.Q. Chen, On the existence of abelian Hadamard difference sets and a new family of difference sets, Finite Fields Appl. 3 (1997) 234–256.
[2] J.A. Davis, J. Jedwab, A unifying construction for difference sets, J. Combin. Theory Ser. A 80 (1997) 13–78.
[3] M. van Eupen, V.D. Tonchev, Linear codes and the existence of a reversible Hadamard difference set in $Z_2 \times Z_2 \times Z_5^4$, J. Combin. Theory Ser. A 79 (1997) 161–167.
[4] J.M. Goethals, J.J. Seidel, Strongly regular graphs derived from combinatorial designs, Canad. J. Math. 22 (1970) 597–614.
[5] I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohner, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer, Berlin, 2001, pp. 196–211.
[6] W.H. Haemers, Strongly regular graphs with maximal energy, Linear Algebra Appl. 429 (2008) 2719–2723.
[7] Y.J. Ionin, M.S. Shrikhande, Combinatorics of Symmetric Designs, Cambridge University Press, 2006.
[8] J.H. Koolen, V. Moulton, Maximal energy graphs, Adv. Appl. Math. 26 (2001) 47–52.
[9] M. Muzychuk, Q. Xiang, Symmetric Bush-type Hadamard matrices of order $4m^4$ exist for all odd m, Proc. Amer. Math. Soc. 134 (2006) 2197–2204.
[10] D. Stevanović, Energy of graphs: A few open problems and some suggestions, a talk given in the ARCC workshop: Spectra of families of matrices described by graphs, digraphs, and sign patterns, Oct. 2006.
[11] R.J. Turyn, A special class of Williamson matrices and difference sets, J. Combin. Theory Ser. A 36 (1984) 111–115.
[12] R.M. Wilson, Q. Xiang, Constructions of Hadamard difference sets, J. Combin. Theory Ser. A 77 (1997) 148–160.
[13] M.Y. Xia, Some infinite classes of special Williamson matrices and difference sets, J. Combin. Theory Ser. A 61 (1992) 230–242.