Journal of Combinatorial Theory, Series A **96**, 180–191 (2001)
doi:10.1006/jcta.2001.3183, available online at http://www.idealibrary.com on IDEAL®

# A Four-Class Association Scheme

Gary L. Ebert

*Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716*
E-mail: ebert@math.udel.edu

Sebastian Egner and Henk D. L. Hollmann

*Philips Research Laboratories, Prof. Holstlaan 4, 5656 AA Eindhoven, The Netherlands*
E-mail: sebastian.egner@natlab.research.philips.com, henk.d.l.hollmann@philips.com

and

Qing Xiang

*Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716*
E-mail: xiang@math.udel.edu

*Communicated by the Managing Editors*

Received August 15, 2000; published online June 18, 2001

We show the existence of a four-class association scheme defined on the unordered pairs of distinct points from $PG(1, q^2)$, for $q \geqslant 4$ a power of 2, thereby proving a conjecture of D. de Caen and E. van Dam (Fissioned triangular schemes via the cross-ratio, *European J. Combin.* **22** (2001), 297–301). This is a fusion of certain relations in the fission scheme $FT(q^2+1)$ obtained from the triangular association scheme. Combining three relations in the above four-class association scheme yields a strongly regular graph, which we show is isomorphic to one constructed by Brouwer and Wilbrink using hyperbolic solid sections of the parabolic quadric in $PG(4, q)$. © 2001 Academic Press

*Key Words:* association scheme; circle geometry; inversive plane; strongly regular graph.

## 1. INTRODUCTION

In [2], a putative association scheme is presented using unordered pairs of distinct points from $PG(1, q^2)$, where $q \geqslant 4$ is a power of 2, as the underlying set with relations defined in terms of cross-ratios. In this paper, we



0097-3165/01 $35.00

show that these relations may be redefined in terms of configurations in a (classical) inversive plane, and then use this model to verify that the structure is indeed a four-class association scheme. In fact, we show that a more general structure is an association scheme (see Theorem 3.7). We are also able to compute the first eigenmatrix of this scheme, thereby proving the truth of both parts of the conjecture made in [2].

It turns out that one obtains a strongly regular graph on the same set of elements (vertices) by merging three relations in the above four-class association scheme. This strongly regular graph has the parameters

$$\begin{aligned} v &= q^2(q^2+1)/2, & k &= (q+1)(q^2-1), \\ \lambda &= (q-1)(3q+2), & \mu &= 2q(q+1), \end{aligned} \tag{1.1}$$

where $q \geqslant 4$ is a power of 2. Strongly regular graphs with these parameters were constructed before by Brouwer and Wilbrink [1]. We will show that the two graphs are isomorphic when $q \geqslant 4$.

## 2. CIRCLE GEOMETRY AND OVOIDS

Let $n \geqslant 2$ be an integer. Any $3-(n^2+1, n+1, 1)$ design is called an *inversive plane of order n*, and the blocks of this design are often referred to as its *circles*. All known finite inversive planes are "egglike" in the following sense. An *ovoid* $\mathcal{O}$ of $PG(3, q)$, where $q > 2$ is a prime power, is any set of $q^2+1$ points with no three collinear. The classical example of an ovoid in $PG(3, q)$ is an elliptic quadric. At each point $P$ of $\mathcal{O}$ there is an unique tangent plane. All other planes in $PG(3, q)$ meet $\mathcal{O}$ in an *oval*; that is, all such planes meet $\mathcal{O}$ in a set of $q+1$ points, no three collinear. In the classical case where $\mathcal{O}$ is an elliptic quadric, these ovals are conics. If one takes as varieties the points of an ovoid $\mathcal{O}$, takes as blocks the nontangent planar intersections of $\mathcal{O}$, and defines incidence by inclusion, the resulting structure $I(\mathcal{O})$ is easily seen to be an inversive plane of order $q$. When $\mathcal{O}$ is an elliptic quadric in $PG(3, q)$, $I(\mathcal{O})$ is the *Miquelian* (or *classical*) inversive plane $M(q)$. When $q \geqslant 8$ is an odd power of 2 and $\mathcal{O}$ is the Tits ovoid of $PG(3, q)$, $I(\mathcal{O})$ is the Suzuki–Tits inversive plane $S(q)$. These are the only known finite inversive planes (see Chapter 6 of [3] for a general discussion of inversive planes).

It should be noted that there are many other models of the Miquelian inversive plane $M(q)$. In particular, the points of the projective line $PG(1, q^2)$ together with its Baer sublines (isomorphic copies of $PG(1, q)$) as "circles" forms a model for $M(q)$. Thus one frequently identifies the points of $M(q)$ with $\mathbb{F}_{q^2} \cup \{\infty\}$, using parametric coordinates for $PG(1, q^2)$. In this model one particular circle is represented by $\mathbb{F}_q \cup \{\infty\}$, and all other circles are obtained as images of this base circle under the linear fractional mappings $x \mapsto \frac{ax+b}{cx+d}$, where $a, b, c, d \in \mathbb{F}_{q^2}$ with $ad - bc \neq 0$, and the usual conventions on the symbol $\infty$ are taken. Using this model we see that $P\Gamma L(2, q^2)$ acts on $M(q)$ as an automorphism group.

