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NEGATIVE LATIN SQUARE TYPE PARTIAL DIFFERENCE SETS IN NONELEMENTARY ABELIAN 2-GROUPS

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Abstract

Combining results on quadrics in projective geometries with an algebraic interplay between finite fields and Galois rings, the first known family of partial difference sets with negative Latin square type parameters is constructed in nonelementary abelian groups, the groups $\mathbb{Z}_4^{2k} \times \mathbb{Z}_2^{4\ell-4k}$ for all k when ℓ is odd and for all $k < \ell$ when ℓ is even. Similarly, partial difference sets with Latin square type parameters are constructed in the same groups for all k when ℓ is even and for all $k < \ell$ when ℓ is odd. These constructions provide the first example where the non-homomorphic bijection approach outlined by Hagita and Schmidt can produce difference sets in groups that previously had no known constructions. Computer computations indicate that the strongly regular graphs associated to the partial difference sets are not isomorphic to the known graphs, and it is conjectured that the family of strongly regular graphs will be new.

1. *Introduction*

A k-element subset D of a finite multiplicative group G of order v is called a (v, k, λ, μ) *partial difference set* in G provided that the multiset of 'differences' $\{d_1 d_2^{-1} \mid d_1, d_2 \in D,\ \ d_1 \neq d_2\}$ contains each nonidentity element of D exactly λ times and each nonidentity element in $G \backslash D$ exactly μ times. Partial difference sets are equivalent to strongly regular graphs with a regular automorphism group, and they are connected to projective two-weight codes and two-intersection sets in projective spaces over finite fields. See [**19**] or [**4**] for background on these alternative approaches.

Ma's survey [**19**] identifies several families of partial difference sets. Among these are partial difference sets with parameters $(n^2,\ r(n-\epsilon),\ \epsilon n + r^2 - 3\epsilon r,\ r^2 - \epsilon r)$ for $\epsilon = \pm 1$. When $\epsilon = 1$, the partial difference set is called a *Latin square type partial difference set*, and when $\epsilon = -1$, the partial difference set is called a *negative Latin square type partial difference set*. Early results in this area can be found in the paper by Bailey and Jungnickel [**1**]. We will focus on the case where $n = 4^\ell$ and $r = 4^{\ell-1} + \epsilon$. Partial difference sets with these parameters have been constructed using quadratic forms over $\mathbb{F}_4$, the field with four elements, as we will mention in Subsection 1.2. This construction, as is true of most of the constructions the authors are aware of, is associated to elementary abelian groups.

Several authors, including those of [**6**, **7**, **13**, **14**, **18**, **21**], have used Galois rings, and more generally finite local rings, to construct partial difference sets in groups that are not elementary abelian. As far as we know, all partial difference sets constructed by finite local rings except those in [**7**] have Latin square parameters. After a presentation on one of these constructions, Mikhail Klin and William Martin [**15**] asked whether local ring construction can produce partial difference sets

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with negative Latin square type parameters. In this paper, we construct the first known family of partial difference sets with negative Latin square type parameters in nonelementary abelian groups by using Galois rings, and therefore answer the aforementioned question in the affirmative.

Recent work by Mathon [**20**] on maximal arcs in projective planes motivated the authors to revisit their work [**7**] on Denniston parameter partial difference sets. We recognized that one of the Galois ring constructions in that paper could be obtained using a natural map from the finite field construction. The mapping is not an isomorphism, but it is a bijection, and there is a relationship between the character sums in the two group rings. We extend that observation to higher dimensional projective spaces. Hagita and Schmidt [**10**] recently proposed using bijections between group rings with the property that character sums are preserved, and that is the approach we take in this paper. Wolfmann [**23**] used a similar approach to constructing Hadamard difference sets in nonelementary abelian groups via bijections between groups; like Hagita and Schmidt's paper, the examples found in Wolfmann's paper involved groups which were already known to contain difference sets. This paper uses similar techniques to construct partial difference sets in groups having no known previous constructions, the first time this approach has led to something new. (We note that Bruck [**3**] used a bijection between group rings to construct difference sets in nonabelian groups. His mapping did not preserve character sums in the same way as the Hagita–Schmidt approach.)

We will limit our attention to abelian groups in this paper. In that context, a (complex) character of an abelian group is a homomorphism from the group to the multiplicative group of complex roots of unity. The *principal character* is the character mapping every element of the group to 1. All other characters are called *nonprincipal*. Starting with the important work of Turyn [**22**], character sums have been a powerful tool in the study of difference sets of all types. The following lemma states how character sums can be used to verify that a subset of a group is a partial difference set.

LEMMA 1.1. *Let G be an abelian group of order v and D be a k-subset of G such that $\{d^{-1} \mid d \in D\} = D$ and $1 \notin D$. Then D is a (v, k, λ, μ) partial difference set in G if and only if, for any complex character χ of G,*

$$\sum_{d\in D} \chi(d) = \begin{cases} k & \text{if } \chi \text{ is principal on } G \\ \frac{(\lambda-\mu)\pm\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} & \text{if } \chi \text{ is nonprincipal on } G. \end{cases}$$

1.1. *Projective two-intersection sets*

Let $\mathrm{PG}(m-1, q)$ denote the desarguesian $(m-1)$-dimensional projective space over the finite field $\mathbb{F}_q$, where q is a power of a prime p, and let $\mathbb{F}_q^m$ be the m-dimensional vector space associated with $\mathrm{PG}(m-1, q)$. A *projective* (n, m, h_1, h_2) *set* $\mathcal{O} = \{\langle y_1\rangle, \langle y_2\rangle, \ldots, \langle y_n\rangle\}$ is a proper, non-empty set of n points of the projective space $\mathrm{PG}(m-1, q)$ with the property that every hyperplane meets $\mathcal{O}$ in h_1 or h_2 points. Define $\Omega = \{v \in \mathbb{F}_q^m \setminus \{0\} \mid \langle v\rangle \in \mathcal{O}\}$ to be the set of vectors in $\mathbb{F}_q^m$ corresponding to $\mathcal{O}$, that is, $\Omega = \mathbb{F}_q^* \mathcal{O}$.

For $(w_1, w_2, \ldots, w_m) \in \mathbb{F}_q^m$, we define a character of the additive group of $\mathbb{F}_q^m$ as follows:

$$\chi_{(w_1,w_2,\ldots,w_m)} : (v_1, v_2, \ldots, v_m) \longmapsto \xi_p^{\sum_{i=1}^m \mathrm{tr}(w_i v_i)}, \quad (v_1, v_2, \ldots, v_m) \in \mathbb{F}_q^m,$$

where ξ_p is a complex primitive pth root of unity and tr is the trace from $\mathbb{F}_q$ to $\mathbb{F}_p$. It is easy to see that $\chi_{(w_1,w_2,\ldots,w_m)}$, $(w_1,w_2,\ldots,w_m)\in\mathbb{F}_q^m$, are all the characters of the additive group of $\mathbb{F}_q^m$.

For any nontrivial additive character $\chi_{(w_1,w_2,\ldots,w_m)}$ of $\mathbb{F}_q^m$, we have

$$\begin{aligned}\chi_{(w_1,w_2,\ldots,w_m)}(\Omega) &= (q-1)|(w_1,w_2,\ldots,w_m)^{\perp}\cap\{y_1,y_2,\ldots,y_n\}|\\ &\quad+(-1)\left(n-|(w_1,w_2,\ldots,w_m)^{\perp}\cap\{y_1,y_2,\ldots,y_n\}|\right)\\ &= q|(w_1,w_2,\ldots,w_m)^{\perp}\cap\{y_1,y_2,\ldots,y_n\}|-n,\end{aligned}$$

where $(w_1,w_2,\ldots,w_m)^{\perp}=\{\langle(x_1,x_2,\ldots,x_m)\rangle\in\mathrm{PG}(m-1,q)\mid\sum_{i=1}^m w_ix_i=0\}$. Using this formula for $\chi_{(w_1,w_2,\ldots,w_m)}(\Omega)$ and Lemma 1.1, one can prove the following lemma.

LEMMA 1.2. *Let $\mathcal{O}$ and Ω be defined as above. Then $\mathcal{O}$ is a projective (n,m,h_1,h_2) set in $\mathrm{PG}(m-1,q)$ if and only if $\chi_{(w_1,w_2,\ldots,w_m)}(\Omega)=qh_1-n$ or qh_2-n, for every nontrivial additive character $\chi_{(w_1,w_2,\ldots,w_m)}$, $(w_1,w_2,\ldots,w_m)\in\mathbb{F}_q^m$. In other words, $\mathcal{O}$ is a projective (n,m,h_1,h_2) set in $\mathrm{PG}(m-1,q)$ if and only if Ω is a $(q^m,(q-1)n,\lambda,\mu)$ partial difference set in the elementary abelian group $(\mathbb{F}_q^m,+)$, where $\lambda=(q-1)n+(qh_1-n)(qh_2-n)+q(h_1+h_2)-2n$, and $\mu=(q-1)n+(qh_1-n)(qh_2-n)$.*

1.2. *Quadratic forms*

Let $\mathbb{F}_q$ be the field of q elements, where q is a prime power, and let V be an m-dimensional vector space over $\mathbb{F}_q$. A function $Q:V\longrightarrow\mathbb{F}_q$ is called a *quadratic form* if the following hold.

(i) $Q(\alpha v)=\alpha^2Q(v)$ for all $\alpha\in\mathbb{F}_q$ and $v\in V$.

(ii) The function $B:V\times V\longrightarrow\mathbb{F}_q$ defined by $B(v_1,v_2)=Q(v_1+v_2)-Q(v_1)-Q(v_2)$ is bilinear.

We call Q *nonsingular* if the subspace W with the property that Q vanishes on W and $B(w,v)=0$ for all $v\in V$ and $w\in W$ is the zero subspace. If the field $\mathbb{F}_q$ has odd characteristic, then Q is nonsingular if and only if B is nondegenerate; but this may not be true when $\mathbb{F}_q$ has characteristic 2, because in that case Q may not be zero on the radical $\mathrm{Rad}(V)=\{w\in V\mid B(w,v)=0 \text{ for all } v\in V\}$. However, if V is an even-dimensional vector space over an even-characteristic field $\mathbb{F}_q$, then Q is nonsingular if and only if B is nondegenerate (cf. [**5**, p. 14]).

