



ELSEVIER

Contents lists available at ScienceDirect

Journal of Combinatorial Theory,
Series A

www.elsevier.com/locate/jcta



Cameron–Liebler line classes with parameter

$$x = \frac{q^2-1}{2}$$



Tao Feng^{a,1}, Koji Momihara^{b,2}, Qing Xiang^c

^a Department of Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, PR China

^b Faculty of Education, Kumamoto University, 2-40-1 Kurokami, Kumamoto 860-8555, Japan

^c Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

ARTICLE INFO

Article history:

Received 25 June 2014

Available online 2 March 2015

Keywords:

Cameron–Liebler line class

Klein correspondence

Projective two-intersection set

Affine two-intersection set

Strongly regular graph

Gauss sum

ABSTRACT

In this paper, we give an algebraic construction of a new infinite family of Cameron–Liebler line classes with parameter $x = \frac{q^2-1}{2}$ for $q \equiv 5$ or $9 \pmod{12}$, which generalizes the examples found by Rodgers in [26] through a computer search. Furthermore, in the case where q is an even power of 3, we construct the first infinite family of affine two-intersection sets in $AG(2, q)$, which is closely related to our Cameron–Liebler line classes.

© 2015 Elsevier Inc. All rights reserved.

E-mail addresses: tfeng@zju.edu.cn (T. Feng), momihara@educ.kumamoto-u.ac.jp (K. Momihara), xiang@math.udel.edu (Q. Xiang).

¹ Research supported in part by Fundamental Research Fund for the Central Universities of China, Zhejiang University K.P. Chao's High Technology Development Foundation, the National Natural Science Foundation of China under Grant Nos. 11201418 and 11422112, and the Research Fund for Doctoral Programs from the Ministry of Education of the People's Republic of China under Grant No. 20120101120089.

² Research supported by JSPS under Grant-in-Aid for Young Scientists (B) 25800093 and Scientific Research (C) 24540013.

<http://dx.doi.org/10.1016/j.jcta.2015.02.004>

0097-3165/© 2015 Elsevier Inc. All rights reserved.

1. Introduction

Cameron–Liebler line classes were first introduced by Cameron and Liebler [6] in their study of collineation groups of $\text{PG}(3, q)$ having the same number of orbits on points and lines of $\text{PG}(3, q)$. Later on it was found that these line classes have many connections to other geometric and combinatorial objects, such as blocking sets of $\text{PG}(2, q)$, projective two-intersection sets in $\text{PG}(5, q)$, two-weight linear codes, and strongly regular graphs. In the last few years, Cameron–Liebler line classes have received considerable attention from researchers in both finite geometry and algebraic combinatorics; see, for example, [7,20,21,26,11,10]. In [6], the authors gave several equivalent conditions for a set of lines of $\text{PG}(3, q)$ to be a Cameron–Liebler line class; Penttila [23] gave a few more of such characterizations. We will use one of these characterizations as the definition of a Cameron–Liebler line class. Let \mathcal{L} be a set of lines of $\text{PG}(3, q)$ with $|\mathcal{L}| = x(q^2 + q + 1)$, x a nonnegative integer. We say that \mathcal{L} is a *Cameron–Liebler line class with parameter x* if every spread of $\text{PG}(3, q)$ contains x lines of \mathcal{L} . Clearly the complement of a Cameron–Liebler line class with parameter x in the set of all lines of $\text{PG}(3, q)$ is a Cameron–Liebler line class with parameter $q^2 + 1 - x$. So without loss of generality we may assume that $x \leq \frac{q^2+1}{2}$ when discussing Cameron–Liebler line classes of parameter x .

Let (P, π) be any non-incident point–plane pair of $\text{PG}(3, q)$. Following [23], we define $\text{star}(P)$ to be the set of all lines through P , and $\text{line}(\pi)$ to be the set of all lines contained in the plane π . We have the following trivial examples:

- (1) The empty set gives a Cameron–Liebler line class with parameter $x = 0$;
- (2) Each of $\text{star}(P)$ and $\text{line}(\pi)$ gives a Cameron–Liebler line class with parameter $x = 1$;
- (3) $\text{star}(P) \cup \text{line}(\pi)$ gives a Cameron–Liebler line class with parameter $x = 2$.

Cameron–Liebler line classes are rare. It was once conjectured [6, p. 97] that the above trivial examples and their complements are all of the Cameron–Liebler line classes. The first counterexample to this conjecture was given by Drudge [9] in $\text{PG}(3, 3)$, and it has parameter $x = 5$. Later Bruen and Drudge [4] generalized Drudge’s example into an infinite family with parameter $x = \frac{q^2+1}{2}$ for all odd q . This represents the only known infinite family of nontrivial Cameron–Liebler line classes before our work. Govaerts and Penttila [13] gave a sporadic example with parameter $x = 7$ in $\text{PG}(3, 4)$. Recent work by Rodgers suggests that there are probably more infinite families of Cameron–Liebler line classes awaiting to be discovered. In [26], Rodgers obtained new Cameron–Liebler line classes with parameter $x = \frac{q^2-1}{2}$ for $q \equiv 5$ or $9 \pmod{12}$ and $q < 200$. In his thesis [25], Rodgers also reported new examples with parameters $x = \frac{(q+1)^2}{3}$ for $q \equiv 2 \pmod{3}$ and $q < 150$ as joint work with his collaborators. These examples motivated us to find new general constructions of Cameron–Liebler line classes.

On the nonexistence side, Govaerts and Storme [14] first showed that there are no Cameron–Liebler line classes in $\text{PG}(3, q)$ with parameter $2 < x \leq q$ when q is prime. Then De Beule, Hallez and Storme [7] excluded parameters $2 < x \leq q/2$ for all val-

ues q . Next Metsch [20] proved the non-existence of Cameron–Liebler line classes with parameter $2 < x \leq q$, and subsequently improved this result by showing the nonexistence of Cameron–Liebler line classes with parameter $2 < x < q\sqrt[3]{\frac{q}{2}} - \frac{2}{3}q$ [21]. The latter result represents the best asymptotic nonexistence result to date. It seems reasonable to believe that for any fixed $0 < \epsilon < 1$ and constant $c > 0$ there are no Cameron–Liebler line classes with $2 < x < cq^{2-\epsilon}$ for sufficiently large q . Very recently, Gavrilyuk and Metsch [10] proved a modular equality which eliminates almost half of the possible values x for a Cameron–Liebler line class with parameter x . We refer to [21] for a comprehensive survey of the known nonexistence results.

In the present paper we construct a new infinite family of Cameron–Liebler line classes with parameter $x = \frac{q^2-1}{2}$ for $q \equiv 5$ or $9 \pmod{12}$. This family of Cameron–Liebler line classes generalizes the examples found by Rodgers in [26] through a computer search. Furthermore, in the case where q is an even power of 3, we construct the first infinite family of affine two-intersection sets, which is closely related to the newly constructed Cameron–Liebler line classes. The first step of our construction follows the same idea as in [26]. That is, we prescribe an automorphism group for the Cameron–Liebler line classes that we intend to construct; as a consequence, the Cameron–Liebler line classes will be unions of orbits of the prescribed automorphism group on the set of lines of $\text{PG}(3, q)$. The main difficulty with this approach is how to choose orbits properly so that their union is a Cameron–Liebler line class. We overcome this difficulty by giving an explicit choice of orbits so that their union gives a Cameron–Liebler line class with the required parameters. The details are given in Section 4.