For each circle $C$ in $M(q)$ there is a unique automorphism $\phi_C$ of $M(q)$ which has order 2 and whose fixed points are precisely the points of $C$ (see [3]). This involution is called *inversion with respect to* $C$, and distinct points $P$ and $Q$ in $M(q)$ are called *conjugate* with respect to $C$ if $\phi_C(P) = Q$. Given any two distinct points $P$ and $Q$ of $M(q)$, the remaining $q^2 - 1$ points may be partitioned into $q - 1$ mutually disjoint circles in exactly one way (see [7] for $q$ even and [6] for $q$ odd). These $q - 1$ circles are precisely the circles in $M(q)$ for which $P$ and $Q$ form a conjugate pair. It is interesting to note that not every circle of the Suzuki–Tits inversive plane $S(q)$ has an associated inversion.

Primarily, we will be using the egglike model $I(\mathcal{O})$ for our inversive planes, and thus we close this section by gathering for future use a few well-known facts concerning ovoids in $PG(3, q)$. Proofs of all these facts may be found in Chapters 15 and 16 of [4]. Through each point $P$ on an ovoid $\mathcal{O}$, the $q + 1$ lines through $P$ in the tangent plane $\pi_P$ to $\mathcal{O}$ at $P$ are the only tangent lines to $\mathcal{O}$ through $P$. Any point $Q$ not on $\mathcal{O}$ also has exactly $q + 1$ tangent lines passing through it. These $q + 1$ lines through $Q$ are coplanar if and only if $q$ is even. That is, for even $q$, the plane through $Q$ containing these $q + 1$ tangent lines meets $\mathcal{O}$ in an oval, and $Q$ is the *nucleus* of that oval. Moreover, for even $q$, if one associates each tangent plane with its point of contact and associates every other plane with the nucleus of the oval obtained by intersecting the plane with $\mathcal{O}$, this correspondence determines a *null polarity* of $PG(3, q)$. In this case the tangent lines to $\mathcal{O}$ are self-polar, while the secant lines and exterior lines to $\mathcal{O}$ get interchanged by the (null) polarity. More precisely, the polar line of a secant line $\ell$ is the intersection of the tangent planes at the two points of $\ell \cap \mathcal{O}$.

Recall that there is a one-to-one correspondence between the lines of $PG(3, q)$ and the points of the Klein (hyperbolic) quadric $\mathscr{K}$ in $PG(5, q)$. This correspondence is given by the Plücker coordinates of a line in 3-space (see [4, pp. 29–31]). Two distinct lines of $PG(3, q)$ meet if and only if the corresponding points on $\mathscr{K}$ are orthogonal; that is, if and only if the corresponding points on $\mathscr{K}$ determine a ruling line of $\mathscr{K}$. If $P$ is a point of $\mathscr{K}$, the tangent (or polar) hyperplane to $\mathscr{K}$ at $P$ meets $\mathscr{K}$ in a cone with vertex $P$ and base a 3-dimensional hyperbolic quadric. Every other hyperplane in $PG(5, q)$ meets $\mathscr{K}$ in a 4-dimensional parabolic quadric. Using Plücker coordinates, one sees that the $(q^2 + 1)(q + 1)$ tangent lines to an ovoid $\mathcal{O}$ in $PG(3, q)$, $q$ even, correspond to a nontangent hyperplane section of the Klein quadric $\mathscr{K}$; that is, the tangent lines to $\mathcal{O}$ correspond to a 4-dimensional parabolic quadric $\mathscr{P}(4, q)$ when $q$ is even.

## 3. A FOUR-CLASS ASSOCIATION SCHEME

Consider the set $\mathscr{X}$ of all 2-subsets $\{a, b\}$ of the projective line $PG(1, q^2)$, where $q \geqslant 4$ is a power of 2. As in [2], we define the following relations for two distinct elements $\{a, b\}$, $\{c, d\}$ in $\mathscr{X}$.

- $S_1$: $|\{a, b\} \cap \{c, d\}| = 1$.
- $S_2$: $\{a, b\} \cap \{c, d\} = \varnothing$, and the cross-ratio $\rho = \rho(a, b; c, d)$ satisfies $\rho^{q-1} = 1$.
- $S_3$: $\{a, b\} \cap \{c, d\} = \varnothing$, and the cross-ratio $\rho = \rho(a, b; c, d)$ satisfies $\rho^{q+1} = 1$.
- $S_4$: all other possibilities.

De Caen and Van Dam [2] conjectured the above four relations together with the diagonal relation $S_0$ form a four-class association scheme on $\mathscr{X}$. They arrived at this conjecture by merging relations in a fission scheme $FT(q^2+1)$ of the triangular association scheme (cf. [2]).