Let Q be a nonsingular quadratic form on an m-dimensional vector space V over $\mathbb{F}_q$. If m is odd, then Q is equivalent to a quadratic form $x_1x_2+x_3x_4+\ldots+x_{m-2}x_{m-1}+cx_m^2$ for some scalar $c\in\mathbb{F}_q$ and it is called a *parabolic* quadratic form. If m is even, then Q is equivalent to either a quadratic form $x_1x_2+x_3x_4+\ldots+x_{m-1}x_m$ (called *hyperbolic*, or type $+1$) or $x_1x_2+x_3x_4+\ldots+x_{m-3}x_{m-2}+p(x_{m-1},x_m)$, where $p(x_{m-1},x_m)$ is an irreducible quadratic form in two indeterminates (called *elliptic*, or type -1).

The *quadric* of the projective space $\mathrm{PG}(m-1,q)$ corresponding to a quadratic form Q is the point set $\mathcal{Q}=\{\langle v\rangle\in\mathrm{PG}(m-1,q)\mid Q(v)=0\}$. The following theorems about the intersections of hyperplanes with quadrics in $\mathrm{PG}(m-1,q)$ are well known, see for example [**2**, p. 151; **4**; **9**].

THEOREM 1.3. *Let $\mathcal{Q}$ be a nonsingular elliptic quadric in* $\mathrm{PG}(2\ell-1,q)$. *Then the hyperplanes of* $\mathrm{PG}(2\ell-1,q)$ *intersect $\mathcal{Q}$ in sets of two sizes a and b, where*

$$a = 1 + \frac{q(q^{\ell-1}+1)(q^{\ell-2}-1)}{q-1}, \quad b = \frac{q^{2\ell-2}-1}{q-1}.$$

If we use Ω to denote the set of nonzero vectors in $\mathbb{F}_q^{2\ell}$ corresponding to $\mathcal{Q}$, then for any nontrivial additive character χ of $\mathbb{F}_q^{2\ell}$, we have $\chi(\Omega) = (q^{\ell-1}-1) - q^{\ell}$ or $(q^{\ell-1}-1)$ according to whether the hyperplane of $\mathrm{PG}(2\ell-1,q)$ *corresponding to χ meets $\mathcal{Q}$ in a or b points. That is, Ω is a $(q^{2\ell}, (q^{\ell}+1)(q^{\ell-1}-1), q^{2\ell-2} - q^{\ell-1}(q-1) - 2, q^{2\ell-2} - q^{\ell-1})$ negative Latin square type partial difference set in the additive group of $\mathbb{F}_q^{2\ell}$.*

REMARK 1.4. (1) Let $\mathcal{H}$ be a hyperplane of a projective space $\mathrm{PG}(m-1,q)$, and denote by W a point outside $\mathcal{H}$. If $\mathcal{Q}$ is a nonsingular quadric in $\mathcal{H}$ then the set

$$\mathcal{C} = \bigcup_{X \in \mathcal{Q}} (WX)$$

is called a *cone* with vertex W over $\mathcal{Q}$. Here WX means the set of points on the line through W and X.

(2) The hyperplanes meeting a nonsingular elliptic quadric $\mathcal{Q}$ in $\mathrm{PG}(2\ell-1,q)$ in sets of size a are called *tangent* hyperplanes; such a hyperplane meets $\mathcal{Q}$ in a cone over a nonsingular elliptic quadric in $\mathrm{PG}(2\ell-3,q)$. Any nontangent hyperplane meets $\mathcal{Q}$ in a nonsingular parabolic quadric in that hyperplane.

(3) Similarly, let Ω be the set of nonzero vectors in $\mathbb{F}_q^{2\ell}$ corresponding to a nonsingular hyperbolic quadric in $\mathrm{PG}(2\ell-1,q)$. Then for any nontrivial additive character χ of $\mathbb{F}_q^{2\ell}$, we have $\chi(\Omega) = q^{\ell} - (q^{\ell-1}+1)$ or $-(q^{\ell-1}+1)$. That is, Ω is a $(q^{2\ell}, (q^{\ell}-1)(q^{\ell-1}+1), q^{2\ell-2} + q^{\ell-1}(q-1) - 2, q^{2\ell-2} + q^{\ell-1})$ Latin square type partial difference set in the additive group of $\mathbb{F}_q^{2\ell}$.

Even though parabolic quadrics do not give rise to partial difference sets, we will also need their intersection patterns with hyperplanes.

THEOREM 1.5. *Let $\mathcal{Q}'$ be a nonsingular (parabolic) quadric in* $\mathrm{PG}(2\ell-2,q)$ *(so the size of $\mathcal{Q}'$ is $(q^{2\ell-2}-1)/(q-1)$). Then the hyperplanes of* $\mathrm{PG}(2\ell-2,q)$ *intersect $\mathcal{Q}'$ in sets of three sizes t, h and e with respective multiplicities T, H and E, where*

$$t = \frac{q^{2\ell-3}-1}{q-1}, \quad T = \frac{q^{2\ell-2}-1}{q-1},$$
$$e = t - q^{\ell-2}, \quad E = \frac{q^{2\ell-2}-q^{\ell-1}}{2},$$
$$h = t + q^{\ell-2}, \quad H = \frac{q^{2\ell-2}+q^{\ell-1}}{2}.$$

REMARK 1.6. (1) The T hyperplanes with intersection size t are called *tangent* hyperplanes to $\mathcal{Q}'$, such a hyperplane meets $\mathcal{Q}'$ in a cone over a parabolic quadric. Each of the E hyperplanes with intersection size e meets $\mathcal{Q}'$ in a nonsingular elliptic quadric in that hyperplane, and each of the H hyperplanes with intersection size h meets $\mathcal{Q}'$ in a nonsingular hyperbolic quadric in that hyperplane.

(2) If we use Ω' to denote the set of nonzero vectors in $\mathbb{F}_q^{2\ell-1}$ corresponding to $\mathcal{Q}'$, then for any nontrivial additive character χ of $\mathbb{F}_q^{2\ell-1}$, we have $\chi(\Omega') = -1$, $-1-q^{\ell-1}$, or $-1+q^{\ell-1}$. (Here the character values -1, $-1-q^{\ell-1}$, and $-1+q^{\ell-1}$ correspond respectively to cone section, elliptic section and hyperbolic section.)

As in the study of all other types of difference sets, one of the central problems in the study of partial difference sets is that for a given parameter set, which groups of the appropriate order contain a partial difference set with these parameters. As far as we know, no examples are known of negative Latin square type partial difference sets in nonelementary abelian groups, and this paper will construct the first such partial difference sets. We will focus on the case $q = 4$, with $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2 = \alpha + 1\}$. We define a quadratic form on $\mathbb{F}_4^{2\ell}$:

$$\begin{aligned} Q_{\ell,j}(x_1, x_2, \dots, x_{2\ell}) = {} & \left(\alpha x_1^2 + x_1x_2 + x_2^2\right) + \left(\alpha x_3^2 + x_3x_4 + x_4^2\right) + \dots \\ & + \left(\alpha x_{2j-1}^2 + x_{2j-1}x_{2j} + x_{2j}^2\right) + x_{2j+1}x_{2j+2} \\ & + x_{2j+3}x_{2j+4} + \dots + x_{2\ell-1}x_{2\ell}. \end{aligned}$$

Lemma 1.7. *Let $Q_{\ell,j}(x_1, x_2, \dots, x_{2\ell})$ be the quadratic form on $\mathbb{F}_4^{2\ell}$ defined above. Then the following hold.*

(a) *When $j \geqslant 2$, $Q_{\ell,j}$ is projectively equivalent to $Q_{\ell,j-2}$.*

(b) *When j is odd, $Q_{\ell,j}$ is elliptic. When j is even, $Q_{\ell,j}$ is hyperbolic.*

(c) *$Q_{\ell,j}$ is nonsingular.*

(d) *$Q_{\ell,j}(0, x_2, x_3, \dots, x_{2\ell})$ is a nonsingular parabolic quadratic form in $2\ell - 1$ indeterminates.*

(e) *$Q_{\ell,j}(x_1, x_2, \dots, x_{2i-1}, x_{2i-1} + x_{2i}, \dots, x_{2j-1}, x_{2j}, x_{2j+1}, \dots, x_{2\ell}) = Q_{\ell,j}(x_1, x_2, \dots, x_{2\ell})$ for any $1 \leqslant i \leqslant j$.*

Proof. (a) The mapping $x_{2j-3} \longmapsto (x'_{2j-3} + x'_{2j-1} + x'_{2j})$, $x_{2j-2} \longmapsto (\alpha x'_{2j-3} + x'_{2j-2})$, $x_{2j-1} \longmapsto (x'_{2j-1} + x'_{2j})$, $x_{2j} \longmapsto (\alpha x'_{2j-3} + x'_{2j-2} + x'_{2j})$, all other $x_i \longmapsto x'_i$ is an invertible linear transformation changing $Q_{\ell,j}$ to $Q_{\ell,j-2}$ as required.

(b) If j is even, then $Q_{\ell,j}$ is projectively equivalent to $Q_{\ell,0} = x_1x_2 + x_3x_4 + \dots + x_{2\ell-1}x_{2\ell}$, which is hyperbolic. Similarly, if j is odd, then $Q_{\ell,j}$ is projectively equivalent to $Q_{\ell,1} = \alpha x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_5x_6 + \dots + x_{2\ell-1}x_{2\ell}$, which is elliptic.

(c) Let $B(x, x')$ be the bilinear form associated with $Q_{\ell,j}$. Straightforward computations show that

$$B(x, x') = x_1x'_2 + x_2x'_1 + \dots + x_{2\ell-1}x'_{2\ell} + x_{2\ell}x'_{2\ell-1},$$

which is nondegenerate. Hence $Q_{\ell,j}$ is nonsingular.

