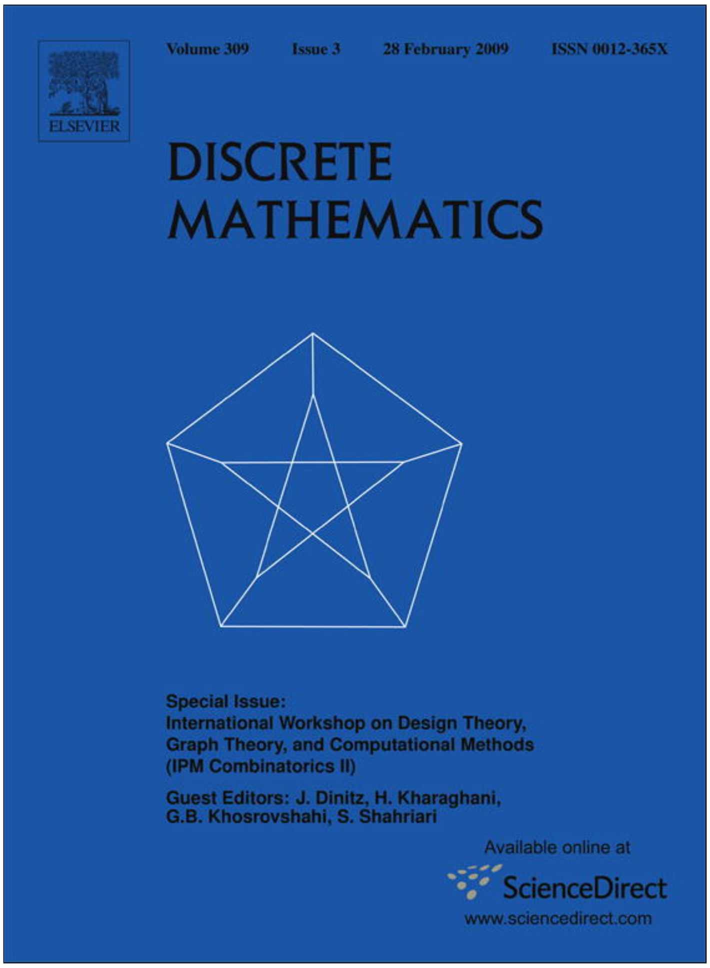
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# On the dimensions of the binary codes of a class of unitals

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ABSTRACT

Let $U_\beta$ be the special Buekenhout-Metz unital in $\mathrm{PG}(2, q^2)$, formed by a union of $q$ conics, where $q = p^e$ is an odd prime power. It can be shown that the dimension of the binary code of the corresponding unital design $\mathcal{U}_\beta$ is less than or equal to $q^3 + 1 - q$. Baker and Wantz conjectured that equality holds. We prove that the aforementioned dimension is greater than or equal to $q^3(1 - \frac{1}{p}) + \frac{q^2}{p}$.



## 1. Introduction

A *unital* is a 2-$(m^3 + 1, m + 1, 1)$ design, where $m \geq 2$. All known unitals with parameters $(m^3 + 1, m + 1, 1)$ have $m$ equal to a prime power, except for one example with $m = 6$ constructed by Mathon [9], and independently by Bagchi and Bagchi [3]. In this note, we will only consider unitals embedded in $\mathrm{PG}(2, q^2)$, i.e., unitals coming from a set of $q^3 + 1$ points of $\mathrm{PG}(2, q^2)$ which meets every line of $\mathrm{PG}(2, q^2)$ in either 1 or $q+1$ points. (Sometimes, a point set of size $q^3+1$ of $\mathrm{PG}(2, q^2)$ with the above line intersection properties is called a unital, too.) A classical example of such unitals is *the Hermitian unital* $\mathcal{U} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ and $\mathcal{B}$ are the set of absolute points and the set of non-absolute lines of a unitary polarity of $\mathrm{PG}(2, q^2)$, respectively.