The paper is organized as follows. In Section 2, we review basic properties of and facts on Cameron–Liebler line classes; furthermore, we collect auxiliary results on characters of finite fields, which are needed in the proof of our main theorem. In Section 3, we introduce a subset of \mathbb{F}_{q^3} , which we will use in the construction of our Cameron–Liebler line classes, and prove a few properties of the subset. In Section 4, we give an algebraic construction of an infinite family of Cameron–Liebler line classes with $x = \frac{q^2-1}{2}$ for $q \equiv 5$ or $9 \pmod{12}$. In Section 5, we construct the first infinite family of affine two-intersection sets in $\text{AG}(2, q)$, q odd, whose existence was conjectured in the thesis [25] of Rodgers. We close the paper with some concluding remarks.

2. Preliminaries

In this section, we review basic facts on Cameron–Liebler line classes, and collect auxiliary results on characters of finite fields.

2.1. Preliminaries on Cameron–Liebler line classes

It is often advantageous to study Cameron–Liebler line classes in $\text{PG}(3, q)$ by using their images under the Klein correspondence. Let $\mathcal{Q}^+(5, q)$ be the 5-dimensional hyperbolic orthogonal space and x be a nonnegative integer. A subset \mathcal{M} of $\mathcal{Q}^+(5, q)$ is called

an *x-tight set* if for every point $P \in \mathcal{Q}^+(5, q)$, $|P^\perp \cap \mathcal{M}| = x(q + 1) + q^2$ or $x(q + 1)$ according as P is in \mathcal{M} or not, where \perp is the polarity determined by $\mathcal{Q}^+(5, q)$. The geometries of $\text{PG}(3, q)$ and $\mathcal{Q}^+(5, q)$ are closely related through a mapping known as the Klein correspondence which maps the lines of $\text{PG}(3, q)$ bijectively to the points of $\mathcal{Q}^+(5, q)$, cf. [16,22]. Let \mathcal{L} be a set of lines of $\text{PG}(3, q)$ with $|\mathcal{L}| = x(q^2 + q + 1)$, x a non-negative integer, and let \mathcal{M} be the image of \mathcal{L} under the Klein correspondence. Then it is known that \mathcal{L} is a Cameron–Liebler line class with parameter x in $\text{PG}(3, q)$ if and only if \mathcal{M} is an x -tight set of $\mathcal{Q}^+(5, q)$. Moreover, if \mathcal{L} is a Cameron–Liebler line class with parameter x , by [20, Theorem 2.1 (b)], it holds that $|P^\perp \cap \mathcal{M}| = x(q + 1)$ for any point P off $\mathcal{Q}^+(5, q)$; consequently \mathcal{M} is a projective two-intersection set in $\text{PG}(5, q)$ with intersection sizes $h_1 = x(q + 1) + q^2$ and $h_2 = x(q + 1)$, namely each hyperplane of $\text{PG}(5, q)$ intersects \mathcal{M} in either h_1 or h_2 points. We summarize these known facts as follows.

Result 2.1. *Let \mathcal{L} be a set of $x(q^2 + q + 1)$ lines in $\text{PG}(3, q)$, with $0 < x \leq \frac{q^2+1}{2}$, and let \mathcal{M} be the image of \mathcal{L} under the Klein correspondence. Then \mathcal{L} is a Cameron–Liebler line class with parameter x if and only if \mathcal{M} is an x -tight set in $\mathcal{Q}^+(5, q)$; moreover, in the case when \mathcal{L} is a Cameron–Liebler line class, we have*

$$|P^\perp \cap \mathcal{M}| = \begin{cases} x(q + 1) + q^2, & \text{if } P \in \mathcal{M}, \\ x(q + 1), & \text{otherwise.} \end{cases}$$

A (v, k, λ, μ) *strongly regular graph* is a simple undirected regular graph on v vertices with valency k satisfying the following: for any two adjacent (resp. nonadjacent) vertices x and y there are exactly λ (resp. μ) vertices adjacent to both x and y . It is known that a graph with valency k , not complete or edgeless, is strongly regular if and only if its adjacency matrix has exactly two restricted eigenvalues. Here, we say that an eigenvalue of the adjacency matrix is *restricted* if it has an eigenvector perpendicular to the all-ones vector.

One of the most effective methods for constructing strongly regular graphs is by the Cayley graph construction. Let G be a finite abelian group and D be an inverse-closed subset of $G \setminus \{0\}$. We define a graph $\text{Cay}(G, D)$ with the elements of G as its vertices; two vertices x and y are adjacent if and only if $x - y \in D$. The graph $\text{Cay}(G, D)$ is called a *Cayley graph* on G with connection set D . The eigenvalues of $\text{Cay}(G, D)$ are given by $\psi(D)$, $\psi \in \widehat{G}$, where \widehat{G} is the group consisting of all complex characters of G , cf. [3, §1.4.9]. Using the aforementioned spectral characterization of strongly regular graphs, we see that $\text{Cay}(G, D)$ with connection set $D (\neq \emptyset, G)$ is strongly regular if and only if $\psi(D)$, $\psi \in \widehat{G} \setminus \{1\}$, take exactly two values, say α_1 and α_2 with $\alpha_1 > \alpha_2$. We note that if $\text{Cay}(G, D)$ is strongly regular with two restricted eigenvalues α_1 and α_2 , then the set $\{\psi \in \widehat{G} \mid \psi(D) = \alpha_1\}$ also forms a connection set of a strongly regular Cayley graph on \widehat{G} ; this set is called the *dual* of D . For basic properties of strongly regular graphs, see [3, Chapter 9]. For known constructions of strongly regular Cayley graphs and their

connections to two-weight linear codes, partial difference sets, and finite geometry, see [3, p. 133] and [5,19].