(d) According to [**12**, Theorem 22.2.1], we define the matrix $A = [a_{ik}]$, where $a_{ii} = 2a_i$, $a_{ki} = a_{ik}$ for $i < k$. Here $a_1 = 1$, $a_2 = \alpha$, $a_3 = 1$, $a_4 = \alpha$, $a_5 = 1, \dots,$ $a_{2j-2} = \alpha$, $a_{2j-1} = 1$, $a_{2j} = \dots = a_{2\ell-1} = 0$, and $a_{12} = a_{13} = a_{14} = \dots = 0$, $a_{23} = 1$, $a_{45} = 1$, etc. View A as a matrix over $\mathbb{Z}$, and view α as an indeterminate for the time being. Compute $\Delta = \frac{1}{2}\det(A) = (4\alpha - 1)^{j-1}(-1)^{\ell-j}$. Now view Δ modulo 2, we have $\Delta \neq 0$, by [**12**, Theorem 22.2.1(i)], this shows that $Q_{\ell,j}(0, x_2, x_3, \dots, x_{2\ell})$ is nonsingular, and it is necessarily parabolic (note that the associated bilinear form for this quadratic form is degenerate, so we need the more sophisticated argument to demonstrate nonsingularity).

(e) This is by straightforward computation, which we omit. □

We note that Lemma 1.7(e) will be used in many character sum computations to get a sum of 0 over the pair of elements $(x_1, x_2, \ldots, x_{2i-1}, x_{2i}, \ldots, x_{2\ell})$ and $(x_1, x_2, \ldots, x_{2i-1}, x_{2i-1} + x_{2i}, \ldots, x_{2\ell})$.

1.3. *Galois ring preliminaries*

We need to recall the basics of Galois rings. Interested readers are referred to Hammons *et al.* [**11**] for more details. We will only use Galois rings over $\mathbb{Z}_4$. A *Galois ring over* $\mathbb{Z}_4$ *of degree* t, $t \geqslant 2$, denoted $\mathrm{GR}(4,t)$, is the quotient ring $\mathbb{Z}_4[x]/\langle\Phi(x)\rangle$, where $\Phi(x)$ is a basic primitive polynomial in $\mathbb{Z}_4[x]$ of degree t. Hensel's lemma implies that such polynomials exist. If ξ is a root of $\Phi(x)$ in $\mathrm{GR}(4,t)$, then $\mathrm{GR}(4,t) = \mathbb{Z}_4[\xi]$ and the multiplicative order of ξ is $2^t - 1$. In this paper, we will only need $\mathrm{GR}(4,2)$, and that has the basic primitive polynomial $\Phi(x) = x^2 + x + 1$.

The ring $R = \mathrm{GR}(4,t)$ is a finite local ring with unique maximal ideal $2R$, and $R/2R$ is isomorphic to the finite field $\mathbb{F}_{2^t}$. If we denote the natural epimorphism from R to $R/2R \cong \mathbb{F}_{2^t}$ by π, then $\pi(\xi)$ is a primitive element of $\mathbb{F}_{2^t}$.

The set $\mathcal{T} = \{0, 1, \xi, \xi^2, \ldots, \xi^{2^t-2}\}$ is a complete set of coset representatives of $2R$ in R. This set is usually called a *Teichmüller system* for R. The restriction of π to $\mathcal{T}$ is a bijection from $\mathcal{T}$ to $\mathbb{F}_{2^t}$, and we refer to this bijection as $\pi_{\mathcal{T}}$. An arbitrary element β of R has a unique 2-adic representation

$$\beta = \beta_1 + 2\beta_2,$$

where $\beta_1, \beta_2 \in \mathcal{T}$. Combining $\pi_{\mathcal{T}}$ with this 2-adic representation and specializing to the case of $\mathrm{GR}(4,2)$, we get a bijection F_k from $\mathbb{F}_4^{2\ell}$ to $(\mathrm{GR}(4,2))^k \times \mathbb{F}_4^{2\ell-2k}$ defined by

$$\begin{aligned} F_k : (&x_1, x_2, \ldots, x_{2k-1}, x_{2k}, \ldots, x_{2\ell}) \\ &\longmapsto \big(\pi_{\mathcal{T}}^{-1}(x_1) + 2\pi_{\mathcal{T}}^{-1}(x_2), \ldots, \pi_{\mathcal{T}}^{-1}(x_{2k-1}) + 2\pi_{\mathcal{T}}^{-1}(x_{2k}), x_{2k+1}, x_{2k+2}, \ldots, x_{2\ell}\big). \end{aligned}$$

The inverse of this map is the map F_k^{-1} from $(\mathrm{GR}(4,2))^k \times \mathbb{F}_4^{2\ell-2k}$ to $\mathbb{F}_4^{2\ell}$,

$$\begin{aligned} F_k^{-1} : (&\xi_1 + 2\xi_2, \ldots, \xi_{2k-1} + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell}) \\ &\longmapsto (\pi_{\mathcal{T}}(\xi_1), \pi_{\mathcal{T}}(\xi_2), \ldots, \pi_{\mathcal{T}}(\xi_{2k-1}), \pi_{\mathcal{T}}(\xi_{2k}), \xi_{2k+1}, \xi_{2k+2}, \ldots, \xi_{2\ell}). \end{aligned}$$

To simplify the notation we will usually omit the subindex in the bijection $\pi_{\mathcal{T}}^{-1} : \mathbb{F}_{2^t} \longrightarrow \mathcal{T}$. (Thus from now on, π^{-1} means the inverse of the bijection $\pi_{\mathcal{T}} : \mathcal{T} \longrightarrow \mathbb{F}_{2^t}$.) We will show in the next section how we can use F_k as a character-sum-preserving bijection to construct a partial difference set in a nonelementary abelian group, namely the additive group of $(\mathrm{GR}(4,2))^k \times \mathbb{F}_4^{2\ell-2k}$.

The *Frobenius map* f from R to itself is the ring automorphism $f : \beta_1 + 2\beta_2 \longmapsto \beta_1^2 + 2\beta_2^2$. This map is used to define the *trace* Tr from R to $\mathbb{Z}_4$, namely, $\mathrm{Tr}(\beta) = \beta + \beta^f + \ldots + \beta^{f^{t-1}}$, for $\beta \in R$. We note here that the Galois ring trace $\mathrm{Tr} : R \longrightarrow \mathbb{Z}_4$ is related to the finite field trace $\mathrm{tr} : \mathbb{F}_{2^m} \longrightarrow \mathbb{F}_2$ via

$$\mathrm{tr} \circ \pi = \pi \circ \mathrm{Tr}. \tag{1.1}$$

As a consequence, we have $\sqrt{-1}^{\mathrm{Tr}(2x)} = \sqrt{-1}^{2\mathrm{Tr}(x)} = (-1)^{\mathrm{Tr}(x)} = (-1)^{\pi \circ \mathrm{Tr}(x)} = (-1)^{\mathrm{tr}(\pi(x))}$, for all $x \in R$. The trace of a Galois ring can be used to define all of the additive characters of the ring, as demonstrated in the following well-known lemma.

LEMMA 1.8. *Let ψ be an additive character of R. Then there is a $\beta \in R$ so that $\psi(x) = \sqrt{-1}^{\mathrm{Tr}(\beta x)}$ for all $x \in R$.*

Since we can write $\beta = \beta_1 + 2\beta_2$ for $\beta \in \mathrm{GR}(4,2)$, where $\beta_k \in \mathcal{T},\ k = 1,2$, we will use the notation $\psi_{\beta} = \psi_{\beta_1+2\beta_2}$ to indicate the ring element used to define the character

$$x \longmapsto \sqrt{-1}^{\mathrm{Tr}(\beta x)}.$$

If $\beta_1 = 0$ but $\beta_2 \neq 0$, then $\psi_{2\beta_2}$ is a character of order 2 and $\psi_{2\beta_2}$ is principal on $2R$. If $\beta_1 \neq 0$, then $\psi_{\beta_1+2\beta_2}$ is a character of order 4 and $\psi_{\beta_1+2\beta_2}$ is nonprincipal on $2R$. Characters of

$$(\mathrm{GR}(4,2))^k \times \mathbb{F}_4^{2\ell-2k}$$

will be written

$$\Psi_k = \psi_{(\beta_1+2\beta_2,\ldots,\beta_{2k-1}+2\beta_{2k})} \otimes \chi_{(w_{2k+1},w_{2k+2},\ldots,w_{2\ell})} \tag{1.2}$$

for $\beta_i \in \mathcal{T},\ 1 \leqslant i \leqslant 2k$, and $w_i \in \mathbb{F}_4, 2k+1 \leqslant i \leqslant 2\ell$.

2. *Construction of partial difference sets in nonelementary abelian 2-groups*

We are now ready to state the main result of this paper. For $0 \leqslant k \leqslant j \leqslant \ell,\ \ell \geqslant 1$, define

$$D_{\ell,j,k} = \big\{F_k(x_1,x_2,\ldots,x_{2\ell}) \,|\, (x_1,x_2,\ldots,x_{2\ell}) \in \mathbb{F}_4^{2\ell}\backslash\{0\}, Q_{\ell,j}(x_1,x_2,\ldots,x_{2\ell}) = 0\big\}. \tag{2.1}$$

That is,

$$\begin{aligned} D_{\ell,j,k} = \big\{&(\xi_1+2\xi_2,\ldots,\xi_{2k-1}+2\xi_{2k},\xi_{2k+1},\ldots,\xi_{2\ell}) \in \big(\mathrm{GR}(4,2)^k \times \mathbb{F}_4^{2\ell-2k}\big)\backslash\{0\} \,| \\ &Q_{\ell,j}\big(F_k^{-1}(\xi_1+2\xi_2,\ldots,\xi_{2k-1}+2\xi_{2k},\xi_{2k+1},\ldots,\xi_{2\ell})\big) = 0\big\}. \end{aligned}$$

We can think of $D_{\ell,j,k}$ as a 'lifting' of the set of nonzero vectors $(x_1,x_2,\ldots,x_{2\ell}) \in \mathbb{F}_4^{2\ell}$ satisfying $Q_{\ell,j}(x_1,x_2,\ldots,x_{2\ell}) = 0$.