The Hermitian unital is a special example of a large class of unitals embedded in $\mathrm{PG}(2, q^2)$, called the *Buekenhout-Metz unitals*. We refer the reader to [5] for a survey of results on these unitals. A subclass of the Buekenhout-Metz unitals which received some attention can be defined as follows.

Let $q = p^e$ be an **odd** prime power, where $e \geq 1$, let $\beta$ be a primitive element of $\mathbb{F}_{q^2}$, and for $r \in \mathbb{F}_q$ let $C_r = \{(1, y, \beta y^2 + r) \mid y \in \mathbb{F}_{q^2}\} \cup \{(0, 0, 1)\}$. We define

$$U_\beta = \bigcup_{r \in \mathbb{F}_q} C_r.$$

Note that each $C_r$ is a conic in $\mathrm{PG}(2, q^2)$, and any two distinct $C_r$ have only the point $P_\infty = (0, 0, 1)$ in common. Hence $|U_\beta| = q^3 + 1$. It can be shown that every line of $\mathrm{PG}(2, q^2)$ meets $U_\beta$ in either 1 or $q + 1$ points (see [1,7]). One immediately obtains a unital (design) $\mathcal{U}_\beta$ from $U_\beta$: The *points* of $\mathcal{U}_\beta$ are the points of $U_\beta$, and the *blocks* of $\mathcal{U}_\beta$ are the intersections of the secant lines with $U_\beta$. In this note, we are interested in the binary code $\mathcal{C}_2(\mathcal{U}_\beta)$ of this design, i.e., the $\mathbb{F}_2$-subspace spanned by the characteristic vectors of the blocks of $\mathcal{U}_\beta$ in $\mathbb{F}_2^{U_\beta}$.

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The following proposition and its proof are due to Baker and Wantz [6,10]. To state the proposition, we use $v^S$ to denote the characteristic vector of a subset $S$ in $U_\beta$.

**Proposition 1.1** (*Baker and Wantz*). *The vectors $v^{C_r}$, $r \in \mathbb{F}_q$, form a linearly independent set of vectors in $\mathcal{C}_2(\mathcal{U}_\beta)^\perp$.*

**Proof.** A binary vector $v$ lies in $\mathcal{C}_2(\mathcal{U}_\beta)^\perp$, if and only if, each block of the design $\mathcal{U}_\beta$ meets the support of $v$ in an even number of points. If a block of $\mathcal{U}_\beta$ goes through $P_\infty$, then it meets every $C_r$ in two points; if a block of $\mathcal{U}_\beta$ does not go through $P_\infty$, then it meets every $C_r$ in either 0 or 2 points. Hence $v^{C_r} \in \mathcal{C}_2(\mathcal{U}_\beta)^\perp$, for every $r \in \mathbb{F}_q$. The $q$ conics $C_r$ have only the point $P_\infty$ in common. Thus, $v^{C_r}$, $r \in \mathbb{F}_q$, are linearly independent. The proof is complete. $\square$

An immediate corollary of Proposition 1.1 is that $\dim \mathcal{C}_2(\mathcal{U}_\beta)^\perp \geq q$. Hence $\dim \mathcal{C}_2(\mathcal{U}_\beta) \leq q^3 + 1 - q$. Baker and Wantz [6,10] made the following conjecture.

**Conjecture 1.2** (*Baker and Wantz*). *The 2-rank of $\mathcal{U}_\beta$ is $q^3 + 1 - q$. That is, $\dim \mathcal{C}_2(\mathcal{U}_\beta) = q^3 + 1 - q$.*