Let \mathcal{L} be a Cameron–Liebler line class with parameter x in $\text{PG}(3, q)$ and let $\mathcal{M} \subset \mathcal{Q}^+(5, q)$ be the image of \mathcal{L} under the Klein correspondence. By Result 2.1, \mathcal{M} is a projective two-intersection set in $\text{PG}(5, q)$. By [5], we can construct a corresponding strongly regular Cayley graph as follows. First define $D := \{\lambda v : \lambda \in \mathbb{F}_q^*, \langle v \rangle \in \mathcal{M}\}$, which is a subset of $(\mathbb{F}_q^6, +)$. Then the Cayley graph with vertex set $(\mathbb{F}_q^6, +)$ and connection set D is strongly regular. Its restricted eigenvalues can be determined as follows. Let ψ be a nonprincipal additive character of \mathbb{F}_q^6 . Then ψ is principal on a unique hyperplane P^\perp for some $P \in \text{PG}(5, q)$. We have

$$\begin{aligned} \psi(D) &= \sum_{\langle v \rangle \in \mathcal{M}} \sum_{\lambda \in \mathbb{F}_q^*} \psi(\lambda v) = \sum_{\langle v \rangle \in \mathcal{M}} (q[\langle v \rangle \in P^\perp] - 1) \\ &= -|\mathcal{M}| + q|P^\perp \cap \mathcal{M}| = \begin{cases} -x + q^3, & \text{if } P \in \mathcal{M}, \\ -x, & \text{otherwise,} \end{cases} \end{aligned}$$

where $[\langle v \rangle \in P^\perp]$ is the Kronecker delta function taking value 1 if $\langle v \rangle \in P^\perp$ and value 0 otherwise. Conversely, for each hyperplane P^\perp of $\text{PG}(5, q)$, we can find a nonprincipal character ψ that is principal on P^\perp , and the size of $P^\perp \cap \mathcal{M}$ can be computed from $\psi(D)$. Therefore, the character values of D reflect the intersection sizes of \mathcal{M} with the hyperplanes of $\text{PG}(5, q)$. To summarize, we have the following result.

Result 2.2. *Let \mathcal{L} be a set of $x(q^2 + q + 1)$ lines in $\text{PG}(3, q)$, with $0 < x \leq \frac{q^2+1}{2}$, and let \mathcal{M} be the image of \mathcal{L} under the Klein correspondence. Define*

$$D := \{\lambda v : \lambda \in \mathbb{F}_q^*, \langle v \rangle \in \mathcal{M}\} \subset (\mathbb{F}_q^6, +).$$

Then \mathcal{L} is a Cameron–Liebler line class with parameter x if and only if $|D| = (q^3 - 1)x$ and for any $P \in \text{PG}(5, q)$

$$\psi(D) = \begin{cases} -x + q^3, & \text{if } P \in \mathcal{M}, \\ -x, & \text{otherwise,} \end{cases}$$

where ψ is any nonprincipal character of \mathbb{F}_q^6 that is principal on the hyperplane P^\perp .

Following [26] we now introduce a model of the hyperbolic quadric $\mathcal{Q}^+(5, q)$, which will facilitate our algebraic construction. Let $E = \mathbb{F}_{q^3}$ and $F = \mathbb{F}_q$. We view $E \times E$ as a 6-dimensional vector space over F . For a nonzero vector $v \in E \times E$, we use $\langle v \rangle$ to denote the projective point in $\text{PG}(5, q)$ corresponding to the one-dimensional subspace over F spanned by v . Define a quadratic form $Q : E \times E \rightarrow F$ by

$$Q((x, y)) = \text{Tr}(xy), \quad \forall (x, y) \in E \times E,$$

where Tr is the relative trace from E to F (that is, for any $x \in E$, $\text{Tr}(x) = x + x^q + x^{q^2}$). The quadratic form Q is clearly nondegenerate and $Q((x, 0)) = 0$ for all $x \in E$. So $\{((x, 0)) \mid x \in E^*\}$ is a totally isotropic plane with respect to Q . It follows that the quadric defined by Q has Witt index 3, and so is hyperbolic. This quadric will be our model for $\mathcal{Q}^+(5, q)$. Note that for a point $P = ((x_0, y_0))$, its polar hyperplane P^\perp is given by $P^\perp = \{((x, y)) : \text{Tr}(xy_0 + x_0y) = 0\}$.

Let ψ_E and ψ_F be the canonical additive characters of E and F , respectively. Then each additive character of $E \times E$ has the form

$$\psi_{a,b}((x, y)) = \psi_E(ax + by) = \psi_F(\text{Tr}(ax + by)), \quad (x, y) \in E \times E, \tag{2.1}$$

where $(a, b) \in E \times E$. Since $\psi_{a,b}$ is principal on the hyperplane $\{((x, y)) : \text{Tr}(ax + by) = 0\}$, the character sum condition in [Result 2.2](#) can be more explicitly rewritten as

$$\psi_{a,b}(D) = \begin{cases} -x + q^3, & \text{if } (b, a) \in D, \\ -x, & \text{otherwise.} \end{cases} \tag{2.2}$$

2.2. Preliminaries on characters of finite fields

In this subsection, we will collect some auxiliary results on Gauss sums. We assume that the reader is familiar with the basic theory of characters of finite fields as can be found in Chapter 5 of [\[18\]](#).

For a multiplicative character χ and the canonical additive character ψ of \mathbb{F}_q , define the *Gauss sum* by

$$G(\chi) = \sum_{x \in \mathbb{F}_q^*} \chi(x)\psi(x).$$

The following are some basic properties of Gauss sums:

- (i) $G(\chi)\overline{G(\chi)} = q$ if χ is nonprincipal;
- (ii) $G(\chi^{-1}) = \chi(-1)\overline{G(\chi)}$;
- (iii) $G(\chi) = -1$ if χ is principal.

Let γ be a fixed primitive element of \mathbb{F}_q and k a positive integer dividing $q - 1$. For $0 \leq i \leq k - 1$ we set $C_i^{(k,q)} = \gamma^i \langle \gamma^k \rangle$. These are called the *kth cyclotomic classes* of \mathbb{F}_q . The *Gauss periods* associated with these cyclotomic classes are defined by $\psi(C_i^{(k,q)}) := \sum_{x \in C_i^{(k,q)}} \psi(x)$, $0 \leq i \leq k - 1$, where ψ is the canonical additive character of \mathbb{F}_q . By orthogonality of characters, the Gauss periods can be expressed as a linear combination of Gauss sums:

$$\psi(C_i^{(k,q)}) = \frac{1}{k} \sum_{j=0}^{k-1} G(\chi^j)\chi^{-j}(\gamma^i), \quad 0 \leq i \leq k - 1, \tag{2.3}$$

where χ is any fixed multiplicative character of order k of \mathbb{F}_q . For example, if $k = 2$, we have

$$\psi(C_i^{(2,q)}) = \frac{-1 + (-1)^i G(\eta)}{2}, \quad 0 \leq i \leq 1, \tag{2.4}$$

where η is the quadratic character of \mathbb{F}_q .

The following theorem on Eisenstein sums will be used in the proof of our main theorem in Section 4.

Theorem 2.3. (See [27, Theorem 1].) *Let χ be a nonprincipal multiplicative character of \mathbb{F}_{q^m} and χ' be its restriction to \mathbb{F}_q . Choose a system L of coset representatives of \mathbb{F}_q^* in $\mathbb{F}_{q^m}^*$ in such a way that L can be partitioned into two parts:*

$$L_0 = \{x : \text{Tr}(x) = 0\} \text{ and } L_1 = \{x : \text{Tr}(x) = 1\},$$

where Tr is the relative trace from \mathbb{F}_{q^m} to \mathbb{F}_q . Then,

$$\sum_{x \in L_1} \chi(x) = \begin{cases} G(\chi)/G(\chi'), & \text{if } \chi' \text{ is nonprincipal,} \\ -G(\chi)/q, & \text{otherwise.} \end{cases}$$

We will also need the Hasse–Davenport product formula, which is stated below.