To show that the above four relations do indeed yield an association scheme, we redefine the relations in terms of the Miquelian inversive plane $M(q)$, where the $q^2+1$ points of $M(q)$ are identified with the points of $PG(1, q^2)$, using the second model for $M(q)$ discussed above. Throughout this section $q \geqslant 4$ will be a power of 2.

**Proposition 3.1.** *Two unordered pairs $\{a, b\}$ and $\{c, d\}$ of distinct points of $M(q)$ are in relation $S_2$ if and only if the four points $a, b, c, d$ are distinct and concircular.*

*Proof.* We will use the model for $M(q)$ arising from $PG(1, q^2)$. In particular, we identify the points of $M(q)$ with $\mathbb{F}_{q^2} \cup \{\infty\}$. Assume that $\{a, b\}$ and $\{c, d\}$ are in relation $S_2$. Then $a, b, c, d$ are four distinct points of $M(q)$. Since $\mathrm{Aut}(M(q)) \cong P\Gamma L(2, q^2)$ contains $PGL(2, q^2)$, which is triply transitive on the points of $M(q)$ and preserves cross-ratio, we may assume $a = 0, b = \infty$, and $c = 1$. Thus $\rho = \rho(a, b; c, d) = 1/d$, which according to relation $S_2$ implies that $d \in \mathbb{F}_q^*$. However, the unique circle in $M(q)$ containing 0, 1 and $\infty$ is $\mathbb{F}_q \cup \{\infty\}$, and thus $a, b, c, d$ are four distinct concircular points.

Conversely, suppose $a, b, c, d$ are four distinct concircular points of $M(q)$. Again we may assume without loss of generality that $a = 0, b = \infty$, and $c = 1$. Then as above, necessarily, $d \in \mathbb{F}_q^*$ and thus $\rho^{q-1} = 1$. That is, $\{a, b\}$ and $\{c, d\}$ are in relation $S_2$. ∎

**Proposition 3.2.** *Two unordered pairs $\{a, b\}$ and $\{c, d\}$ of distinct points of $M(q)$ are in relation $S_3$ if and only if there is a circle containing $\{c, d\}$ whose inversion has $\{a, b\}$ as a conjugate pair of points.*

*Proof.* Assume first that $\{a, b\}$ and $\{c, d\}$ are in relation $S_3$. Then, as in the proof of the previous proposition, we may assume that $a = 0$, $b = \infty$, and $c = 1$. Hence $\rho = \rho(a, b; c, d) = 1/d \in \mathbb{F}_{q^2} \setminus \{0, 1\}$, with $d^{q+1} = 1$. One easily checks that the mapping $\phi: z \mapsto 1/z^q$ is an inversion interchanging $a$ and $b$ whose circle of fixed points contains $c$ and $d$.

Conversely, assume that $C$ is some circle containing $c$ and $d$ such that $\phi_C(a) = b$. In particular, $a$, $b$, $c$ and $d$ are four distinct points. Without loss of generality, we may assume $a = 0$, $b = \infty$, $c = 1$ and $\rho = \rho(a, b; c, d) = 1/d \in \mathbb{F}_q{}^2 \setminus \{0, 1\}$. Using the transitive action of $PGL(2, q^2)$ on the circles of $M(q)$, we see that the unique circle through 1 whose inversion interchanges 0 and $\infty$ is $C = \{x \in \mathbb{F}_{q^2} : x^{q+1} = 1\}$. Since $d \in C$, this implies that $d^{q+1} = 1$, and therefore $\{a, b\}$ and $\{c, d\}$ are in relation $S_3$. ∎

*Remark.* Since there is a circle containing $\{c, d\}$ with $\{a, b\}$ as a conjugate pair if and only if there is a circle containing $\{a, b\}$ with $\{c, d\}$ as a conjugate pair, it is clear that $S_3$ is indeed a symmetric relation.

It is now a relatively easy matter to show that the above four relations $S_1$, $S_2$, $S_3$ and $S_4$ form an association scheme and to compute the intersection parameters. We do this by switching models and now representing $M(q)$ as the egglike inversive plane $I(\mathcal{O})$, where $\mathcal{O}$ is an elliptic quadric in $PG(3, q)$. Thus an unordered pair $\{a, b\}$ of distinct points in $M(q)$ gets identified with the secant line to $\mathcal{O}$ passing through $a$ and $b$. Four points $a$, $b$, $c$ and $d$ of $M(q)$ being concircular is equivalent to these four points of $\mathcal{O}$ being coplanar. If $\perp$ denotes the null polarity of $PG(3, q)$ determined by $\mathcal{O}$, the $q - 1$ (mutually disjoint) circles for which $\{a, b\}$ is a conjugate pair are obtained by intersecting $\mathcal{O}$ with the nontangent planes through $\ell^\perp$, where $\ell$ is the secant line meeting $\mathcal{O}$ in $\{a, b\}$. We thus reformulate the relations $S_1$, $S_2$, $S_3$ and $S_4$ as relations on distinct secant lines $\ell$ and $m$ of $\mathcal{O}$.