Wantz [10] verified Conjecture 1.2 in the cases where $q = 3, 5, 7$, and 9 by using a computer and MAGMA [4]. Gary Ebert [6] popularized the above conjecture of Baker and Wantz in a talk in Oberwolfach in 2001. See also [11] for a description of the above conjecture. Of course, the conjecture is equivalent to saying that $\dim \mathcal{C}_2(\mathcal{U}_\beta)^\perp = q$. So it suffices to show that $\{v^{C_r} \mid r \in \mathbb{F}_q\}$ spans $\mathcal{C}_2(\mathcal{U}_\beta)^\perp$. That is, we need to show that if $S \subset U_\beta$ and $S$ meets every block of $\mathcal{U}_\beta$ in an even number of points, then $S$ is a union of some $C_r$'s, or a union of some $C_r$'s with $P_\infty$ deleted. We have not been able to prove this equivalent version of the conjecture. What we could prove is a lower bound on $\dim \mathcal{C}_2(\mathcal{U}_\beta)$ as stated in the abstract. The main idea in our proofs is to realize a shortened code of $\mathcal{C}_2(\mathcal{U}_\beta)$ as an ideal in a certain group algebra of the elementary abelian $p$-group of order $q^3$. We hope that the current note will stimulate further research on this conjecture.

## 2. A lower bound on the dimension of $\mathcal{C}_2(\mathcal{U}_\beta)$

We first consider the automorphisms of $\mathcal{U}_\beta$. Let

$$G = \{\theta \in \mathrm{PGL}(3, q^2) \mid \theta(U_\beta) = U_\beta\}$$

be the linear collineation group of $\mathrm{PG}(2, q^2)$ fixing $U_\beta$ as a set. It was shown by Baker and Ebert [2] that

$$G = T \rtimes \mathbb{Z}_{2(q-1)},$$

where $T$ is an elementary abelian group of order $q^3$, and $\mathbb{Z}_{2(q-1)}$ is a cyclic group of order $2(q-1)$. The group $G$ certainly is also an automorphism group of the design $\mathcal{U}_\beta$ since any element of $G$ maps a secant line of $U_\beta$ to a secant line of $U_\beta$. In fact, the group $T$ above acts regularly on $U_\beta \setminus \{P_\infty\}$. Explicitly,

$$T = \left\{ \begin{pmatrix} 1 & t & \beta t^2 \\ 0 & 1 & 2\beta t \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, t \in \mathbb{F}_{q^2}, r \in \mathbb{F}_q \right\} \cong (\mathbb{F}_{q^2}, +) \times (\mathbb{F}_q, +).$$

In the rest of the paper, we will use $T(t, r)$, $t \in \mathbb{F}_{q^2}$, $r \in \mathbb{F}_q$, to denote the element

$$\begin{pmatrix} 1 & t & \beta t^2 \\ 0 & 1 & 2\beta t \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of $T$.

The coordinates of the code $\mathcal{C}_2(\mathcal{U}_\beta)$ are labeled by the points in $U_\beta$. Deleting the coordinate labeled by $P_\infty$ from all codewords of $\mathcal{C}_2(\mathcal{U}_\beta)$, we get a shortened (or punctured) code $\mathcal{C}_2(\mathcal{U}_\beta)'$, which has the same dimension over $\mathbb{F}_2$ as $\mathcal{C}_2(\mathcal{U}_\beta)$ since $v^{\{P_\infty\}} \notin \mathcal{C}_2(\mathcal{U}_\beta)$. Since $T$ acts regularly on $U_\beta \setminus \{P_\infty\}$, we may identify the coordinates of $\mathcal{C}_2(\mathcal{U}_\beta)'$ with the elements of $T$. Under this identification, the point $(1, t, \beta t^2 + r)$ of $U_\beta$ correspond to the group element $T(t, r)$ since

$$(1, 0, 0) \cdot T(t, r) = (1, t, \beta t^2 + r).$$

After the above identification, the code $\mathcal{C}_2(\mathcal{U}_\beta)'$ becomes an ideal of the group algebra $\mathbb{F}_2[T]$. Now we can use the characters of $T$ to help compute the dimension of $\mathcal{C}_2(\mathcal{U}_\beta)'$.