Theorem 2.4. (See [2, Theorem 11.3.5].) *Let θ be a multiplicative character of order $\ell > 1$ of \mathbb{F}_q . For every nonprincipal multiplicative character χ of \mathbb{F}_q ,*

$$G(\chi) = \frac{G(\chi^\ell)}{\chi^\ell(\ell)} \prod_{i=1}^{\ell-1} \frac{G(\theta^i)}{G(\chi\theta^i)}.$$

The Stickelberger theorem on the prime ideal factorization of Gauss sums gives us p -adic information on Gauss sums. We will need this theorem to prove a certain divisibility result later on. Let p be a prime, $q = p^f$, and let ξ_{q-1} be a complex primitive $(q - 1)$ th root of unity. Fix any prime ideal \mathfrak{P} in $\mathbb{Z}[\xi_{q-1}]$ lying over p . Then $\mathbb{Z}[\xi_{q-1}]/\mathfrak{P}$ is a finite field of order q , which we identify with \mathbb{F}_q . Let $\omega_{\mathfrak{P}}$ be the Teichmüller character on \mathbb{F}_q , i.e., an isomorphism

$$\omega_{\mathfrak{P}} : \mathbb{F}_q^* \rightarrow \{1, \xi_{q-1}, \xi_{q-1}^2, \dots, \xi_{q-1}^{q-2}\}$$

satisfying

$$\omega_{\mathfrak{P}}(\alpha) \pmod{\mathfrak{P}} = \alpha, \tag{2.5}$$

for all α in \mathbb{F}_q^* . The Teichmüller character $\omega_{\mathfrak{P}}$ has order $q - 1$. Hence it generates all multiplicative characters of \mathbb{F}_q .

Let \mathcal{P} be the prime ideal of $\mathbb{Z}[\xi_{q-1}, \xi_p]$ lying above \mathfrak{P} . For an integer a , let

$$s(a) = \nu_{\mathcal{P}}(G(\omega_{\mathfrak{P}}^{-a})),$$

where $\nu_{\mathcal{P}}$ is the \mathcal{P} -adic valuation. Thus $\mathcal{P}^{s(a)} \parallel G(\omega_{\mathfrak{P}}^{-a})$. The following evaluation of $s(a)$ is due to Stickelberger (see [2, p. 344]).

Theorem 2.5. *Let p be a prime and $q = p^f$. For an integer a not divisible by $q - 1$, let $a_0 + a_1p + a_2p^2 + \dots + a_{f-1}p^{f-1}$, $0 \leq a_i \leq p - 1$, be the p -adic expansion of the reduction of a modulo $q - 1$. Then*

$$s(a) = a_0 + a_1 + \dots + a_{f-1},$$

that is, $s(a)$ is the sum of the p -adic digits of the reduction of a modulo $q - 1$.

3. The subset X and its properties

Notation 3.1. Let q be a prime power with $q \equiv 5$ or $9 \pmod{12}$ so that $\gcd(q - 1, q^2 + q + 1) = 1$. Write $E = \mathbb{F}_{q^3}$, $F = \mathbb{F}_q$, and let ω be a fixed primitive element of E . For any $x \in E^*$, we use $\log_{\omega}(x)$ to denote the integer $0 \leq i \leq q^3 - 2$ such that $x = \omega^i$. We write $N = q^2 + q + 1$, and let w_1 be an element of order N in E (for example, take $w_1 = \omega^{q-1}$). For $x \in F^*$, we define the sign of x , $\text{sgn}(x) \in F$, by

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x \text{ is a square,} \\ -1, & \text{otherwise.} \end{cases} \tag{3.1}$$

We also define $\text{sgn}(0) := 0$.

It is the purpose of this section to introduce a subset $X \subset E$ and prove a few results on X , which we will need in the construction of our Cameron–Liebler line classes.

Viewing E as a 3-dimensional vector space over \mathbb{F}_q , we will use E as the underlying vector space of $\text{PG}(2, q)$. The points of $\text{PG}(2, q)$ are $\langle \omega^i \rangle$, $0 \leq i \leq q^3 - 2$, and the lines of $\text{PG}(2, q)$ are

$$L_u := \{ \langle x \rangle : \text{Tr}(\omega^u x) = 0 \}, \tag{3.2}$$

where $0 \leq u \leq q^3 - 2$. Of course, $\langle \omega^i \rangle = \langle \omega^{i+jN} \rangle$ and $L_u = L_{u+jN}$, for any i, j and u . Note that since $\gcd(q - 1, q^2 + q + 1) = 1$, we can also take $\langle w_1^i \rangle$, $0 \leq i \leq q^2 + q$, as the points of $\text{PG}(2, q)$.

Define a quadratic form $f : E \rightarrow F$ by $f(x) := \text{Tr}(x^2)$, where Tr is the relative trace from E to F . The associated bilinear form $B : E \times E \rightarrow F$ is given by $B(x, y) = 2 \text{Tr}(xy)$. It is clear that B is nondegenerate. Therefore f defines a nondegenerate conic $\mathcal{Q} = \{ \langle x \rangle \mid f(x) = 0 \}$ in $\text{PG}(2, q)$, which contains $q + 1$ points. Consequently each line l of $\text{PG}(2, q)$

meets \mathcal{Q} in 0, 1 or 2 points, and l is called a *passant*, *tangent* or *secant line* accordingly. Also it is known that each point $P \in \text{PG}(2, q) \setminus \mathcal{Q}$ is on either 0 or 2 tangent lines to \mathcal{Q} , and P is called an *interior* or *exterior point* accordingly, cf. [22, p. 158].

Lemma 3.2. *With the above notation, we have the following:*

- (1) *The tangent lines to \mathcal{Q} are given by L_u with $\text{Tr}(\omega^{2u}) = 0$, $0 \leq u \leq q^3 - 2$.*
- (2) *The polarity of $\text{PG}(2, q)$ induced by \mathcal{Q} interchanges $\langle \omega^u \rangle$ and L_u , where $\text{Tr}(\omega^{2u}) = 0$, $0 \leq u \leq q^3 - 2$, and maps exterior (resp. interior) points to secant (resp. passant) lines.*
- (3) *For any point $P = \langle v \rangle$ off \mathcal{Q} , P is an exterior (resp. interior) point if and only if $f(v)$ has some fixed nonzero sign ϵ (resp. $-\epsilon$).*

Proof. (1) and (2) are well-known facts, and we refer the reader to [15] for the proofs. (3) is clear from [22, p. 166] in the proof of Theorem 4.3.1 therein. \square

Now we define the following subset of \mathbb{Z}_N :

$$I_{\mathcal{Q}} := \{i : 0 \leq i \leq N - 1, \text{Tr}(w_1^{2i}) = 0\} = \{d_0, d_1, \dots, d_q\}, \tag{3.3}$$

where the elements are numbered in any (unspecified) order. That is, $\mathcal{Q} = \{\langle w_1^{d_i} \rangle \mid 0 \leq i \leq q\}$.