THEOREM 2.1. *For j odd, $1 \leqslant k \leqslant j \leqslant \ell$, the set $D_{\ell,j,k}$ is a $(4^{2\ell},\ (4^\ell+1)(4^{\ell-1}-1),\ 4^{2\ell-2}-3\cdot 4^{\ell-1}-2,\ 4^{2\ell-2}-4^{\ell-1})$ partial difference set in $\mathbb{Z}_4^{2k}\times\mathbb{Z}_2^{4\ell-4k}$. For j even, $1 \leqslant k \leqslant j \leqslant \ell$, the set $D_{\ell,j,k}$ is a $(4^{2\ell},\ (4^\ell-1)(4^{\ell-1}+1),\ 4^{2\ell-2}+3\cdot 4^{\ell-1}-2,\ 4^{2\ell-2}+4^{\ell-1})$ partial difference set in $\mathbb{Z}_4^{2k}\times\mathbb{Z}_2^{4\ell-4k}$.*

This theorem immediately leads to the following corollary, which lists the first known negative Latin square type partial difference sets in nonelementary abelian groups.

COROLLARY 2.2. *There are $(4^{2\ell},\ (4^\ell+1)(4^{\ell-1}-1),\ 4^{2\ell-2}-3\cdot 4^{\ell-1}-2,\ 4^{2\ell-2}-4^{\ell-1})$ negative Latin square type partial difference sets in $\mathbb{Z}_4^{2k}\times\mathbb{Z}_2^{4\ell-4k}$ for every $k \leqslant \ell$ except possibly $\mathbb{Z}_4^{2\ell}$ for ℓ even.*

By Lemma 1.1, in order to prove Theorem 2.1, we demonstrate that all of the nonprincipal characters of $(\mathrm{GR}(4,2))^k \times \mathbb{F}_4^{2\ell-2k}$ have a sum over $D_{\ell,j,k}$ of $-4^{\ell-1}-1\pm 2\cdot 4^{\ell-1}$ for j odd and $4^{\ell-1}-1\pm 2\cdot 4^{\ell-1}$ for j even. By Theorem 1.3 and Remark 1.4(3), the set $F_k^{-1}(D_{\ell,j,k}) = D_{\ell,j,0}$ is a partial difference set in the elementary abelian group of order $4^{2\ell}$, so it will have character sums equal to those we are expecting for $D_{\ell,j,k}$. The following sequence of lemmas will indicate a connection between the character sums over $D_{\ell,j,k}$ in the additive group of $(\mathrm{GR}(4,2))^k \times \mathbb{F}_4^{2\ell-2k}$ and the character sums over $F_k^{-1}(D_{\ell,j,k})$ in the additive group of $\mathbb{F}_4^{2\ell}$. We first consider the characters of order 2 of the additive group of $(\mathrm{GR}(4,2))^k \times \mathbb{F}_4^{2\ell-2k}$.

Lemma 2.3. *Let Ψ_k be a character of $(\mathrm{GR}(4,2))^k \times \mathbb{F}_4^{2\ell-2k}$ defined by (1.2) with $\beta_{2i-1}=0$ for $1 \leqslant i \leqslant k$. Then*

$$\Psi_k(D_{\ell,j,k}) = \chi_{(\pi(\beta_2),0,\pi(\beta_4),0,\ldots,\pi(\beta_{2k}),0,w_{2k+1},w_{2k+2},\ldots,w_{2\ell})}(F_k^{-1}(D_{\ell,j,k})).$$

Proof. Let $(\xi_1+2\xi_2, \xi_3+2\xi_4, \ldots, \xi_{2k-1}+2\xi_{2k}, \xi_{2k+1}, \xi_{2k+2}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k}$. The character value of this element is

$$\begin{aligned}
&\Psi_k(\xi_1+2\xi_2, \xi_3+2\xi_4, \ldots, \xi_{2k-1}+2\xi_{2k}, \xi_{2k+1}, \xi_{2k+2}, \ldots, \xi_{2\ell}) \\
&\quad = \sqrt{-1}^{\,\mathrm{Tr}(\sum_{i=1}^{k} 2\beta_{2i}\xi_{2i-1})}(-1)^{\mathrm{tr}(\sum_{i'=2k+1}^{2\ell} w_{i'}\xi_{i'})} \\
&\quad = (-1)^{\mathrm{tr}(\sum_{i=1}^{k} \pi(\beta_{2i})\pi(\xi_{2i-1}) + \sum_{i'=2k+1}^{2\ell} w_{i'}\xi_{i'})} \\
&\quad = \chi_{(\pi(\beta_2),0,\ldots,\pi(\beta_{2k}),0,w_{2k+1},w_{2k+2},\ldots,w_{2\ell})}(\pi(\xi_1), \pi(\xi_2), \ldots, \pi(\xi_{2k}), \xi_{2k+1}, \ldots, \xi_{2\ell}).
\end{aligned}$$

The second equality uses the fact that $\sqrt{-1}^{\,\mathrm{Tr}(2\beta_{2i}\xi_{2i-1})} = (-1)^{\mathrm{tr}(\pi(\beta_{2i}\xi_{2i-1}))}$ as mentioned earlier in the discussion on trace. (Here Tr is the trace from $\mathrm{GR}(4,2)$ to $\mathbb{Z}_4$, and tr is the trace from $\mathbb{F}_4$ to $\mathbb{F}_2$.) This proves the lemma. $\square$

Thus, the character sums $\Psi_k(D_{\ell,j,k})$ associated to characters Ψ_k of order 2 will have the correct sum. In order to prove Theorem 2.1, we only need compute $\Psi_k(D_{\ell,j,k})$, where Ψ_k has order 4.

Our strategy for proving Theorem 2.1 is as follows. We will first prove Theorem 2.1 in the case $k=1$, then prove the whole theorem by strong induction on k. We start by computing the character sum $\Psi_1(D_{\ell,j,1})$, where $\Psi_1 = \psi_{\beta_1+2\beta_2} \otimes \chi_{(w_3,w_4,\ldots,w_{2\ell})}$ is a character of order 4, namely $\beta_1 \neq 0$. We will need the following definitions. Let

$$\Omega_0 = \{(2\xi_2, \xi_3, \ldots, \xi_{2\ell}) \in \mathrm{GR}(4,2) \times \mathbb{F}_4^{2\ell-2} \backslash \{0\} \mid Q_{\ell,j}\big(F_1^{-1}(2\xi_2, \xi_3, \ldots, \xi_{2\ell})\big) = 0\}, \tag{2.2}$$

and let

$$O_0 = F_1^{-1}(\Omega_0) = \{(0, \pi(\xi_2), \xi_3, \ldots, \xi_{2\ell}) \in \mathbb{F}_4^{2\ell} \backslash \{0\} \mid Q_{\ell,j}(0, \pi(\xi_2), \xi_3, \ldots, \xi_{2\ell}) = 0\}. \tag{2.3}$$

We observe that $\Psi_1(\Omega_0) = \chi_{(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})}(O_0)$. The next lemma shows their common sum when $\beta_1 \neq 0$.

Lemma 2.4. *Suppose that $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})}$ is a character of $\mathbb{F}_4^{2\ell}$ with $\beta_1, \beta_2 \in \mathcal{T}$ and $\beta_1 \neq 0$, and let O_0 be as above. Then*

$$\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})}(O_0) = -1 \pm 4^{\ell-1}.$$

Proof. For convenience, let

$$O_0' = \{(x_2, x_3, \ldots, x_{2\ell}) \in \mathbb{F}_4^{2\ell-1} \backslash \{0\} \mid Q_{\ell,j}(0, x_2, \ldots, x_{2\ell}) = 0\}.$$

By the definition of O_0, we have

$$\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})}(O_0) = \chi'_{(\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O'_0).$$

Since $Q_{\ell,j}(0,x_2,x_3,\ldots,x_{2\ell})$ is a nonsingular parabolic quadratic form in $2\ell-1$ variables (cf. Lemma 1.7(d)), the corresponding quadric in $\mathrm{PG}(2\ell-2,4)$ has three intersection sizes with the hyperplanes, leading to three distinct character values of $-1\pm4^{\ell-1}$ or -1 (see Theorem 1.5, and Remark 1.6). To prove the lemma, we need to show that the condition $\beta_1\neq0$ excludes all characters that have a sum of -1 over O'_0. To do that, we observe that there are $4^{2\ell-2}-1$ nonprincipal characters satisfying $\pi(\beta_1)=0$, which is the same number of characters that have a sum of -1 over O'_0 since the number of tangent hyperplanes to a nonsingular parabolic quadratic form in $\mathrm{PG}(2\ell-2,4)$ is $(4^{2\ell-2}-1)/(4-1)$ (cf. Theorem 1.5). We will show that all of these characters $\chi'_{(0,w_3,\ldots,w_{2\ell})}$ do indeed have a sum of -1 over O'_0, implying that

$$\chi'_{(\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O'_0), \qquad \pi(\beta_1)\neq0,$$

are equal to $-1\pm4^{\ell}$.

For any nontrivial character $\chi'_{(0,w_3,\ldots,w_{2\ell})}$, we have

$$\begin{aligned}\chi'_{(0,w_3,\ldots,w_{2\ell})}(O'_0) &= \sum_{x_2^2+x_3x_4+\ldots+x_{2\ell-1}x_{2\ell}=0}(-1)^{\mathrm{tr}(w_3x_3+\ldots+w_{2\ell}x_{2\ell})}\\ &= \sum_{(0,0,\ldots,0)\neq(x_3,x_4,\ldots,x_{2\ell})\in\mathbb{F}_4^{2\ell-2}}(-1)^{\mathrm{tr}(w_3x_3+\ldots+w_{2\ell}x_{2\ell})}\sum_{x_2^2=x_3x_4+\ldots+x_{2\ell-1}x_{2\ell}}1.\end{aligned}$$

Note that the inner sum in the last summation actually has only one term, so

$$\chi'_{(0,w_3,\ldots,w_{2\ell})}(O'_0) = \sum_{(0,0,\ldots,0)\neq(x_3,x_4,\ldots,x_{2\ell})\in\mathbb{F}_4^{2\ell-2}}(-1)^{\mathrm{tr}(w_3x_3+\ldots+w_{2\ell}x_{2\ell})} = -1.$$

As $(w_3,\ldots,w_{2\ell})$ runs through all nonzero $(2\ell-2)$-tuples, we get -1 as the character sum of O'_0 $(4^{2\ell-2}-1)$ times; these account for *all* the tangent hyperplanes. Hence the lemma follows. □

The following lemma considers the case j odd and $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O_0) = -1-4^{\ell-1}$, keeping in mind that the character sum over the entire set $F_1^{-1}(D_{\ell,j,1})$ is $-1-4^{\ell-1}\pm2\cdot4^{\ell-1}$ since $F_1^{-1}(D_{\ell,j,1})$ with j odd is a negative Latin square type partial difference set in the additive group of $\mathbb{F}_4^{2\ell}$ (cf. Theorem 1.3).