First of all, we need to extend the field over which the code $\mathcal{C}_2(\mathcal{U}_\beta)$ is defined. Let $K = \mathbb{F}_{2^m}$, where $m = \mathrm{ord}_p(2)$ is the order of 2 modulo $p$ (i.e., $m$ is the smallest positive integer such that $2^m \equiv 1 \pmod{p}$). So $K$ contains a primitive $p$th root of unity $\xi_p$. We consider the code $\mathcal{C}_K(\mathcal{U}_\beta)$ and puncture it at $P_\infty$ to get $\mathcal{C}_K(\mathcal{U}_\beta)'$, which will be denoted by $M$ for simplicity of notation. The code $M$ is an ideal of the group algebra $K[T]$, and

$$\dim_K(M) = \dim_{\mathbb{F}_2}(\mathcal{C}_2(\mathcal{U}_\beta)').$$

Therefore, Conjecture 1.2 is equivalent to the statement that

$$\dim_K(M) = q^3 + 1 - q.$$

Since $M$ is an ideal of $K[T]$, and $T$ is abelian, it is well known [8, p. 277] that

$$\dim_K(M) = |\{\chi \in \hat{T} \mid Me_\chi \neq 0\}|,$$

where $\hat{T}$ is the group of characters $\chi : T \to K^*$ of $T$, and

$$e_\chi = \frac{1}{|T|} \sum_{g \in T} \chi(g^{-1})g$$

are primitive idempotents of $K[T]$. We also mention that for any $h \in T$ and any $\chi \in \hat{T}$,

$$h \cdot e_\chi = \chi(h)e_\chi. \tag{2.1}$$

Since $T \cong (\mathbb{F}_{q^2}, +) \times (\mathbb{F}_q, +)$, every character $\chi$ of $T$ can be written as $(\psi_a, \lambda_b) : T \to K^*$, where $a \in \mathbb{F}_{q^2}$, $b \in \mathbb{F}_q$,

$$\psi_a : x \mapsto \xi_p^{\mathrm{Tr}_{q^2/p}(ax)}, \quad x \in \mathbb{F}_{q^2},$$

and

$$\lambda_b : y \mapsto \xi_p^{\mathrm{Tr}_{q/p}(by)}, \quad y \in \mathbb{F}_q.$$

Here $\mathrm{Tr}_{q^2/p}$ (resp. $\mathrm{Tr}_{q/p}$) is the trace from $\mathbb{F}_{q^2}$ (resp. $\mathbb{F}_q$) to $\mathbb{F}_p$. (We note in passing that $\mathrm{Tr}_{q^2/p}(a/2) = \mathrm{Tr}_{q/p}(a)$ for all $a \in \mathbb{F}_q$, a fact which will be used in the proof of Theorem 2.4.) Hence we need to count the number of pairs $(a, b) \in \mathbb{F}_{q^2} \times \mathbb{F}_q$ such that

$$Me_{(\psi_a, \lambda_b)} \neq 0.$$

To this end, we need to write down the blocks of the unital design $\mathcal{U}_\beta$ more explicitly.

We first recall some properties of $U_\beta$, which will be used to describe the blocks of $\mathcal{U}_\beta$. The proofs of these properties can be found in [1,2,7].

- Among the $q^2 + 1$ lines through $P_\infty$, $q^2$ of them are secant to $U_\beta$, and one is tangent to $U_\beta$. The secant lines through $P_\infty$ are $[t, 1, 0]$, where $t \in \mathbb{F}_{q^2}$, and the unique tangent line through $P_\infty$ is $[1, 0, 0]$.
- A secant line to $U_\beta$, not through $P_\infty$ must pass through $(0, 1, \alpha)$ for some $\alpha \in \mathbb{F}_{q^2}$. Moreover, for every $\alpha \in \mathbb{F}_{q^2}$, there are $q^2 - q$ secant line through $(0, 1, \alpha)$.
- The line $[t, -\alpha, 1]$, $t, \alpha \in \mathbb{F}_{q^2}$, through $(0, 1, \alpha)$ is secant to $U_\beta$ if and only if $t \notin \mathbb{F}_q + \frac{\alpha^2}{4\beta}$. (This can be seen as follows: The line $[t, -\alpha, 1]$ is tangent to $U_\beta$ if and only if it is tangent to some conic $C_r$, $r \in \mathbb{F}_q$, which in turn is equivalent to $\alpha^2 - 4\beta(t + r) = 0$. The last condition is simply saying that $t \in \mathbb{F}_q + \frac{\alpha^2}{4\beta}$.)