- $S_1$: $\ell$ meets $m$ in a point of $\mathcal{O}$.
- $S_2$: $\ell$ meets $m$ in a point of $PG(3, q) \setminus \mathcal{O}$.
- $S_3$: $\ell^\perp \cap m \neq \emptyset$ (or, equivalently, $\ell \cap m^\perp \neq \emptyset$).
- $S_4$: all other possibilities.

It should be noted that these relations are symmetric and partition the unordered pairs $\{\ell, m\}$ of distinct secant lines to any ovoid $\mathcal{O}$ in $PG(3, q)$, $q$ even, whether or not $\mathcal{O}$ is an elliptic quadric. For instance, to show that $\ell^\perp \cap m \neq \emptyset$ and $\ell \cap m \neq \emptyset$ together are inconsistent, let $\pi$ denote the plane determined by the intersecting lines $\ell^\perp$ and $m$. Then $\pi \cap \mathcal{O}$ is some oval $\Omega$, as $m$ is secant to $\mathcal{O}$, and $\pi^\perp$ is the nucleus $N$ of this oval $\Omega$. But $\ell^\perp \subset \pi$ and thus $N \in \ell$, implying $\ell \cap \pi = N$. If $\ell \cap m \neq \emptyset$, then necessarily $\ell \cap m = N$, and the nucleus $N$ lies on the secant line $m$ to the oval $\Omega$, a contradiction.

Thus we no longer assume that $\mathcal{O}$ is an elliptic quadric, but only assume that $\mathcal{O}$ is an ovoid of $PG(3, q)$, $q$ even. If we can show the above relations determine an association scheme, we not only prove the conjectures made in [2] are true, but also obtain another association scheme (with the same parameters) from the Tits ovoid when $q \geqslant 8$ is an odd power of 2. That is, we do not need circle inversions to prove that we have an association scheme. Throughout the remainder of the paper, $\perp$ will denote the null polarity of $PG(3, q)$ obtained from the ovoid $\mathcal{O}$.

PROPOSITION 3.3. *Let $\ell$ be a secant line of $\mathcal{O}$. Then the number $n_i$ of secant lines $m$ for which $\ell$ and $m$ are related by $S_i$, $i = 1, 2, 3, 4$, is given as follows*:

(1) $n_1 = 2(q^2 - 1)$,

(2) $n_2 = (q^2 - 1)(q - 2)/2$,

(3) $n_3 = (q^2 - 1)\, q/2$,

(4) $n_4 = (q^2 - 1)(q - 2)\, q/2$.

*Proof.* Let $\ell \cap \mathcal{O} = \{a, b\}$, and say that $m \cap \mathcal{O} = \{c, d\}$. To compute $n_1$, one must choose one point from $\{c, d\}$ to be either $a$ or $b$, and then the other point to be any one of the $q^2 - 1$ points of $\mathcal{O} \backslash \{a, b\}$. Thus $n_1 = 2(q^2 - 1)$. This is clearly independent of the given secant line $\ell$. Similarly, there are $q + 1$ planes $\pi$ passing through $\ell$, and one can choose $\binom{q-1}{2}$ pairs of distinct points from $(\pi \cap \mathcal{O}) \backslash \{a, b\}$. Each such pair determines a secant line $m$ meeting $\ell$ in some point not on $\mathcal{O}$, and there are no other choices for such secant lines. As any two distinct planes through $\ell$ meet only in $\ell$, we have $n_2 = (q^2 - 1)(q - 2)/2$. As $\ell^\perp \cap \mathcal{O} = \varnothing$, there are two tangent planes and $q - 1$ oval planes passing through the line $\ell^\perp$. For any such oval plane $\sigma$, one can choose $\binom{q+1}{2}$ pairs of distinct points on $\sigma \cap \mathcal{O}$. Each such pair determines a secant line $m$ meeting $\ell^\perp$, and there are no other choices for such secant lines. Thus $n_3 = (q^2 - 1)\, q/2$. The parameter $n_4$ is thus uniquely determined, independent of the choice of $\ell$, as there are a total of $\binom{q^2+1}{2}$ secant lines to $\mathcal{O}$. ∎

PROPOSITION 3.4. *The intersection parameter $p^3_{23}$ is well-defined, and its value is equal to* $\frac{(2q-1)(q-2)}{2}$.