LEMMA 2.5. *Let j be odd and $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}$ be a character of $\mathbb{F}_4^{2\ell}$ with $\beta_1,\beta_2\in\mathcal{T}$, $\beta_1\neq0$. If $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O_0)=-1-4^{\ell-1}$, then*

$$\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}\big(F_1^{-1}(D_{\ell,j,1})\backslash O_0\big)=\pm2\cdot4^{\ell-1}.$$

Proof. This is obvious from the comments before the lemma. □

The analogous lemma for the j even case is as follows.

LEMMA 2.6. *Let j be even and $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}$ be a character of $\mathbb{F}_4^{2\ell}$ with $\beta_1,\beta_2\in\mathcal{T}$, $\beta_1\neq0$. If $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O_0)=-1+4^{\ell-1}$, then*

$$\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}\big(F_1^{-1}(D_{\ell,j,1})\backslash O_0\big)=\pm2\cdot4^{\ell-1}.$$

Proof. Note that when j is even, the character sum over the entire set $F_1^{-1}(D_{\ell,j,1})$ is $-1+4^{\ell-1} \pm 2\cdot 4^{\ell-1}$ since $F_1^{-1}(D_{\ell,j,1})$ with j even is a Latin square type partial difference set in the additive group of $\mathbb{F}_4^{2\ell}$. The conclusion of the lemma is obvious from this observation. □

We now consider the other case from Lemma 2.4, namely that $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O_0) = -1+4^{\ell-1}$ in the case when j is odd, and $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O_0) = -1-4^{\ell-1}$ in the case when j is even.

Lemma 2.7. *Let j be odd and let $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}$ be a character of $\mathbb{F}_4^{2\ell}$ with $\beta_1,\beta_2 \in \mathcal{T}$, and $\beta_1 \neq 0$. If $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O_0) = -1+4^{\ell-1}$, then*

$$\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}\big(F_1^{-1}(D_{\ell,j,1})\backslash O_0\big) = 0.$$

Proof. Since j is odd, $F_1^{-1}(D_{\ell,j,1})$ corresponds to a nonsingular elliptic quadric in $\mathrm{PG}(2\ell-1,4)$; hence its character values $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(F_1^{-1}(D_{\ell,j,1}))$ are $-1-4^{\ell-1}\pm 2\cdot 4^{\ell-1}$. Assume to the contrary that $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(F_1^{-1}(D_{\ell,j,1})\backslash O_0) \neq 0$. By the assumption that $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(O_0) = -1+4^{\ell-1}$, we have

$$\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}(F_1^{-1}(D_{\ell,j,1})) = -1-4^{\ell-1}-2\cdot 4^{\ell-1},$$

that is, the hyperplane $\mathcal{H} : \pi(\beta_2)x_1+\pi(\beta_1)x_2+w_3x_3+\ldots+w_{2\ell}x_{2\ell} = 0$ meets $\mathcal{Q}_{\ell,j}$ in a cone $\mathcal{C}$ with vertex W over a nondegenerate elliptic quadric in $\mathrm{PG}(2\ell-3,4)$. (Here $\mathcal{Q}_{\ell,j}$ is the elliptic quadric in $\mathrm{PG}(2\ell-1,4)$ defined by $Q_{\ell,j}$.) We write $\mathcal{H}\cap\mathcal{Q}_{\ell,j} = \mathcal{C}$.

Let $\mathcal{H}_1$ denote the hyperplane $x_1 = 0$. We know that $\mathcal{H}_1 \cap \mathcal{Q}_{\ell,j} = \mathcal{Q}'_{\ell,j}$ is a nonsingular parabolic quadric with equation $x_2^2+\alpha x_3^2+x_3x_4+x_4^2+\ldots+\alpha x_{2j-1}^2+x_{2j-1}x_{2j}+x_{2j}^2+x_{2j+1}x_{2j+2}+\ldots+x_{2\ell-1}x_{2\ell}$ (cf. Lemma 1.7(d)).

Now consider $\mathcal{H}\cap\mathcal{Q}'_{\ell,j}$ (this will determine the character value $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})}(O_0)$). We have

$$\mathcal{H}\cap\mathcal{Q}'_{\ell,j} = \mathcal{H}_1\cap(\mathcal{H}\cap\mathcal{Q}_{\ell,j}) = \mathcal{H}_1\cap\mathcal{C}.$$

Note that $\mathcal{C}$ is a cone over an elliptic quadric, so any hyperplane not through the vertex W meets $\mathcal{C}$ in an elliptic quadric [**2**, p. 177]. Hence if we can show that $\mathcal{H}_1$ does not go through the vertex W, then we know that $\mathcal{H}_1\cap\mathcal{C} = \mathcal{H}\cap\mathcal{Q}'_{\ell,j}$ is not hyperbolic; thus $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})}(O_0) \neq -1+4^{\ell-1}$, which is a contradiction.

The cone $\mathcal{C}$ is a degenerate quadric with a one-dimensional radical being its vertex W. We compute the radical of $\mathcal{C} = \mathcal{H}\cap\mathcal{Q}_{\ell,j}$. Let $B(X,X') = Q_{\ell,j}(X+X')-Q_{\ell,j}(X)-Q_{\ell,j}(X')$ be the bilinear form associated with $Q_{\ell,j}$, where $X = (x_1,x_2,\ldots,x_{2\ell})$ and $X' = (x'_1,x'_2,\ldots,x'_{2\ell})$. Then

$$\begin{aligned}\mathrm{Rad}(\mathcal{H}\cap\mathcal{Q}_{\ell,j}) &= \{X \mid B(X,X') = 0 \text{ for all } X'\in\mathcal{H}\}\\ &= \{(x_1,x_2,\ldots,x_{2\ell}) \mid x_1x'_2+x_2x'_1+\ldots+x_{2\ell-1}x'_{2\ell}+x_{2\ell}x'_{2\ell-1} = 0\\ &\qquad \text{for all } (x'_1,x'_2,\ldots,x'_{2\ell})\in\mathcal{H}\}\\ &= \{\epsilon(\pi(\beta_1),\pi(\beta_2),w_4,w_3,\ldots,w_{2\ell},w_{2\ell-1}) \mid \epsilon\in\mathbb{F}_4^*\}.\end{aligned}$$

Therefore the vertex of $\mathcal{C}$ is $W = (\pi(\beta_1),\pi(\beta_2),w_4,w_3,\ldots,w_{2\ell},w_{2\ell-1})$. Since $\beta_1 \neq 0$, we see that $\mathcal{H}_1 : x_1 = 0$ does not go through W. The proof is now complete. □

The analogous lemma for the j even case is given below.

LEMMA 2.8. *Let j be even and let $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\dots,w_{2\ell})}$ be a character of $\mathbb{F}_4^{2\ell}$ with $\beta_1,\beta_2\in\mathcal{T}$, and $\beta_1\neq 0$. If $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\dots,w_{2\ell})}(O_0)=-1-4^{\ell-1}$, then*

$$\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\dots,w_{2\ell})}\big(F_1^{-1}(D_{\ell,j,1})\backslash O_0\big)=0.$$

The proof of this lemma is completely parallel to that of Lemma 2.7. We omit the details.

The preceding five lemmas are enough to prove Theorem 2.1 in the case $k=1$ for j both odd and even (see the proof below). For the purpose of doing induction on k, we define, for any integers j,k, $2\leqslant k<j\leqslant\ell$, the following sets:

$$\begin{aligned}
\Upsilon_{k-1,k}&=\{(\xi_1+2\xi_2,\dots,0+2\xi_{2k-2},0+2\xi_{2k},\xi_{2k+1},\dots,\xi_{2\ell})\in D_{\ell,j,k}\},\\
\Upsilon_{k-1}&=\{(\xi_1+2\xi_2,\dots,0+2\xi_{2k-2},\xi_{2k-1}+2\xi_{2k},\xi_{2k+1},\dots,\xi_{2\ell})\\
&\qquad\in D_{\ell,j,k},\xi_{2k-1}\neq 0\},\\
\Upsilon_{k}&=\{(\xi_1+2\xi_2,\dots,\xi_{2k-3}+2\xi_{2k-2},0+2\xi_{2k},\xi_{2k+1},\dots,\xi_{2\ell})\\
&\qquad\in D_{\ell,j,k},\xi_{2k-3}\neq 0\},\\
\Upsilon&=\{(\xi_1+2\xi_2,\dots,\xi_{2k-3}+2\xi_{2k-2},\xi_{2k-1}+2\xi_{2k},\xi_{2k+1},\dots,\xi_{2\ell})\\
&\qquad\in D_{\ell,j,k},\xi_{2k-1}\neq 0,\xi_{2k-3}\neq 0\},\\
U_{k-1,k}&=\{(\xi_1+2\xi_2,\dots,0,\pi(\xi_{2k-2}),0,\pi(\xi_{2k}),\xi_{2k+1},\dots,\xi_{2\ell})\in D_{\ell,j,k-2}\},\\
U_{k-1}&=\{(\xi_1+2\xi_2,\dots,0,\pi(\xi_{2k-2}),\pi(\xi_{2k-1}),\pi(\xi_{2k}),\xi_{2k+1},\dots,\xi_{2\ell})\\
&\qquad\in D_{\ell,j,k-2},\pi(\xi_{2k-1})\neq 0\},\\
U_{k}&=\{(\xi_1+2\xi_2,\dots,\pi(\xi_{2k-3}),\pi(\xi_{2k-2}),0,\pi(\xi_{2k}),\xi_{2k+1},\dots,\xi_{2\ell})\\
&\qquad\in D_{\ell,j,k-2},\pi(\xi_{2k-3})\neq 0\},\\
U&=\{(\xi_1+2\xi_2,\dots,\pi(\xi_{2k-3}),\pi(\xi_{2k-2}),\pi(\xi_{2k-1}),\pi(\xi_{2k}),\xi_{2k+1},\dots,\xi_{2\ell})\\
&\qquad\in D_{\ell,j,k-2},\pi(\xi_{2k-1})\neq 0,\pi(\xi_{2k-3})\neq 0\}.
\end{aligned}$$

We observe that $D_{\ell,j,k}=\Upsilon_{k-1,k}\cup\Upsilon_{k-1}\cup\Upsilon_k\cup\Upsilon$ and $D_{\ell,j,k-2}=U_{k-1,k}\cup U_{k-1}\cup U_k\cup U$. The following lemma connects the character sum over $D_{\ell,j,k}$ with the character sum over $D_{\ell,j,k-2}$.