The unital design $\mathcal{U}_\beta$ has a total of $q^2(q^2 - q + 1)$ blocks, which fall into two types. The type I blocks are the intersections of the $q^2$ secant lines through $P_\infty$ with $U_\beta$. These are

$$U_\beta \cap [t, 1, 0] = \{(1, -t, \beta t^2 + r) \mid r \in \mathbb{F}_q\} \cup \{P_\infty\},$$

where $t \in \mathbb{F}_{q^2}$. We may identify $(U_\beta \cap [t, 1, 0]) \setminus \{P_\infty\}$ with the group ring element

$$B_{t,\infty} := \sum_{r \in \mathbb{F}_q} T(-t, r) \in K[T]. \tag{2.2}$$

The type II blocks are the intersections of the secant lines through $(0, 1, \alpha)$ with $U_\beta$, with $q^2 - q$ of them for each $\alpha \in \mathbb{F}_{q^2}$. These blocks are

$$U_\beta \cap [t, -\alpha, 1] = \{(1, y, \beta y^2 + r) \mid r \in \mathbb{F}_q, y \in \mathbb{F}_{q^2}, t - \alpha y + \beta y^2 + r = 0\},$$

where $\alpha \in \mathbb{F}_{q^2}$ and $t \in \mathbb{F}_{q^2} \setminus (\mathbb{F}_q + \frac{\alpha^2}{4\beta})$. We may identify the above block with the group ring element

$$B_{t,\alpha} := \sum_{y \in \mathbb{F}_{q^2}, r = -t + \alpha y - \beta y^2 \in \mathbb{F}_q} T(y, r) \in K[T]. \tag{2.3}$$

Therefore we have a complete description of the blocks of the unital design $\mathcal{U}_\beta$.

**Lemma 2.1.** *With the above notation, $Me_{(\psi_a, \lambda_0)} \neq 0$, for all $a \in \mathbb{F}_{q^2}$.*

**Proof.** We will show that $B_{t,\infty} \cdot e_{(\psi_a,\lambda_0)} \neq 0$, where $B_{t,\infty}$ is defined in (2.2). By (2.1), we have $h \cdot e_\chi = \chi(h)e_\chi$ for any $h \in T$ and any $\chi \in \hat{T}$. So we need to show that $(\psi_a, \lambda_0)(B_{t,\infty}) \neq 0$.

$$\begin{aligned}(\psi_a, \lambda_0)(B_{t,\infty}) &= \sum_{r\in\mathbb{F}_q} \psi_a(-t)\lambda_0(r) \\ &= \sum_{r\in\mathbb{F}_q} \psi_a(-t) \\ &= q \cdot \psi_a(-t).\end{aligned}$$

Since $q$ is odd, and $\psi_a(-t)$ is a root of unity in $K$, we see that $(\psi_a, \lambda_0)(B_{t,\infty}) \neq 0$. The proof is complete. □

**Lemma 2.2.** *With the above notation, $Me_{(\psi_0,\lambda_b)} = 0$, for all nonzero $b \in \mathbb{F}_q$.*

**Proof.** The ideal $M$ is generated by two types of elements $B_{t,\infty}$ and $B_{t,\alpha}$, which correspond to the two types of blocks of $\mathcal{U}_\beta$. We will show that the character $(\psi_0, \lambda_b)$, $b \neq 0$, is zero on both types of generating elements.

For any type I element $B_{t,\infty}$, $t \in \mathbb{F}_{q^2}$, in (2.2), we have

$$\begin{aligned}(\psi_0, \lambda_b)(B_{t,\infty}) &= \sum_{r\in\mathbb{F}_q} \psi_0(-t)\lambda_b(r) \\ &= \sum_{r\in\mathbb{F}_q} \lambda_b(r) \\ &= 0,\end{aligned}$$

since $b$ is nonzero.