Let β be any element of F^* such that $\text{sgn}(\beta) = \epsilon$, with ϵ as defined in Lemma 3.2. For $d_0 \in I_{\mathcal{Q}}$, we define

$$X := \{w_1^{d_i} \text{Tr}(w_1^{d_0+d_i}) : 1 \leq i \leq q\} \cup \{2\beta w_1^{d_0}\} \tag{3.4}$$

and

$$\bar{X} := \{\log_{\omega}(x) \pmod{2N} : x \in X\} \subset \mathbb{Z}_{2N}. \tag{3.5}$$

Lemma 3.3. *Let d_i, d_j, d_k be three distinct elements of $I_{\mathcal{Q}}$. Then the sign of*

$$B(w_1^{d_i}, w_1^{d_j})B(w_1^{d_i}, w_1^{d_k})B(w_1^{d_j}, w_1^{d_k})$$

is equal to the sign of $f(v)$ for any exterior point $\langle v \rangle$. In other words, $2 \text{Tr}(w_1^{d_i+d_j}) \times \text{Tr}(w_1^{d_i+d_k}) \text{Tr}(w_1^{d_j+d_k})$ has sign ϵ , where ϵ is the same as in part (3) of Lemma 3.2. In particular, $\epsilon = 1$.

Proof. Since \mathcal{Q} is a conic, $w_1^{d_i}, w_1^{d_j}, w_1^{d_k}$ are linearly independent over F and thus form a basis of E over F . The Gram matrix of the bilinear form B with respect to this basis is equal to

$$\begin{pmatrix} B(w_1^{d_i}, w_1^{d_i}) & B(w_1^{d_i}, w_1^{d_j}) & B(w_1^{d_i}, w_1^{d_k}) \\ B(w_1^{d_j}, w_1^{d_i}) & B(w_1^{d_j}, w_1^{d_j}) & B(w_1^{d_j}, w_1^{d_k}) \\ B(w_1^{d_k}, w_1^{d_i}) & B(w_1^{d_k}, w_1^{d_j}) & B(w_1^{d_k}, w_1^{d_k}) \end{pmatrix} \tag{3.6}$$

which is symmetric with diagonal entries equal to 0. Its determinant is equal to

$$2B(w_1^{d_i}, w_1^{d_j})B(w_1^{d_i}, w_1^{d_k})B(w_1^{d_j}, w_1^{d_k}).$$

Let $P = \langle v \rangle$ be an exterior point, say, P is the intersection of the tangent lines through $\langle w_1^{d_i} \rangle$ and $\langle w_1^{d_j} \rangle$. Then $B(v, w_1^{d_i}) = B(v, w_1^{d_j}) = 0$, and the Gram matrix with respect to the basis $v, w_1^{d_i}, w_1^{d_j}$ has determinant $-2f(v)B(w_1^{d_i}, w_1^{d_j})^2$. Since $q \equiv 1 \pmod{4}$, -1 is a square in \mathbb{F}_q^* . By [22, p. 262], the two determinants have the same sign. This proves the first part of the lemma. It remains to prove that $\epsilon = 1$. We observe that the matrix (3.6) can be written as $2MM^T$ with

$$M = \begin{pmatrix} w_1^{d_i} & w_1^{qd_i} & w_1^{q^2d_i} \\ w_1^{d_j} & w_1^{qd_j} & w_1^{q^2d_j} \\ w_1^{d_k} & w_1^{qd_k} & w_1^{q^2d_k} \end{pmatrix}.$$

We claim that the determinant of M , $\det(M)$, is in F . To see this, applying the Frobenius automorphism $\sigma : x \mapsto x^q$ of $Gal(E/F)$ to M entry-wise, we get

$$\sigma(\det(M)) = \det(\sigma(M)) = \begin{vmatrix} w_1^{qd_i} & w_1^{q^2d_i} & w_1^{d_i} \\ w_1^{qd_j} & w_1^{q^2d_j} & w_1^{d_j} \\ w_1^{qd_k} & w_1^{q^2d_k} & w_1^{d_k} \end{vmatrix} = \det(M).$$

The claim that $\epsilon = 1$ now follows from this fact and the first part of the lemma. \square

Lemma 3.4. *With the above notation, if we use any other d_i in place of d_0 in the definition of X , then the resulting set X' satisfies that $\overline{X'} \equiv \overline{X} \pmod{2N}$ or $\overline{X'} \equiv \overline{X} + N \pmod{2N}$.*

Proof. Without loss of generality, we assume that we use d_1 in place of d_0 in the definition of X , and obtain $X' = \{w_1^{d_i} \text{Tr}(w_1^{d_1+d_i}) : 0 \leq i \leq q, i \neq 1\} \cup \{2\beta w_1^{d_1}\}$. We have the following observations.

- (1) For $i \neq 0, 1$, that is, $2 \leq i \leq q$, the sign of the quotient of $w_1^{d_i} \text{Tr}(w_1^{d_0+d_i})$ and $w_1^{d_i} \text{Tr}(w_1^{d_1+d_i})$ is a constant: their quotient $\text{Tr}(w_1^{d_0+d_i}) \text{Tr}(w_1^{d_1+d_i})^{-1}$ lies in F , and its sign is clearly equal to that of $\text{Tr}(w_1^{d_0+d_i}) \text{Tr}(w_1^{d_1+d_i})$; this sign is equal to $\text{sgn}(2\text{Tr}(w_1^{d_0+d_1}))$ by Lemma 3.3.
- (2) The sign of the quotient of $2\beta w_1^{d_1}$ and $w_1^{d_1} \text{Tr}(w_1^{d_0+d_1})$ is equal to $\text{sgn}(2\beta \times \text{Tr}(w_1^{d_1+d_0})) = \text{sgn}(2\text{Tr}(w_1^{d_1+d_0}))$, since we have chosen β such that $\text{sgn}(\beta) = \epsilon = 1$. Similarly the sign of the quotient of $2\beta w_1^{d_0}$ and $w_1^{d_0} \text{Tr}(w_1^{d_1+d_0})$ is equal to $\text{sgn}(2\text{Tr}(w_1^{d_1+d_0}))$.

The above observations imply that there is a pairing of the elements of X' and X , say, $p : X' \rightarrow X$ a bijection, such that $x'/p(x')$ are nonzero squares in F for all $x' \in X'$ or $x'/p(x')$ are nonsquares in F for all $x' \in X'$. Upon taking logarithm and modulo $2N$ we get the conclusion of the lemma. \square

Remark 3.5. It is clear that $d'_0 := qd_0 \pmod{N}$ is an element of $I_{\mathcal{Q}}$. In Lemma 3.4, consider the special case where we replace d_0 by d'_0 in the definition of X , and denote the resulting set by X' . We observe that

$$2 \operatorname{Tr}(w_1^{d'_0+d_0}) = \operatorname{Tr}(w_1^{d_0})^2 - \operatorname{Tr}(w_1^{2d_0}) - \operatorname{Tr}(w_1^{2d_0})^q = (\operatorname{Tr}(w_1^{d_0}))^2.$$

Consequently $\operatorname{sgn}(2 \operatorname{Tr}(w_1^{d'_0+d_0})) = 1$. It follows that $\overline{X'} \equiv \overline{X} \pmod{2N}$ in this particular case. By definition, we have $\overline{X'} \equiv q\overline{X} \pmod{N}$, so we have shown that the subset \overline{X} is invariant under multiplication by q . This fact will be needed in the next section when we discuss automorphism groups of the newly constructed Cameron–Liebler line classes.