LEMMA 2.9. *For $k\geqslant 2$, let Ψ_k be defined as in (1.2), and let*

$$\Psi_{k-2}=\psi_{(\beta_1+2\beta_2,\dots,\beta_{2k-5}+2\beta_{2k-4})}\otimes\chi_{(\pi(\beta_{2k-2}),\pi(\beta_{2k-3}),\pi(\beta_{2k}),\pi(\beta_{2k-1}),w_{2k+1},\dots,w_{2\ell})}.$$

If β_{2k-3} and β_{2k-1} are both nonzero, then $\Psi_k(\Upsilon_{k-1,k})=\Psi_{k-2}(U_{k-1,k})$, $\Psi_k(\Upsilon_{k-1})=-\Psi_{k-2}(U_{k-1})$, $\Psi_k(\Upsilon_k)=-\Psi_{k-2}(U_k)$, and $\Psi_k(\Upsilon)=\Psi_{k-2}(U)$.

Proof. It is straightforward to see that $\Psi_k(\Upsilon_{k-1,k})=\Psi_{k-2}(U_{k-1,k})$. We will show that $\Psi_k(\Upsilon_{k-1})=-\Psi_{k-2}(U_{k-1})$; the other arguments are similar. We use Lemma 1.7(e) to organize our sum to take advantage of pairs of the form $\Psi_k(\xi_1+2\xi_2,\dots,\xi_{2k-1}+2\xi_{2k},\xi_{2k+1},\dots,\xi_{2\ell})+\Psi_k(\xi_1+2\xi_2,\dots,\xi_{2k-1}+2(\xi_{2k-1}+\xi_{2k}),\xi_{2k+1},\dots,\xi_{2\ell})=\Psi_k(\xi_1+2\xi_2,\dots,\xi_{2k-1}+2\xi_{2k},\xi_{2k+1},\dots,\xi_{2\ell})(1+(-1)^{\mathrm{tr}(\pi(\beta_{2k-1})\pi(\xi_{2k-1}))})$. This last term will be 0 unless $\pi(\xi_{2k-1})=\pi(\beta_{2k-1}^{-1})$. We note that we will have the same term

$$\big(1+(-1)^{\mathrm{tr}(\pi(\beta_{2k-1})\pi(\xi_{2k-1}))}\big)$$

in the $\Psi_{k-2}(U_{k-1})$ sum. Thus, this character sum, while originally over all elements of Υ_{k-1} or U_{k-1}, will only be over elements with $\pi(\xi_{2k-1})=\pi(\beta_{2k-1}^{-1})$. We can factor out the term

$$i^{\mathrm{Tr}(\beta_{2k-1}\xi_{2k-1})} = i^{\mathrm{Tr}(\beta_{2k-1}\beta_{2k-1}^{-1})} = i^{\mathrm{Tr}(1)} = -1$$

from the $\Psi_k(\Upsilon_{k-1})$ sum, and what is left will be $\Psi_{k-2}(U_{k-1})$. Thus, $\Psi_k(\Upsilon_{k-1}) = -\Psi_{k-2}(U_{k-1})$. □

We will use strong induction, and hence we will assume that the character sum $\Psi_{k-2}(D_{\ell,j,k-2})$ will have the correct values. The following lemma shows how we can get a correct sum for $\Psi_k(\Upsilon_{k-1})$ and $\Psi_k(\Upsilon_k)$ when $\beta_{2k-3} = \beta_{2k-1} \neq 0$.

LEMMA 2.10. *If $\beta_{2k-3} = \beta_{2k-1} \neq 0$, then $\Psi_k(\Upsilon_{k-1}) = \Psi_k(\Upsilon_k) = 0$.*

Proof. Suppose that $\beta_{2k-3} = \beta_{2k-1} \neq 0$. We will show that $\Psi_k(\Upsilon_{k-1}) = 0$; the other case is similar. Using the fact that we only have to sum over elements of Υ_{k-1} with $\xi_{2k-1} = \beta_{2k-1}^{-1}$, and the fact that $\pi(\xi_{2k-2}) = \alpha^2\pi(\xi_1) + \pi(\xi_1)^2\pi(\xi_2)^2 + \pi(\xi_2) + \ldots + \pi(\xi_{2k-4}) + \alpha^2\pi(\xi_{2k-1}) + \pi(\xi_{2k-1})^2\pi(\xi_{2k})^2 + \pi(\xi_{2k}) + \ldots + \alpha^2\xi_{2j-1} + \xi_{2j-1}^2\xi_{2j}^2 + \xi_{2j} + \xi_{2j+1}^2\xi_{2j+2}^2 + \ldots + \xi_{2\ell-1}^2\xi_{2\ell}^2$ from the quadratic form, we get

$$\begin{aligned}
&\Psi_k(\Upsilon_{k-1})\\
&= \sum_{\xi_{2k-3}=0} i^{\mathrm{Tr}(\sum_{i=1}^{k}\beta_{2i-1}\xi_{2i-1})}(-1)^{\mathrm{tr}(\sum_{i=1}^{k}(\pi(\beta_{2i-1}\xi_{2i})+\pi(\beta_{2i}\xi_{2i-1}))+w_{2k+1}\xi_{2k+1}+\ldots+w_{2\ell}\xi_{2\ell})}\\
&= \sum_{\xi_{2k-3}=0} i^{\mathrm{Tr}(\sum_{i=1}^{k-2}\beta_{2i-1}\xi_{2i-1}+\beta_{2k-1}\xi_{2k-1})}(-1)^{\mathrm{tr}(\sum_{i=1}^{k-2}(\pi(\beta_{2i-1}\xi_{2i})+\pi(\beta_{2i}\xi_{2i-1})))}\\
&\quad\cdot(-1)^{\mathrm{tr}(\pi(\beta_{2k-3})\pi(\xi_{2k-2}))}\\
&\quad\cdot(-1)^{\mathrm{tr}(\pi(\beta_{2k-2})(0)+\pi(\beta_{2k-1}\xi_{2k})+\pi(\beta_{2k}\beta_{2k-1}^{-1})+w_{2k+1}\xi_{2k+1}+\ldots+w_{2\ell}\xi_{2\ell})}.
\end{aligned}$$

This last sum ranges over arbitrary values for ξ_i except for $i = 2k-3$ and $i = 2k-1$, where the values are fixed. We can rearrange the sum so there is an inner sum over all possible values of ξ_{2k}. If we do that, this inner sum will be

$$\sum_{\xi_{2k}}(-1)^{\mathrm{tr}((\pi(\beta_{2k-1})+\pi(\beta_{2k-3})+\pi(\beta_{2k-3}^2\beta_{2k-1}^2))\pi(\xi_{2k}))}.$$

Since $\beta_{2k-3} = \beta_{2k-1}$, this sum reduces to $\sum_{\xi_{2k}}(-1)^{\mathrm{tr}(\pi(\beta_{2k-1})\pi(\xi_{2k}))}$, and this sum is 0, proving the lemma. □

From Lemmas 2.9 and 2.10 we conclude that $\Psi_k(D_{\ell,j,k}) = \Psi_{k-2}(D_{\ell,j,k-2})$ when $\beta_{2k-3} = \beta_{2k-1} \neq 0$. Induction will tell us the value of $\Psi_{k-2}(D_{\ell,j,k-2})$, taking care of this case.

The final piece we need to prove the main theorem is to compute the value of $\Psi_k(\Upsilon_{k-1})$ when $\beta_{2k-3} \neq \beta_{2k-1}$, where both β_{2k-3} and β_{2k-1} are nonzero. We have the same inner sum as in the proof of the last lemma, but the inner sum in this case, namely $\sum_{\xi_{2k}}(-1)^{\mathrm{tr}((\pi(\beta_{2k-1})+\pi(\beta_{2k-3})+\pi(\beta_{2k-3}^2\beta_{2k-1}^2))\pi(\xi_{2k}))}$ will be 4 since $\pi(\beta_{2k-1}) + \pi(\beta_{2k-3}) + \pi(\beta_{2k-3}^2\beta_{2k-1}^2) = 0$ for all possible choices. We can extend this idea to each pair $(\xi_{2i-1}, \xi_{2i}), i \leqslant k-2$ to get 'inner sums' of

$$\begin{aligned}
&\sum_{\xi_{2i-1},\xi_{2i}} i^{\mathrm{Tr}(\beta_{2i-1}\xi_{2i-1})}\\
&\qquad\cdot(-1)^{\mathrm{tr}((\pi(\beta_{2i-1}\xi_{2i})+\pi(\beta_{2i}\xi_{2i-1})+\pi(\beta_{2k-3})(\alpha^2\pi(\xi_{2i-1})+\pi(\xi_{2i-1}^2\xi_{2i}^2)+\pi(\xi_{2i}))))}.
\end{aligned}$$