For any type II element $B_{t,\alpha}$ in (2.3), we have

$$\begin{aligned}(\psi_0, \lambda_b)(B_{t,\alpha}) &= \sum_{r\in\mathbb{F}_q, y\in\mathbb{F}_{q^2}, r=-(\beta y^2-\alpha y+t)} \psi_0(y)\lambda_b(r) \\ &= \sum_{r\in\mathbb{F}_q, y\in\mathbb{F}_{q^2}, r=-(\beta y^2-\alpha y+t)} \lambda_b(r) \\ &= 0,\end{aligned}$$

since two distinct $y \in \mathbb{F}_{q^2}$ give rise to the same $r \in \mathbb{F}_q$. The proof is complete. □

By the above two lemmas, we see that Conjecture 1.2 is equivalent to

**Conjecture 2.3.** *For nonzero $a \in \mathbb{F}_{q^2}$ and nonzero $b \in \mathbb{F}_q$, one has*

$$Me_{(\psi_a,\lambda_b)} \neq 0.$$

Up to now we have only been able to prove some partial results on this latter conjecture.

**Theorem 2.4.** *Let $a \in \mathbb{F}_{q^2}^*$ and $b \in \mathbb{F}_q^*$. If $\mathrm{Tr}_{q^2/p}(\frac{a^2}{2b\beta}) \neq 0$, then $Me_{(\psi_a,\lambda_b)} \neq 0$.*

**Proof.** Let $t, \alpha \in \mathbb{F}_{q^2}$ with $\Delta := t - \frac{\alpha^2}{4\beta} \notin \mathbb{F}_q$. Then $[t, -\alpha, 1]$ is a secant line to $U_\beta$, and

$$U_\beta \cap [t, -\alpha, 1] = \{(1, y, \beta y^2 + r) \mid y \in \mathbb{F}_{q^2}, r = -t + \alpha y - \beta y^2 \in \mathbb{F}_q\}.$$

This set can be identified with the group ring element

$$B_{t,\alpha} = \sum_{y\in A_{t,\alpha}} T(y, -t + \alpha y - \beta y^2) \in K[T],$$

where $A_{t,\alpha} = \{y \in \mathbb{F}_{q^2} \mid t - \alpha y + \beta y^2 \in \mathbb{F}_q\}$.

Now let $\mu \in \mathbb{F}_{q^2}^*$ such that $\mu^2 \in \mathbb{F}_q^*$ (explicitly, $\mu \in \langle \beta^{\frac{q+1}{2}} \rangle$, the subgroup of order $2(q-1)$ of $\mathbb{F}_{q^2}^*$). Then $[t\mu^2, -\alpha\mu, 1]$ is also a secant line to $U_\beta$, and

$$U_\beta \cap [t\mu^2, -\alpha\mu, 1] = \{(1, z, \beta z^2 + r) \mid z \in \mathbb{F}_{q^2}, r = -t\mu^2 + \alpha\mu z - \beta z^2 \in \mathbb{F}_q\}.$$

This set can be identified with the group ring element

$$B_{t\mu^2,\alpha\mu} = \sum_{y\in A_{t,\alpha}} T(\mu y, -\mu^2(t - \alpha y + \beta y^2)) \in K[T].$$

Since $\Delta \notin \mathbb{F}_q$, the set $(\mathbb{F}_q - \Delta)$ contains $\frac{q+1}{2}$ nonsquares of $\mathbb{F}_{q^2}$, say $n_1, n_2, \ldots, n_{\frac{q+1}{2}}$ (see Lemma 5.2 in [7]). Now we can write down the elements of $A_{t,\alpha}$ explicitly. Note that $t - \alpha y + \beta y^2 \in \mathbb{F}_q$, if and only if, $\beta(y - \frac{\alpha}{2\beta})^2 \in \mathbb{F}_q - \Delta$, which in turn is equivalent to $\beta(y - \frac{\alpha}{2\beta})^2 = n_i$ for some $i$, $1 \le i \le (q+1)/2$. Therefore, we have $y \in A_{t,\alpha}$ if and only if $y = \frac{\alpha}{2\beta} \pm \sqrt{\beta^{-1} n_i}$, $1 \le i \le (q+1)/2$. It follows that

$$B_{t\mu^2,\alpha\mu} = \sum_{i=1}^{\frac{q+1}{2}} T\left(\frac{\alpha\mu}{2\beta} \pm \sqrt{\beta^{-1} n_i \mu^2}, -\mu^2(n_i + \Delta)\right).$$