*Proof.* Let $\ell$ and $m$ be secant lines to $\mathcal{O}$ such that $\ell^\perp \cap m \neq \varnothing$, and hence $\ell \cap m^\perp \neq \varnothing$ as well. We must count the number of secant lines $n$ such that $n$ meets $\ell$ in a point not on $\mathcal{O}$ and $n \cap m^\perp \neq \varnothing$. Let $X = \ell \cap m^\perp$. Since $m^\perp \cap \mathcal{O} = \varnothing$, $X \notin \mathcal{O}$ and thus $X$ lies on $\frac{1}{2}(q^2 - q)$ secant lines to $\mathcal{O}$, one of which is $\ell$. Any such secant, other than $\ell$, is a valid choice for $n$. All other choices for $n$ must meet $\ell$ in a point which does not equal $X$ and which does not lie on $\mathcal{O}$. Let $R$ be any one of the $q-2$ such points of $\ell$, and consider the plane $\pi = \langle R, m^{\perp} \rangle = \langle \ell, m^{\perp} \rangle$. Since $R$ lies on the secant line $\ell$ to the oval $\Omega = \pi \cap \mathcal{O}$, $R$ is not the nucleus of $\Omega$ and thus $R$ lies on $\frac{1}{2}q$ secants to $\Omega$, one of which is $\ell$. As each such secant must necessarily meet $m^{\perp}$, we get $\frac{1}{2}q - 1$ choices for $n$ for each such $R$. Allowing $R$ to vary and observing that there is no double counting, we see that $p_{23}^{3} = \frac{1}{2}(q^2 - q - 2) + \frac{1}{2}(q-2)^2 = \frac{1}{2}(2q-1)(q-2)$. Clearly $p_{23}^{3}$ is independent of the choice of $\ell$ and $m$. ∎

PROPOSITION 3.5. *The intersection parameter $p_{33}^{2}$ is well-defined, and it is equal to* $\frac{q(2q-1)}{2}$.

*Proof.* Let $\ell$ and $m$ be distinct secant lines of $\mathcal{O}$ meeting in a point $P \notin \mathcal{O}$. We must count the number of secant lines $n$ with $n^{\perp} \cap \ell \neq \emptyset$ and $n^{\perp} \cap m \neq \emptyset$. Since a line $n$ is secant if and only if $n^{\perp}$ is exterior, we must count the number of exterior lines to $\mathcal{O}$ meeting both $\ell$ and $m$. Since $\ell \cap m = P \notin \mathcal{O}$, there are $\frac{1}{2}(q^2 + q)$ exterior lines through $P$, and each of these lines is a valid choice for $n^{\perp}$. Any other exterior line meeting $\ell$ and $m$ must lie in the plane $\pi = \langle \ell, m \rangle$. Since $\pi \cap \mathcal{O}$ is an oval $\Omega$, we start by choosing a point $R$ on $\ell$ which is not equal to $P$ and does not lie on $\Omega$. There are $q-2$ choices for $R$. Since any such point $R$ lies on the secant line $\ell$ to $\Omega$, $R$ is not the nucleus of $\Omega$ and hence lies on $\frac{1}{2}q$ exterior lines to $\Omega$, all of which meet $m$ and are necessarily exterior to $\mathcal{O}$. Thus we see that $p_{33}^{2} = \frac{1}{2}q(q-2) + \frac{1}{2}(q^2 + q) = \frac{1}{2}q(2q-1)$, independent of the choice of $\ell$ and $m$. ∎

PROPOSITION 3.6. *The intersection parameter $p_{22}^{4}$ is well-defined, and it is equal to* $\frac{q^2 - 5q + 8}{2}$.

*Proof.* Let $\ell$ and $m$ be distinct secant lines of $\mathcal{O}$ such that $\ell \cap m = \emptyset$ and $\ell^{\perp} \cap m = \emptyset$, thereby also implying that $\ell \cap m^{\perp} = \emptyset$. We count the number of secant lines $n$ to $\mathcal{O}$ which meet each of $\ell$ and $m$ in a point not on $\mathcal{O}$. Any such line $n$ must lie in a plane through $m$ and pass through the intersection of that plane with $\ell$. Let $m \cap \mathcal{O} = \{X, Y, \}$, $A = Y^{\perp} \cap \ell$, and $B = X^{\perp} \cap \ell$. Since $X^{\perp} \cap Y^{\perp} = XY = m$ and $m \cap \ell = \emptyset$, necessarily $A \neq B$. First consider the plane $\pi_1 = \langle A, m \rangle$. If $\Omega_1$ denotes the oval $\pi_1 \cap \mathcal{O}$, then the nucleus of $\Omega_1$ is $\pi_1^{\perp} = m^{\perp} \cap \pi_1$. Hence, since $m^{\perp} \cap \ell = \emptyset$, $A$ is not the nucleus of $\Omega_1$, and thus $A$ lies on a unique tangent line of $\Omega_1$. Since $A \in Y^{\perp}$, AY is tangent to $\mathcal{O}$ and hence must be the unique tangent line to $\Omega_1$ through $A$. Therefore the $\frac{1}{2}q$ secant lines to $\Omega_1$ through $A$ include $AX$ but not $AY$. That is, we obtain exactly $\frac{1}{2}q - 1$ choices for $n$ in $\pi_1$, as our secant line $n$ must meet $m$ in a point not on $\mathcal{O}$. We similarly obtain precisely $\frac{1}{2}q - 1$ choices for $n$ in the plane $\pi_2 = \langle B, m \rangle$.