We now prove some properties of the set X which will be needed in the next section. Let S (resp. N) be the set of nonzero squares (resp. nonsquares) of F , and write

$$s = \sum_{x \in S} \psi_F(x), \quad n = \sum_{x \in N} \psi_F(x),$$

where ψ_F is the canonical additive character of F . For $0 \leq u \leq q^3 - 2$, we define the exponential sums

$$T_u := \sum_{i \in \overline{X}} \sum_{x \in S} \psi_F(\operatorname{Tr}(\omega^{u+i})x)$$

To simplify notation, we often write $T_u = \sum_{i \in \overline{X}} \psi_F(\operatorname{Tr}(\omega^{u+i})S)$. Note that $i \in \overline{X}$ if and only if $i \equiv \log_{\omega}(x) \pmod{2N}$ for some $x \in X$, which in turn is equivalent to $x = \omega^{i+2Nj}$ for some integer j . Since ω^{2Nj} is an element of S , we can view ω^i in the above definition of T_u as coming from X . It follows that

$$T_u = \psi_F(\operatorname{Tr}(2\beta\omega^u w_1^{d_0})S) + \sum_{i=1}^q \psi_F(\operatorname{Tr}(\omega^u w_1^{d_i})\operatorname{Tr}(w_1^{d_i+d_0})S) \tag{3.7}$$

We will evaluate these sums explicitly.

Remark 3.6. The following are some simple observations.

- (1) It is clear that each summand in the right hand side of (3.7) is equal to one of $|S| = \frac{q-1}{2}$, s and n , depending on the sign of the trace term in front of S .
- (2) If we replace X by X' as in Lemma 3.4, then the value of T_u is either unchanged or is equal to T_{u+N} .

Theorem 3.7. Let $T = \{\log_{\omega}(x) \pmod N \mid \text{Tr}(x) = 0\}$ and $T' = \{\log_{\omega}(x) \pmod{2N} \mid \text{Tr}(x) = 1\}$. The exponential sums T_u take the following four values:

$$T_u = \begin{cases} \frac{q-1}{2} + qs \text{ or } \frac{q-1}{2} + qn, & \text{if } 2u \pmod N \in T, \\ -\frac{q+1}{2}, & \text{if } 2u \pmod N \notin T \text{ and } 2u \pmod{2N} \notin T', \\ \frac{q-1}{2}, & \text{if } 2u \pmod N \notin T \text{ and } 2u \pmod{2N} \in T', \end{cases}$$

where s and n are defined as above, that is, $s = \sum_{x \in S} \psi_F(x)$ and $n = \sum_{x \in N} \psi_F(x)$.

Proof. We consider the following three cases according as the line L_u as defined in (3.2) is a tangent, passant, or secant.

Case 1: L_u is a tangent line. In this case, ω^u is a zero of f by (1) of Lemma 3.2, where we recall that $f(x) := \text{Tr}(x^2)$, so $\langle \omega^u \rangle = \langle w_1^{d_k} \rangle$ for some $0 \leq k \leq q$ by the definition of $I_{\mathcal{Q}}$ in Eq. (3.3). Note that u satisfies $2u \pmod N \in T$ by the definition of f and T . In view of (2) of Remark 3.6, we may assume that $d_k = d_0$. (If necessary, replace u by $u + N$, and then the resulting ω^u is still a zero of f , i.e., satisfies $2u \pmod N \in T$.) Now $\text{Tr}(\omega^u w_1^{d_i}) = 0$ if and only if $\langle w_1^{d_i} \rangle$ lies on the tangent line L_u , i.e., $d_i = d_0$. We see that the elements $\text{Tr}(w_1^{d_0+d_i})^2, i \neq 0$, are all nonzero squares, so that $T_u = \frac{q-1}{2} + qs$ or $\frac{q-1}{2} + qn$ by (1) of Remark 3.6. Note that if we replace u by $u + N$, then the value of T_u is replaced by the other in this case. Hence, $\{T_u, T_{u+N}\} = \{\frac{q-1}{2} + qs, \frac{q-1}{2} + qn\}$.

Case 2: L_u is a passant line. In this case, $\langle \omega^u \rangle$ is an interior point and thus $f(\omega^u)$ has sign $-\epsilon (= -1)$ by Lemmas 3.2 and 3.3. Note that u satisfies $2u \pmod N \notin T$ and $2u \pmod{2N} \notin T'$ since $f(\omega^u)$ is a nonsquare of F . Each line through $\langle \omega^u \rangle$ has either 0 or 2 points of \mathcal{Q} , so the points of \mathcal{Q} are partitioned into pairs accordingly. Let $\langle w_1^{d_i} \rangle, \langle w_1^{d_j} \rangle$ be two points of \mathcal{Q} that lie on a secant line through $\langle \omega^u \rangle$.

Since the three points $\langle w_1^{d_i} \rangle, \langle w_1^{d_j} \rangle$ and $\langle \omega^u \rangle$ are collinear, the Gram matrix of the bilinear form B with respect to $w_1^{d_i}, w_1^{d_j}, \omega^u$ is singular. By direct computations we find that the determinant of this Gram matrix is equal to $B(w_1^{d_i}, w_1^{d_j})^2 B(\omega^u, \omega^u) + 2B(w_1^{d_i}, \omega^u)B(w_1^{d_j}, \omega^u)B(w_1^{d_i}, w_1^{d_j})$. It follows that

$$B(w_1^{d_i}, w_1^{d_j})^2 B(\omega^u, \omega^u) + 2B(w_1^{d_i}, \omega^u)B(w_1^{d_j}, \omega^u)B(w_1^{d_i}, w_1^{d_j}) = 0 \tag{3.8}$$

Recall that $f(\omega^u) = \text{Tr}(\omega^{2u})$ has sign $-\epsilon = -1$, and -1 is a square since $q \equiv 1 \pmod 4$. We see from (3.8) that $2 \text{Tr}(w_1^{d_i} \omega^u) \text{Tr}(w_1^{d_j} \omega^u) \text{Tr}(w_1^{d_i+d_j})$ has sign $-\epsilon = -1$. If either d_i or d_j is d_0 , say d_i is d_0 , then $\text{Tr}(2\beta \omega^u w_1^{d_0}) \text{Tr}(\omega^u w_1^{d_j}) \text{Tr}(w_1^{d_j+d_0})$ is a nonsquare since $\beta \in F^*$ has sign $\epsilon = 1$. Hence exactly one of $\text{Tr}(2\beta \omega^u w_1^{d_0})$ and $\text{Tr}(\omega^u w_1^{d_j}) \text{Tr}(w_1^{d_j+d_0})$ is a square and the other is a nonsquare of F . Their corresponding summands in the right hand side of (3.7) thus contribute $s + n = -1$ to T_u . On the other hand, if $d_i, d_j \neq d_0$, by Lemma 3.3, $2 \text{Tr}(w_1^{d_i+d_j}) \text{Tr}(w_1^{d_i+d_0}) \text{Tr}(w_1^{d_j+d_0})$ has sign $\epsilon = 1$, so $\text{Tr}(w_1^{d_i} \omega^u) \text{Tr}(w_1^{d_j} \omega^u) \text{Tr}(w_1^{d_i+d_0}) \text{Tr}(w_1^{d_j+d_0})$ has sign -1 , i.e., it is a nonsquare of F . Again exactly one of $\text{Tr}(w_1^{d_i} \omega^u) \text{Tr}(w_1^{d_i+d_0})$ and $\text{Tr}(w_1^{d_j} \omega^u) \text{Tr}(w_1^{d_j+d_0})$ is a square and the other is a nonsquare of F . Their corresponding summands in the right hand side of (3.7) thus contribute $s + n = -1$ to T_u .