We are assuming that $\beta_{2i-1} \neq 0$, since otherwise we could combine $\Psi_k(D_{\ell,j,k}) = \Psi_{k-1}(D_{\ell,j,k-1})$ with the inductive hypothesis to get the correct character sum. (Note here that $\Psi_{k-1} = \psi_{(\beta_1+2\beta_2,\ldots,\beta_{2k-3}+2\beta_{2k-2})} \otimes \chi_{(\pi(\beta_{2k}),\pi(\beta_{2k-1}),w_{2k+1},\ldots,w_{2\ell})}$.) We can use Lemma 1.7(e) to restrict the possible values for ξ_{2i-1}, and then consider the sums over $\xi_{2i-1} = 0$ and $\xi_{2i-1} = \beta_{2i-1}^{-1}$ separately. These sums are

$$\sum_{\xi_{2i-1}=0,\xi_{2i}} (-1)^{\operatorname{tr}((\pi(\beta_{2i-1})+\pi(\beta_{2k-3}))\pi(\xi_{2i}))}$$

and

$$-(-1)^{\operatorname{tr}(\pi(\beta_{2i}\beta_{2i-1}^{-1})+\alpha^2\pi(\beta_{2k-3}\beta_{2i-1}^{-1}))} \cdot \sum_{\xi_{2i-1}=\beta_{2i-1}^{-1},\xi_{2i}} (-1)^{\operatorname{tr}((\pi(\beta_{2i-1})+\pi(\beta_{2k-3})+\pi(\beta_{2i-1}^2\beta_{2k-3}^2))\pi(\xi_{2i}))}.$$

If $\beta_{2i-1} = \beta_{2k-3}$, then the first sum is 4 and the second sum is 0; if $\beta_{2i-1} \neq \beta_{2k-3}$, then the first sum is 0 and the second sum is ± 4. Thus, in either case, the total sum over the pair (ξ_{2i-1}, ξ_{2i}) is ± 4. When $k < j$, the 'inner sum' for pairs (ξ_{2i-1}, ξ_{2i}), $k+1 \leqslant i \leqslant j$, will be

$$\sum_{\xi_{2i-1},\xi_{2i}} (-1)^{\operatorname{tr}((w_{2i-1}\xi_{2i}+w_{2i}\xi_{2i-1}+\pi(\beta_{2k-3})(\alpha^2\xi_{2i-1}+\xi_{2i-1}^2\xi_{2i}^2+\xi_{2i})))}.$$

This is the character sum of a quadratic form, which is ± 4 (see [**16**, p. 341, Exercise 6.30]).

For $i > j$, we have 'inner sums' of the form

$$\sum_{\xi_{2i-1},\xi_{2i}} (-1)^{\operatorname{tr}(\pi(\beta_{2k-3})^2\xi_{2i-1}\xi_{2i}+w_{2i-1}\xi_{2i-1}+w_{2i}\xi_{2i})}.$$

Again this is the character sum of a quadratic form, which is ± 4. Thus, we have $\ell - 1$ 'inner sums', each of which are ± 4. This proves the following lemma.

LEMMA 2.11. *Suppose that $\beta_{2i-1} \neq 0$ for $i \leqslant k$ and $\beta_{2k-3} \neq \beta_{2k-1}$. Then $\Psi_k(\Upsilon_{k-1}) = \pm 4^{\ell-1}$ and $\Psi_k(\Upsilon_k) = \pm 4^{\ell-1}$.*

The sum $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k)$ is either 0 or $\pm 2 \cdot 4^{\ell-1}$. If the sum is 0, then we can use the same idea as in the comments after Lemma 2.10 to show that $\Psi_k(D_{\ell,j,k}) = \Psi_{k-2}(D_{\ell,j,k-2})$, and induction will tell us that this has the correct sum. The following lemma finishes the computations we need to prove the main theorem.

LEMMA 2.12. *Let Ψ_k be defined as in* (1.2) *and Ψ_{k-2} as in Lemma* 2.9 *with $\beta_{2k-3} \neq 0$ and $\beta_{2k-1} \neq 0$, and suppose that $D_{\ell,j,k-2}$ is a partial difference set* (*with parameters depending on j odd or even*). *If j is odd and $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k) = \pm 2 \cdot 4^{\ell-1}$, then $\Psi_k(\Upsilon_{k-1,k}) + \Psi_k(\Upsilon) = -1 - 4^{\ell-1}$. If j is even and $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k) = \pm 2 \cdot 4^{\ell-1}$, then $\Psi_k(\Upsilon_{k-1,k}) + \Psi_k(\Upsilon) = -1 + 4^{\ell-1}$.*

Proof. By Lemma 2.9, $\Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) = \Psi_{k-2}(D_{\ell,j,k-2}) - \Psi_{k-2}(U_{k-1}) - \Psi_{k-2}(U_k) = \Psi_{k-2}(D_{\ell,j,k-2}) + \Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k)$. If j is odd and $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k) = 2 \cdot 4^{\ell-1}$, then $\Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) = (-1 - 4^{\ell-1} \pm 2 \cdot 4^{\ell-1}) + 2 \cdot 4^{\ell-1}$ (the case $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k) = -2 \cdot 4^{\ell-1}$ is similar, as is the j even case). We will show that $\Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) = -1 + 3 \cdot 4^{\ell-1}$

leads to a contradiction. Define a character Ψ'_{k-2} to have $\beta'_i = \beta_i$ except for the following.

(i) β'_{2k-2} is chosen to satisfy $\mathrm{tr}(\pi(\beta'_{2k-2}\beta^{-1}_{2k-3})) = 1 + \mathrm{tr}(\pi(\beta_{2k-2}\beta^{-1}_{2k-3}))$.
(ii) β'_{2k} is chosen to satisfy $\mathrm{tr}(\pi(\beta'_{2k}\beta^{-1}_{2k-1})) = 1 + \mathrm{tr}(\pi(\beta_{2k}\beta^{-1}_{2k-1}))$.

Neither $\pi(\beta_{2k-2})$ nor $\pi(\beta_{2k})$ affects $\Psi_{k-2}(U_{k-1,k})$, so $\Psi'_{k-2}(U_{k-1,k}) = \Psi_{k-2}(U_{k-1,k})$. Both $\pi(\beta_{2k-2})$ and $\pi(\beta_{2k})$ appear in $\Psi_{k-2}(U)$ in the term $(-1)^{\mathrm{tr}(\pi(\beta_{2k-2}\beta^{-1}_{2k-3}) + \pi(\beta_{2k}\beta^{-1}_{2k-1}))}$. (Here again we use the fact that if $\pi(\xi_{2i-1}) \neq \pi(\beta^{-1}_{2i-1})$ then Lemma 1.7(e) will allow us to pair elements to sum to 0.) The conditions on β'_{2k-2} and β'_{2k} imply that

$$(-1)^{\mathrm{tr}(\pi(\beta'_{2k-2}(\beta'_{2k-3})^{-1})+\pi(\beta'_{2k}(\beta'_{2k-1})^{-1}))} = (-1)^{\mathrm{tr}(\pi(\beta_{2k-2}\beta^{-1}_{2k-3})+\pi(\beta_{2k}\beta^{-1}_{2k-1}))},$$

so $\Psi'_{k-2}(U) = \Psi_{k-2}(U)$. The element β_{2k} appears in $\Psi_{k-2}(U_{k-1})$ in the term

$$(-1)^{\mathrm{tr}(\pi(\beta_{2k}\beta^{-1}_{2k-1}))},$$

but β_{2k-2} does not appear in this sum, implying that $\Psi'_{k-2}(U_{k-1}) = -\Psi_{k-2}(U_{k-1})$. Similarly, $\Psi'_{k-2}(U_k) = -\Psi_{k-2}(U_k)$. Hence

$$\begin{aligned}
\Psi'_{k-2}(D_{\ell,j,k-2}) &= \Psi'_{k-2}(U_{k-1,k}) + \Psi'_{k-2}(U) + \Psi'_{k-2}(U_{k-1}) + \Psi'_{k-2}(U_k)\\
&= \Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) - \Psi_{k-2}(U_{k-1}) - \Psi_{k-2}(U_k)\\
&= \Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) + \Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k).
\end{aligned}$$

If $\Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) = -1 + 3\cdot 4^{\ell-1}$, then $\Psi'_{k-2}(D_{\ell,j,k-2}) = -1 + 5\cdot 4^{\ell-1}$. We assumed that $D_{\ell,j,k-2}$ was a partial differences set, but $-1 + 5\cdot 4^{\ell-1}$ is not a correct character sum for this set, so that contradiction proves the lemma. □

Proof of Theorem 2.1. We proceed by induction on k. For the $k = 1$ case, Lemma 2.3 shows that the characters Ψ_1 of order 2 have the correct character sums, so we only need to consider characters $\Psi_1 = \psi_{\beta_1+2\beta_2} \otimes \chi_{(w_3,w_4,\ldots,w_{2\ell})}$ of order 4 (that is, $\beta_1 \neq 0$). We will only give the detailed arguments in the case j odd using Lemmas 2.5 and 2.7. The case j even is similar (using Lemmas 2.6 and 2.8).