Now

$$\begin{aligned}(\psi_a, \lambda_b)(B_{t\mu^2,\alpha\mu}) &= \sum_{i=1}^{\frac{q+1}{2}} \left(\xi_p^{\mathrm{Tr}_{q^2/p}(\frac{a\alpha\mu}{2\beta} + a\sqrt{\beta^{-1} n_i \mu^2})} + \xi_p^{\mathrm{Tr}_{q^2/p}(\frac{a\alpha\mu}{2\beta} - a\sqrt{\beta^{-1} n_i \mu^2})}\right) \xi_p^{\mathrm{Tr}_{q/p}(-b\mu^2(n_i+\Delta))} \\ &= \xi_p^{\mathrm{Tr}_{q^2/p}(\frac{a\alpha\mu}{2\beta} - \frac{b\Delta\mu^2}{2})} \sum_{i=1}^{\frac{q+1}{2}} \left(\xi_p^{\mathrm{Tr}_{q^2/p}(a\sqrt{\beta^{-1} n_i \mu^2})} + \xi_p^{\mathrm{Tr}_{q^2/p}(-a\sqrt{\beta^{-1} n_i \mu^2})}\right) \xi_p^{\mathrm{Tr}_{q^2/p}(-\frac{b n_i \mu^2}{2})}.\end{aligned}$$

Define

$$S_{t\mu^2,-\alpha\mu} := \sum_{i=1}^{\frac{q+1}{2}} \left(\xi_p^{\mathrm{Tr}_{q^2/p}(a\sqrt{\beta^{-1} n_i \mu^2})} + \xi_p^{\mathrm{Tr}_{q^2/p}(-a\sqrt{\beta^{-1} n_i \mu^2})}\right) \xi_p^{\mathrm{Tr}_{q^2/p}(-\frac{b n_i \mu^2}{2})}.$$

Let $R$ be a complete set of coset representatives of the subgroup $\{1, -1\}$ in $\langle \beta^{\frac{q+1}{2}} \rangle$ (so $|R| = (q-1)$). We will show that

$$\sum_{\mu \in R} S_{t\mu^2,-\alpha\mu} \neq 0. \tag{2.4}$$

From (2.3), we immediately see that there exists some $\mu \in R$ such that $(\psi_a, \lambda_b)(B_{t\mu^2,\alpha\mu}) \neq 0$, which proves the conclusion of the theorem.

First we claim that as $\mu$ runs through $R$ and $i$ runs through $1, 2, \ldots, \frac{q+1}{2}$, $n_i\mu^2$ run through the set $N$ of nonsquares of $\mathbb{F}_{q^2}^*$. The claim can be proved as follows. Clearly, each $n_i\mu^2$ is a nonsquare of $\mathbb{F}_{q^2}^*$. It suffices to show that $n_i\mu^2$, $1 \le i \le \frac{q+1}{2}$ and $\mu \in R$, are all distinct. Assume that $n_i\mu^2 = n_j\lambda^2$, for some $1 \le i, j \le \frac{q+1}{2}$, and some $\mu, \lambda \in R$. Since $n_i, n_j \in \mathbb{F}_q - \Delta$, we set $n_i = x - \Delta$ and $n_j = y - \Delta$, where $x, y \in \mathbb{F}_q$. We have

$$\mu^2 x - \mu^2 \Delta = \lambda^2 y - \lambda^2 \Delta.$$

Noting that $\mu^2, \lambda^2 \in \mathbb{F}_q$ and $\Delta \notin \mathbb{F}_q$, we see that $\mu^2 = \lambda^2$. Since $\mu, \lambda \in R$, we must have $\mu = \lambda$, from which we deduce $n_i = n_j$. The claim is proved.