Summing up we have $T_u = -\frac{q+1}{2}$ in this case. Note that replacing u by $u + N$ does not change the value of T_u in this case. Hence, $T_u = T_{u+N} = -\frac{q+1}{2}$.

Case 3: L_u is a secant line. In this case, $\langle \omega^u \rangle$ is an exterior point, and thus $f(\omega^u)$ has sign $\epsilon(= 1)$ by Lemmas 3.2 and 3.3. Note that u satisfies $2u \pmod N \notin T$ and $2u \pmod{2N} \in T'$ since $f(\omega^u)$ is a square. By (2) of Remark 3.6, we may assume that it is on the tangent line l of $\langle w_1^{d_0} \rangle$. (If necessary, replace u by $u + N$, and then the resulting u still satisfies that $2u \pmod N \notin T$ and $2u \pmod{2N} \in T'$.) For $i \neq 0$, the tangent line through $\langle w_1^{d_i} \rangle$ intersects l at the point $\langle \omega^u - \lambda_i w_1^{d_0} \rangle$ by straightforward calculations, where $\lambda_i = \text{Tr}(\omega^u w_1^{d_i}) \text{Tr}(w_1^{d_i+d_0})^{-1}$. When i ranges from 1 to q , we get all the points of l other than $\langle w_1^{d_0} \rangle$, since each other tangent line intersects l at a distinct point. This implies that the values λ_i are all distinct, and hence $\{\lambda_i : 1 \leq i \leq q\}$ is equal to F . Since $\text{Tr}(\omega^u w_1^{d_i}) \text{Tr}(w_1^{d_i+d_0})$ has the same sign as λ_i , we see that the q terms in the second sum in the right hand side of (3.7) are $\frac{q-1}{2}$ once, s with multiplicity $\frac{q-1}{2}$, and n with multiplicity $\frac{q-1}{2}$. Also the first summand in the right hand side of (3.7) is equal to $\frac{q-1}{2}$ since $\langle \omega^u \rangle$ is on the tangent line l of $\langle w_1^{d_0} \rangle$. Summing up, in this case, we have $T_u = 2 \cdot \frac{q-1}{2} + \frac{q-1}{2}(s+n) = \frac{q-1}{2}$. Note that the replacing of u by $u + N$ does not change the value of T_u in this case. Hence, $T_u = T_{u+N} = \frac{q-1}{2}$.

This completes the proof of the theorem. \square

Remark 3.8.

- (i) Some parts of the above theorem can be made more precise. We have $T_u = \frac{q-1}{2} + qs$ if and only if

$$u \pmod{2N} \in \begin{cases} \bar{X}, & \text{when } \text{sgn}(2) = 1, \\ \bar{X} + N, & \text{when } \text{sgn}(2) = -1. \end{cases}$$

and $T_u = \frac{q-1}{2} + qn$ if and only if

$$u \pmod{2N} \in \begin{cases} \bar{X}, & \text{when } \text{sgn}(2) = -1, \\ \bar{X} + N, & \text{when } \text{sgn}(2) = 1. \end{cases}$$

To see this, by Eq. (3.7), consider the set

$$\begin{aligned} & \{\text{sgn}(\text{Tr}(\omega^u \cdot c)) : c \in X\} \\ &= \{\text{sgn}(\text{Tr}(\omega^u w_1^{d_i}) \text{Tr}(w_1^{d_0+d_i})) : 1 \leq i \leq q\} \cup \{\text{sgn}(2\beta \text{Tr}(\omega^u w_1^{d_0}))\}. \end{aligned}$$

For $\omega^u \in X$, by Lemma 3.3, we have the following observations:

- (1) If $\omega^u = w_1^{d_j} \text{Tr}(w_1^{d_j+d_0})$, then

$$\text{sgn}(\text{Tr}(\omega^u w_1^{d_i}) \text{Tr}(w_1^{d_0+d_i})) = \text{sgn}(\text{Tr}(w_1^{d_j+d_i}) \text{Tr}(w_1^{d_i+d_0}) \text{Tr}(w_1^{d_j+d_0})) = \text{sgn}(2)$$

and

$$\operatorname{sgn}(2\beta \operatorname{Tr}(\omega^u w_1^{d_0})) = \operatorname{sgn}(2\beta) \operatorname{sgn}(\operatorname{Tr}(w_1^{2d_0})) = 0.$$

(2) If $\omega^u = 2\beta \operatorname{Tr}(w_1^{d_0})$, then

$$\operatorname{sgn}(\operatorname{Tr}(\omega^u w_1^{d_i}) \operatorname{Tr}(w_1^{d_0+d_i})) = \operatorname{sgn}(2\beta \operatorname{Tr}(w_1^{d_i+d_0})^2) = \operatorname{sgn}(2)$$

and

$$\operatorname{sgn}(2\beta \operatorname{Tr}(\omega^u w_1^{d_0})) = \operatorname{sgn}(2\beta) \operatorname{sgn}(\operatorname{Tr}(w_1^{2d_0})) = 0.$$

Summing up, we have $T_u = \frac{q-1}{2} + qs$ (resp. $\frac{q-1}{2} + qn$) when $\operatorname{sgn}(2) = 1$ (resp. $\operatorname{sgn}(2) = -1$). Similarly, for $\omega^u \in \omega^N X$, it holds that $T_u = \frac{q-1}{2} + qs$ (resp. $\frac{q-1}{2} + qn$) when $\operatorname{sgn}(2) = -1$ (resp. $\operatorname{sgn}(2) = 1$). By [Theorem 3.7](#), the converse is also true.

(ii) By [\(2.4\)](#), the values of s and n are given by $s = \frac{-1+G(\eta')}{2}$ and $n = \frac{-1-G(\eta')}{2}$, where η' is the quadratic character of F . Thus, the condition of the remark above can be described as follows: $T_u = \frac{q-1}{2} + q\frac{-1+\eta'(2)G(\eta')}{2}$ (resp. $T_u = \frac{q-1}{2} + q\frac{-1-\eta'(2)G(\eta')}{2}$) if and only if $u \pmod{2N} \in \overline{X}$ (resp. $u \pmod{2N} \in \overline{X} + N$).