All remaining choices for $n$ lie in planes of the form $\langle R, m\rangle$, where $R$ is a point of $\ell \backslash \mathcal{O}$ other than $A$ or $B$. Let $\pi = \langle R, m\rangle$ be any one of these $q-3$ planes. For the same reason as above, $R$ is not the nucleus of the oval $\pi \cap \mathcal{O}$. Since $RX$ and $RY$ are distinct secant lines to this oval, there are $\frac{1}{2}q-2$ secants to $\pi \cap \mathcal{O}$ through $R$ which meet $m$ in a point not on $\mathcal{O}$, and these are precisely the choices for $n$ in this plane. Therefore $p_{22}^4 = 2(\frac{1}{2}q-1)+(q-3)(\frac{1}{2}q-2) = \frac{1}{2}(q^2-5q+8)$, independent of the choice of $\ell$ and $m$. ▮

Similar computations allow us to directly compute the intersection parameters $p_{ij}^k$ for $1 \leqslant k \leqslant 4$ and $1 \leqslant i, j \leqslant 3$. We then compute $n_i - p_{i1}^k - p_{i2}^k - p_{i3}^k = p_{i4}^k = p_{4i}^k$, thereby showing $p_{i4}^k$ is well-defined, and finally determine the constant $p_{44}^k = n_4 - p_{14}^k - p_{24}^k - p_{34}^k$, showing it also is well-defined. These computations verify the conjectures made in [2].

THEOREM 3.7. *The modified relations $S_1, S_2, S_3, S_4$ together with the diagonal relation $S_0$ define an association scheme on the set of secant lines to any ovoid $\mathcal{O}$ in $PG(3, q)$, where $q = 2^f$ with $f \geqslant 2$. The first eigenmatrix of this scheme is*

$$P = \begin{pmatrix} 1 & 2(q^2-1) & (q/2-1)(q^2-1) & q(q^2-1)/2 & q(q/2-1)(q^2-1) \\ 1 & q^2-3 & 2-q & -q & -q(q-2) \\ 1 & -2 & 1-q & 0 & q \\ 1 & -2 & (q/2-1)(q-1) & q(q-1)/2 & -q(q-2) \\ 1 & -2 & q(q-1)/2+1 & -q(q+1)/2 & q \end{pmatrix}.$$

*Proof.* The first eigenmatrix can be easily computed from the intersection matrices $P_0$, $P_1$, $P_2$, $P_3$, and $P_4$, where the $(j, k)$-entry of $P_i$ is the intersection parameter $p_{ij}^k$. Of course $P_0$ is the identity matrix. Computations as in the proofs of the above propositions show that

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ n_1 & q^2-1 & 4 & 4 & 4 \\ 0 & q-2 & 2(q-3) & 2(q-2) & 2(q-2) \\ 0 & q & 2q & 2(q-1) & 2q \\ 0 & q(q-2) & 2q(q-2) & 2q(q-2) & 2r \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & q-2 & 2(q-3) & 2(q-2) & 2(q-2) \\ n_2 & (q/2-1)(q-3) & (q^2-9q/2+6) & (q/2-1)(q-4) & (q^2-5q+8)/2 \\ 0 & q(q/2-1) & q(q/2-2) & (q/2-1)(2q-1) & q(q-3)/2 \\ 0 & q(q-2)^2/2 & q(q^2-5q+8)/2 & q(q/2-1)(q-3) & (q/2-1)\,r \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & q & 2q & 2(q-1) & 2q \\ 0 & q(q/2-1) & q(q/2-2) & (q/2-1)(2q-1) & q(q-3)/2 \\ n_3 & q(q-1)/2 & q(2q-1)/2 & q(q/2-1) & q(q-1)/2 \\ 0 & q^2(q/2-1) & q^2(q-3)/2 & q(q-1)(q/2-1) & qr/2 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & q(q-2) & 2q(q-2) & 2q(q-2) & 2r \\ 0 & q(q-2)^2/2 & q(q^2-5q+8)/2 & q(q/2-1)(q-3) & (q/2-1)\,r \\ 0 & q^2(q/2-1) & q^2(q-3)/2 & q(q-1)(q/2-1) & qr/2 \\ n_4 & qr(q/2-1) & qr(q/2-1) & qr(q/2-1) & q(q^3-4q^2+q+8)/2 \end{pmatrix},$$

where $n_1 = 2(q^2-1)$, $n_2 = (q^2-1)(q-2)/2$, $n_3 = q(q^2-1)/2$, $n_4 = (q^2-1) \cdot (q-2)\,q/2$ and $r = q^2 - 2q - 1$. ∎

COROLLARY 3.8. *The original relations $S_1, S_2, S_3, S_4$ together with the diagonal relation $S_0$ define an association scheme on $\mathscr{X}$, the set of 2-subsets of $PG(1, q^2)$, where $q = 2^f$ with $f \geqslant 2$. The first eigenmatrix of this scheme is the same as that given in Theorem 3.7.*

*Proof.* Take $\mathscr{O}$ to be an elliptic quadric in Theorem 3.7. ∎

*Remark.* When $f \geqslant 3$ is odd and $\mathscr{O}$ is the Tits ovoid, the association scheme obtained from Theorem 3.7 is different from the one obtained in Corollary 3.8. To see that these schemes really are different, note that the subgroup of the automorphism group of the scheme fixing class $S_1$ is essentially the stabilizer of $\mathscr{O}$ in $P\Gamma L(4, q)$, which is not the same for the Tits ovoid as for an elliptic quadric (the former is the Suzuki group $Sz(q)$, having size $(q^2+1)\,q^2(q-1)$, the latter is the orthogonal group $PGO_-(4, q)$, having size $2(q^2+1)\,q^2(q^2-1)$).