We break $\Psi_1(D_{\ell,j,1})$ into three sums,

$$\begin{aligned}
&\Psi_1(D_{\ell,j,1})\\
&= \sum_{(\xi_1+2\xi_2,\xi_3,\ldots,\xi_{2\ell})\in D_{\ell,j,1}} (\sqrt{-1})^{\mathrm{Tr}((\beta_1+2\beta_2)(\xi_1+2\xi_2))}(-1)^{\mathrm{tr}(\sum_{i=3}^{2\ell} w_i\xi_i)}\\
&= \sum_{(\xi_1+2\xi_2,\xi_3,\ldots,\xi_{2\ell})\in D_{\ell,j,1},\mathrm{Tr}(\beta_1\xi_1)=0} (-1)^{\mathrm{tr}(\pi(\beta_2\xi_1+\beta_1\xi_2))}(-1)^{\mathrm{tr}(\sum_{i=3}^{2\ell} w_i\xi_i)}\\
&\quad - \sum_{(\xi_1+2\xi_2,\xi_3,\ldots,\xi_{2\ell})\in D_{\ell,j,1},\mathrm{Tr}(\beta_1\xi_1)=2} (-1)^{\mathrm{tr}(\pi(\beta_2\xi_1+\beta_1\xi_2))}(-1)^{\mathrm{tr}(\sum_{i=3}^{2\ell} w_i\xi_i)}\\
&\quad + \sum_{(\xi_1+2\xi_2,\xi_3,\ldots,\xi_{2\ell})\in D_{\ell,j,1},\mathrm{Tr}(\beta_1\xi_1)=\mathrm{odd}} (\sqrt{-1})^{\mathrm{Tr}(\beta_1\xi_1)}(-1)^{\mathrm{tr}(\pi(\beta_2\xi_1+\beta_1\xi_2))}(-1)^{\mathrm{tr}(\sum_{i=3}^{2\ell} w_i\xi_i)}.
\end{aligned}$$

The third sum above is over elements of $D_{\ell,j,1}$ with the property that $\mathrm{Tr}(\beta_1\xi_1)$ is odd. In that case, the observation following Lemma 1.7 implies that each such element has a matching element so that the pair will combine for a character sum of 0, so the sum over all these elements is 0. We note that this is also true for the sum of $\chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})}$ over the elements of $F_1^{-1}(D_{\ell,j,1})$ with $\mathrm{tr}(\pi(\beta_1)\pi(\xi_1)) = 1$.

For the first sum above, noting that $\mathrm{Tr}(\beta_1\xi_1) = 0$ implies that $\xi_1 = 0$, we see that the first sum is over the set of elements of $D_{\ell,j,1}$ with $\xi_1 = 0$. Hence

$$\begin{aligned}\sum_{(\xi_1+2\xi_2,\xi_3,\dots,\xi_{2\ell})\in D_{\ell,j,1}} & (\sqrt{-1})^{\mathrm{Tr}((\beta_1+2\beta_2)(2\xi_2))}(-1)^{\mathrm{tr}(\sum_{i=3}^{2\ell} w_i\xi_i)} \\ &= \Psi_1(\Omega_0) \\ &= \chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\dots,w_{2\ell})}(O_0),\end{aligned}$$

where Ω_0 and O_0 are defined in (2.2) and (2.3), respectively. By Lemma 2.4, the first sum is equal to $-1 \pm 4^{\ell-1}$.

The second sum above is over elements of $D_{\ell,j,1}$ satisfying $\xi_1 \neq 0$ and $\mathrm{Tr}(\beta_1\xi_1)$ is even. Since both β_1 and ξ_1 are in the Teichmüller set $\mathcal{T}$, their product is in $\mathcal{T}$ as well. The only nonzero element of $\mathcal{T}$ with an even trace is 1, and $\mathrm{Tr}(1) = 2$. This implies that the second sum is

$$S = \sum_{(\beta_1^{-1}+2\xi_2,\xi_3,\dots,\xi_{2\ell})\in D_{\ell,j,1}} \chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\dots,w_{2\ell})}(\pi(\xi_1),\pi(\xi_2),\xi_3,\dots,\xi_{2\ell}).$$

Combining this with the observation after our analysis of the third sum, we see that

$$S = \chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\dots,w_{2\ell})}\big(F_1^{-1}(D_{\ell,j,1})\backslash O_0\big).$$

Hence we finally have

$$\Psi_1(D_{\ell,j,1}) = \chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\dots,w_{2\ell})}(O_0) - \chi_{(\pi(\beta_2),\pi(\beta_1),w_3,\dots,w_{2\ell})}\big(F_1^{-1}(D_{\ell,j,1})\backslash O_0\big). \tag{2.4}$$

Thus, using Lemmas 2.4, 2.5 and 2.7 we get $\Psi_1(D_{\ell,j,1}) = -1 - 4^{\ell-1} \pm 2\cdot 4^{\ell-1}$ as required. This completes the proof of the theorem in the case where $k = 1$.

For the inductive step, suppose that $D_{\ell,j,k'}$ is a partial difference set for all $k-1 \geqslant k' \geqslant 0, \ell \geqslant j \geqslant k > 1$, with the appropriate parameters depending on j odd or even. To show that $D_{\ell,j,k}$ is also a partial difference set, we compute the character sums $\Psi_k(D_{\ell,j,k})$, where

$$\Psi_k = \psi_{(\beta_1+2\beta_2,\dots,\beta_{2k-1}+2\beta_{2k})} \otimes \chi_{(w_{2k+1},\dots,w_{2\ell})}.$$

As in the $k = 1$ case, Lemma 2.3 shows that the characters Ψ_k of order 2 have the correct character sum. For characters Ψ_k of order 4, we consider three cases.

The first case is when there is a $\beta_{2i-1} = 0$ for some $1 \leqslant i \leqslant k$. We can assume without loss of generality that $i = k$, permuting the coordinates if necessary. In this case, $\Psi_k(D_{\ell,j,k}) = \Psi_{k-1}(D_{\ell,j,k-1})$, and the inductive hypothesis implies that this sum is correct (note here that $\Psi_{k-1} = \psi_{(\beta_1+2\beta_2,\dots,\beta_{2k-3}+2\beta_{2k-2})} \otimes \chi_{(\pi(\beta_{2k}),0,w_{2k+1},\dots,w_{2\ell})}$ since $\pi(\beta_{2k-1}) = 0$).

The second case is when none of the β_{2i-1} are 0 for $i \leqslant k$ but there is a pair $\beta_{2i-1} = \beta_{2i'-1}$ for $i, i' \leqslant k,\ i \neq i'$. Again, without loss of generality we can assume that $i = k-1$ and $i' = k$. By applying Lemmas 2.9 and 2.10, we get

$$\begin{aligned}\Psi_k(D_{\ell,j,k}) &= \Psi_k(\Upsilon_{k-1,k}) + \Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k) + \Psi_k(\Upsilon) \\ &= \Psi_k(\Upsilon_{k-1,k}) + \Psi_k(\Upsilon) \\ &= \Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) \\ &= \Psi_{k-2}(D_{\ell,j,k-2}).\end{aligned}$$

Induction tells us that $\Psi_k(D_{\ell,j,k})$ has the correct value.

The final case is when none of the β_{2i-1} are 0 for $i \leqslant k$, and all of the β_{2i-1} are distinct. According to Lemma 2.11, there are three possibilities for $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k)$: 0 or $\pm 2 \cdot 4^{\ell-1}$. If the sum is 0, then the remarks after Lemma 2.10 indicate that $\Psi_k(D_{\ell,j,k}) = \Psi_{k-2}(D_{\ell,j,k-2})$, so induction implies that the character sum has the correct value. When $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k) = \pm 2 \cdot 4^{\ell-1}$, Lemma 2.12 shows that $\Psi_k(\Upsilon_{k-1,k}) + \Psi_k(\Upsilon) = -1 - 4^{\ell-1}$ when j is odd, so $\Psi_k(D_{\ell,j,k}) = -1 - 4^{\ell-1} \pm 2 \cdot 4^{\ell-1}$ as required (the j even case is similar).

Thus, in all cases the character sum $\Psi_k(D_{\ell,j,k})$ is as required, proving the theorem. □

REMARK 2.13. (1) We have checked that lifting a nonsingular elliptic quadric in $\mathrm{PG}(3, q)$, where $q \neq 4$, in the same way as in Theorem 2.1 (namely using a straightforward bijection from multiple copies of the field to the Teichmüller set representation of the Galois ring elements), will not produce a partial difference set in nonelementary abelian 2-group.

(2) Since $D_{2,1,0}$ is a partial difference set in $G_0 = (\mathbb{F}_4^4, +)$ (here $D_{2,1,0}$ corresponds to an elliptic quadric in $\mathrm{PG}(3, 4)$), the Cayley graph $(G_0, D_{2,1,0})$ is a strongly regular graph with vertex set G_0. Using GAP, it is found that the full automorphism group of this graph has order $2^{12} \cdot 3^2 \cdot 5 \cdot 17$. Similarly, since $D_{2,1,1}$ is a partial difference set in $G_1 = \mathbb{Z}_4^2 \times \mathbb{F}_4^2$, the Cayley graph $(G_1, D_{2,1,1})$ is a strongly regular graph with vertex set G_1. Again by using GAP, it is found that the full automorphism group of this graph has order $2^{12} \cdot 3^2 \cdot 5$. Even though the two strongly regular Cayley graphs $(G_0, D_{2,1,0})$ and $(G_1, D_{2,1,1})$ have the same parameters, they have different full automorphism groups, hence they are nonisomorphic. Based on further computations of automorphism groups, we conjecture that the strongly regular Cayley graphs arising from $D_{\ell,j,k}$, $k > 0$, are never isomorphic to the classical strongly regular Cayley graphs arising from quadratic forms over $\mathbb{F}_4$.

(3) The partial difference sets $D_{\ell,j,k}$ and $D_{\ell,j',k}$ are equivalent if $j - j'$ is even. The mapping defined in the proof of Lemma 1.7(a) induces a group automorphism ϕ on $(\mathrm{GR}(4, 2))^k \times \mathbb{F}_4^{2\ell-2k}$ satisfying $\phi(D_{\ell,j,k}) = D_{\ell,j-2,k}$ for $0 \leqslant k \leqslant j - 2$ (ϕ will fix all elements of the group $(\mathrm{GR}(4, 2))^k \times \mathbb{F}_4^{2\ell-2k}$ except x_{2j-3} through x_{2j}, and these are mapped as directed in the proof). We chose to do the proof in the general form since that made the induction easier to read, but we could have stated the result in terms of $D_{\ell,j,j}$ and $D_{\ell,j+1,j}$ and obtained a partial difference set in each equivalence class.

3. *Future directions*

The following list describes possible implications of the results in this paper.

(1) Are there other groups of the appropriate order that support partial difference sets with the same parameters in this paper? In particular, are there other combinations of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ that contain partial difference sets? A good candidate might be to see if $\mathbb{Z}_4^4$ contains a partial difference set with negative Latin square parameters, finishing off the one case we cannot deal with in Corollary 2.2.

(2) Can the 'bijection idea' produce any more sets where the character sums are so well preserved?

(3) Can we do anything in other contexts to get groups such as $\mathbb{Z}_8$ involved in a partial difference set group?

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