For convenience, we will use $S$ to denote the set of nonzero squares of $\mathbb{F}_{q^2}$. So we have

$$\begin{aligned}\sum_{\mu \in R} S_{t\mu^2,-\alpha\mu} &= \sum_{x \in N} \left(\xi_p^{\mathrm{Tr}_{q^2/p}(a\sqrt{\beta^{-1}x})} + \xi_p^{\mathrm{Tr}_{q^2/p}(-a\sqrt{\beta^{-1}x})}\right) \xi_p^{\mathrm{Tr}_{q^2/p}(-\frac{bx}{2})} \\ &= \sum_{y \in S} \left(\xi_p^{\mathrm{Tr}_{q^2/p}(a\sqrt{y})} + \xi_p^{\mathrm{Tr}_{q^2/p}(-a\sqrt{y})}\right) \xi_p^{\mathrm{Tr}_{q^2/p}(\frac{-b\beta y}{2})} \\ &= \sum_{z \in \mathbb{F}_{q^2}^*} \xi_p^{\mathrm{Tr}_{q^2/p}(az - \frac{b\beta z^2}{2})} \\ &= \sum_{z \in \mathbb{F}_{q^2}} \xi_p^{\mathrm{Tr}_{q^2/p}(az - \frac{b\beta z^2}{2})} - 1 \\ &= \xi_p^{\mathrm{Tr}_{q^2/p}(\frac{a^2}{2b\beta})} \sum_{x \in \mathbb{F}_{q^2}} \xi_p^{\mathrm{Tr}_{q^2/p}(-\frac{b\beta}{2}x^2)} - 1.\end{aligned}$$

Note that $p$ is odd and $\mathrm{Tr}_{q^2/p}(-\frac{b\beta}{2}x^2) = \mathrm{Tr}_{q^2/p}(-\frac{b\beta}{2}(-x)^2)$ for any $x \in \mathbb{F}_{q^2}$. As $\xi_p \in K$ and $K$ has characteristic 2, we have

$$\sum_{x \in \mathbb{F}_{q^2}} \xi_p^{\mathrm{Tr}_{q^2/p}(-\frac{b\beta}{2}x^2)} = 1.$$

Hence

$$\sum_{\mu \in R} S_{t\mu^2, -\alpha\mu} = \xi_p^{\mathrm{Tr}_{q^2/p}(\frac{a^2}{2b\beta})} - 1.$$

Therefore, if $\mathrm{Tr}_{q^2/p}(\frac{a^2}{2b\beta}) \neq 0$, then $\sum_{\mu \in R} S_{t\mu^2, -\alpha\mu} \neq 0$. The proof is complete. □

An immediate corollary is the following.

**Corollary 2.5.** $\dim C_2(\mathcal{U}_\beta) \geq q^3(1 - \frac{1}{p}) + \frac{q^2}{p}$.

**Proof.** By Lemma 2.1, we have $q^2$ characters $(\psi_a, \lambda_0)$, $a \in \mathbb{F}_{q^2}$, of $T$ such that $Me_{(\psi_a, \lambda_0)} \neq 0$.

Next, for each $b \in \mathbb{F}_q^*$, the number of $a$'s such that $\mathrm{Tr}_{q^2/p}(\frac{a^2}{2b\beta}) \neq 0$ is $(q^2 - p^{2e-1}) = (q^2 - q^2/p)$. So Theorem 2.4 produces $(q-1)(q^2 - q^2/p)$ characters $(\psi_a, \lambda_b)$ of $T$, such that $Me_{(\psi_a, \lambda_b)} \neq 0$.

Therefore, $\dim C_2(\mathcal{U}_\beta) \geq q^2 + (q-1)(q^2 - q^2/p) = q^3(1 - \frac{1}{p}) + \frac{q^2}{p}$. The proof is complete. □

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