## 4. STRONGLY REGULAR GRAPHS

From the first eigenmatrix $P$ of a four-class scheme constructed as in Section 3, it is easy to see that one gets a strongly regular graph by merging $S_1$, $S_2$ and $S_3$ in the association scheme in Theorem 3.7. We will denote this strongly regular graph by $G(\mathscr{O})$. (Here $\mathscr{O}$ refers to the ovoid in Theorem 3.7.) In this graph, the vertices are the secant lines of $\mathscr{O}$, two distinct secant lines are adjacent in this graph if and only if they are related by $S_1$ or $S_2$ or $S_3$. This was pointed out in [2], modulo their conjectures. The parameters of such a strongly regular graph are

$$v = q^2(q^2+1)/2, \qquad k = (q+1)(q^2-1),$$

$$\lambda = (q-1)(3q+2), \qquad \mu = 2q(q+1),$$

and $r = q^2 - 2q - 1$, $s = -q - 1$, where $q = 2^f$, $f \geqslant 2$. Here, as usual, $r$ is the nontrivial positive eigenvalue, and $s$ is the negative one.

Brouwer and Wilbrink [1, Sect. 7B] also constructed strongly regular graphs with these parameters. Their construction goes as follows. Let $\mathcal{Q}$ be a nonsingular quadric in $PG(4, q)$. Let $V$ be the set of hyperplanes meeting $\mathcal{Q}$ in a hyperbolic quadric. If $x, y \in V$, then $x \sim y$ iff the corresponding hyperbolic sections are tangent (i.e., iff $x \cap y \cap \mathcal{Q}$ consists of two intersecting lines). This defines a strongly regular graph with parameters

$$v = q^2(q^2+1)/2, \qquad k = (q+1)(q^2-1),$$

$$\lambda = (q-1)(3q+2), \qquad \mu = 2q(q+1),$$

where $q$ is an arbitrary prime power. For convenience, we will call this graph the Brouwer–Wilbrink graph, or the BW graph in short.

In this section, we will show that the strongly regular graph $G(\mathcal{O})$ obtained by merging classes $S_1$, $S_2$, and $S_3$ as discussed above is isomorphic to the BW graph whenever $q \geqslant 4$ is a power of 2. This is true independent of the ovoid $\mathcal{O}$ used in the construction of the association scheme. Hence we do not get any new strongly regular graphs.

As mentioned in Section 2, the tangent lines to $\mathcal{O}$ are represented via Plücker coordinates on the Klein quadric $\mathcal{K}$ by a nontangent hyperplane section, which is necessarily a 4-dimensional parabolic quadric $\mathcal{Q} = \mathcal{P}(4, q)$. We use this particular quadric $\mathcal{Q}$ as the underlying quadric for the BW graph. If $\ell$ is any secant line to $\mathcal{O}$, the corresponding point $P_\ell$ on the Klein quadric will not lie on $\mathcal{Q}$, and the tangent hyperplane to $\mathcal{K}$ at $P_\ell$ will meet $\mathcal{Q}$ in a 3-dimensional hyperbolic quadric. If $\dagger$ denotes the null polarity associated with the Klein quadric $\mathcal{K}$, then as $\ell$ varies over all the secant lines to $\mathcal{O}$, $P_\ell^\dagger \cap \mathcal{Q}$ varies over all the hyperbolic solid sections of $\mathcal{Q} = \mathcal{P}(4, q)$. Hence the mapping $\ell \mapsto P_\ell^\dagger \cap \mathcal{Q}$ is a bijection between the vertices of the strongly regular graph $G(\mathcal{O})$ and the vertices of the BW graph. Note that under the Klein correspondence $P_\ell^\dagger \cap \mathcal{Q}$ corresponds to the tangent lines of $\mathcal{O}$ meeting $\ell$.

THEOREM 4.1. *For any ovoid $\mathcal{O}$ in $PG(3, q)$, $q = 2^f$ with $f \geqslant 2$, the strongly regular graph $G(\mathcal{O})$ is isomorphic to the BW graph.*

*Proof.* Since the graphs have the same parameters, it suffices to show that adjacent vertices $\ell$ and $m$ of $G(\mathcal{O})$ are mapped to adjacent vertices

$P_\ell^\dagger \cap \mathcal{Q}$ and $P_m^\dagger \cap \mathcal{Q}$ of the BW graph. That is, we must show that $P_\ell^\dagger \cap P_m^\dagger \cap \mathcal{Q}$ is a degenerate 2-dimensional quadric consisting of two intersecting lines. Using the Klein correspondence, this is equivalent to showing that the tangent lines to $\mathcal{O}$ meeting both $\ell$ and $m$ form two planar pencils, sharing one common line. There are three cases to consider.

Suppose first that $\ell$ and $m$ meet in a point $A$ not on $\mathcal{O}$. Let $\pi$ denote the plane determined by $\ell$ and $m$, and consider the oval $\Omega = \pi \cap \mathcal{O}$. The tangent lines to $\mathcal{O}$ meeting $\ell$ and $m$ are precisely the $q+1$ tangent lines to $\Omega$ in $\pi$, concurrent in the nucleus of $\Omega$, and the $q+1$ tangent lines to $\mathcal{O}$ in $A^\perp$, one of which lies in $\pi$. Hence we obtain two planar pencils sharing one line, as desired. The argument for $\ell$ and $m$ meeting in a point $A$ of $\mathcal{O}$ is essentially the same, the only difference being that $A^\perp$ is now a tangent plane to $\mathcal{O}$ rather than an oval plane.

Finally, consider the case when $\ell^\perp$ meets $m$ in some point $X$. The plane $\pi = \langle m, \ell^\perp \rangle$ meets $\mathcal{O}$ in an oval $\Omega$, whose nucleus is $\pi^\perp = \ell \cap \pi = \ell \cap m^\perp$. In this case the tangent lines to $\mathcal{O}$ meeting $\ell$ and $m$ are precisely the $q+1$ lines in $\pi$ through $\pi^\perp$ and (by symmetry) the $q+1$ lines in $X^\perp = \langle X, \ell \rangle$ through the nucleus $X$ of the oval $\mathcal{O} \cap X^\perp$. Once again, we obtain two planar pencils, sharing one common line. This completes the proof. ∎

*Remark* 4.2. As a side remark, we note that it is easy to compute the clique number of $G(\mathcal{O})$. Namely, let $\pi$ be any nontangent plane to $\mathcal{O}$, so that $\pi^\perp$ is the nucleus of the oval $\pi \cap \mathcal{O}$. There are $\frac{1}{2}(q^2 - q)$ secant lines to $\mathcal{O}$ through $\pi^\perp$ and $\frac{1}{2}(q^2 + q)$ secant lines to $\mathcal{O}$ in $\pi$. It is straightforward to check that any two of these $q^2$ secant lines are in relation $S_1$, $S_2$ or $S_3$, and thus we have a clique of size $q^2$ in $G(\mathcal{O})$. By the Hoffman bound [1, p. 91], the size of a clique $C$ in a $(v, k, \lambda, \mu, r, s)$ strongly regular graph is bounded by $1 + k/(-s)$, where $r$ denotes the nontrivial positive eigenvalue, and $s$ the negative one. Applying this bound to the graph $G(\mathcal{O})$, we see that the maximum size of a clique in this graph is $q^2$. Hence we conclude that the clique number of $G(\mathcal{O})$ is $q^2$ and the Hoffman bound is achieved.

## ACKNOWLEDGMENTS

The research of G.L. Ebert was partially supported by NSA Grant MDA 904-00-1-0029. The research of Q. Xiang was supported in part by NSA Grant MDA 904-99-1-0012, and a grant from the University of Delaware Research Foundation. The authors are extremely grateful to the referee for pointing out the advantage of using the egglike model for the classical inversive plane. This not only greatly simplified our proof of the isomorphism result concerning the BW graph, but also enabled us to construct an association scheme for any ovoid $\mathcal{O}$ in $PG(3, q)$, $q$ even. We also thank Edwin van Dam for carefully reading an early version of this paper.

## REFERENCES

1. A. E. Brouwer and J. H. van Lint, Strongly regular graphs and partial geometries, *in* "Enumeration and Designs" (D. M. Jackson and S. A. Vanstone, Eds.), pp. 85–122, Academic Press, New York.
2. D. de Caen and E. van Dam, Fissioned triangular schemes via the cross-ratio, *European J. Combin.* **22** (2001), 297–301.
3. P. Dembowski, "Finite Geometries," Springer-Verlag, New York/Berlin, 1968.
4. J. W. P. Hirschfeld, "Finite Projective Spaces of Three Dimensions," Oxford Univ. Press, London, 1985.
5. J. W. P. Hirschfeld and J. Thas, "General Galois Geometries," Oxford Univ. Press, London, 1991.
6. W. F. Orr, "The Miquelian Inversive Plane $IP(q)$ and the Associated Projective Planes," Ph.D. thesis, University of Wisconsin, Madison, 1973.
7. J. A. Thas, Flocks of finite egglike inversive planes, *in* "Finite Geometric Structures and Their Applications," Bressanone, Cremonese, Roma, 1973, pp. 189–191.