4. A construction of Cameron–Liebler line classes with parameter $x = \frac{q^2-1}{2}$

In this section, we give the promised construction of Cameron–Liebler line classes with parameter $x = \frac{q^2-1}{2}$. We will use the same notation introduced in previous sections. By [Result 2.1](#), it suffices to construct an x -tight set $\mathcal{M} \in \mathcal{Q}^+(5, q)$ satisfying the hyperplane intersection property specified in [Result 2.1](#). Going from a subset of points in $\operatorname{PG}(5, q)$ to a subset of vectors in $E \times E$, by [Result 2.2](#), it suffices to construct a subset $D \subset E \times E \setminus \{(0, 0)\}$ such that $|D| = x(q^3 - 1)$, $Q((u, v)) = 0$ for all $(u, v) \in D$, and the additive character values of D satisfy [\(2.2\)](#).

The first step of our construction follows the idea in [\[25, p. 37\]](#). That is, we prescribe an automorphism group for the x -tight set that we intend to construct. We will take the model of $\mathcal{Q}^+(5, q)$ as introduced in [Section 2](#). Define the map g on $\mathcal{Q}^+(5, q)$ by

$$g : (x, y) \mapsto (w_1 x, w_1^{-1} y),$$

where $w_1 \in E^*$ has order $N = q^2 + q + 1$. Then the cyclic subgroup $C \leq \operatorname{PGO}^+(6, q)$ generated by g has order $q^2 + q + 1$, and it acts semi-regularly on the points of $\mathcal{Q}^+(5, q)$ (so each orbit contains $q^2 + q + 1$ points). The x -tight set \mathcal{M} will be a union of orbits of C acting on $\mathcal{Q}^+(5, q)$. The main difficulty with this approach lies in coming up with a general choice of orbits of C for all q so that the union of the chosen orbits is an x -tight set. We will use the subset X introduced in [Section 3](#) to help choose orbits for our purpose.

Let \bar{X} be the set as defined in (3.5). The set \bar{X} can be expressed as

$$\bar{X} = 2A \cup (2B + N) \pmod{2N}$$

for some $A, B \subseteq \mathbb{Z}_N$ with $|A| + |B| = q + 1$. Define a subset I_X of \mathbb{Z}_{4N} by

$$\begin{aligned} I_X := & \{4t + Ns \pmod{4N} : t \in A, s = 0, 1\} \\ & \cup \{4t + Ns \pmod{4N} : t \in B, s = 2, 3\}. \end{aligned} \tag{4.1}$$

That is,

$$I_X = 4A \cup (4A + N) \cup (4B + 2N) \cup (4B + 3N).$$

Note that $|I_X| = 2(q + 1)$. Now we use I_X to give the main construction of this paper. Let q be a prime power such that $q \equiv 5$ or $9 \pmod{12}$. Define

$$D := \{(xy, xy^{-1}z\omega^\ell) \mid x \in F^*, y \in C_0^{(q-1, q^3)}, z \in C_0^{(4N, q^3)}, \ell \in I_X\} \subseteq E \times E, \tag{4.2}$$

where $C_0^{(q-1, q^3)} := \langle \omega^{q-1} \rangle$, $C_0^{(4N, q^3)} := \langle \omega^{4N} \rangle$. Clearly we have

$$|D| = (q - 1)(q^2 + q + 1) \frac{q - 1}{4} \cdot 2(q + 1) = \frac{q^2 - 1}{2}(q^3 - 1),$$

and $\lambda D = D$ for all $\lambda \in F^*$. Also it is clear that D is a subset of the hyperbolic quadric $\{(x, y) \in E^2 \mid \text{Tr}(xy) = 0\}$ since $\text{Tr}(\omega^\ell) = 0$ for any $\ell \in I_X$ by the definition of I_X and X . Let \mathcal{M} be the set of projective points in $\text{PG}(5, q)$ corresponding to D . Then $|\mathcal{M}| = \frac{q^2-1}{2}(q^2 + q + 1)$ and $\mathcal{M} \subset \mathcal{Q}^+(5, q)$.

Theorem 4.1. *The line set \mathcal{L} in $\text{PG}(3, q)$ corresponding to \mathcal{M} under the Klein correspondence forms a Cameron–Liebler line class with parameter $x = \frac{q^2-1}{2}$.*

In the rest of this section we give a proof of Theorem 4.1. By the discussions at the beginning of this section, it suffices to show that for all $(0, 0) \neq (a, b) \in E \times E$,

$$\psi_{a,b}(D) = \begin{cases} -\frac{q^2-1}{2} + q^3, & \text{if } (b, a) \in D, \\ -\frac{q^2-1}{2}, & \text{otherwise.} \end{cases}$$

By (2.1) and the definition of D , $\psi_{a,b}(D)$ is expressed as

$$S_{a,b} := \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{j=0}^{\frac{q-1}{4}-1} \sum_{k=0}^{q-2} \psi_E(a\omega^{Nk}\omega^{(q-1)i} + b\omega^{Nk}\omega^{-(q-1)i}\omega^{4Nj+\ell}). \tag{4.3}$$

We evaluate these character sums by considering two cases: (i) $a = 0$ or $b = 0$; and (ii) $a \neq 0$ and $b \neq 0$.

Lemma 4.2. Assume either $a = 0$ or $b = 0$. Then the quadruple sum $S_{a,b}$ in (4.3) is equal to $-\frac{q^2-1}{2}$.

Proof. If $a \neq 0$ and $b = 0$, since $\{\omega^{Nk+(q-1)i} \mid 0 \leq k \leq q-2, 0 \leq i \leq N-1\} = E^*$, we have

$$\begin{aligned} S_{a,b} &= \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{j=0}^{\frac{q-1}{4}-1} \sum_{k=0}^{q-2} \psi_E(a\omega^{Nk}\omega^{(q-1)i}) \\ &= \frac{q-1}{4} |I_X| \sum_{x \in E^*} \psi_E(ax) = -\frac{1}{4}(q-1)|I_X| = -\frac{q^2-1}{2}. \end{aligned}$$

Similarly, if $a = 0$ and $b \neq 0$, we have

$$\begin{aligned} S_{a,b} &= \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{j=0}^{\frac{q-1}{4}-1} \sum_{k=0}^{q-2} \psi_E(b\omega^{Nk}\omega^{-(q-1)i}\omega^{4Nj+\ell}) \\ &= \sum_{\ell \in I_X} \sum_{j=0}^{\frac{q-1}{4}-1} \sum_{x \in E^*} \psi_E(bx\omega^{4Nj+\ell}) = -\frac{1}{4}(q-1)|I_X| = -\frac{q^2-1}{2}. \end{aligned}$$

This completes the proof of the lemma. \square

Next, we consider case (ii): $a \neq 0$ and $b \neq 0$. The quadruple sum $S_{a,b}$ in this case can be essentially reduced to the character sum T_u , which was already evaluated in Theorem 3.7 (and Remark 3.8). The computations involved in reducing $S_{a,b}$ to T_u are routine but complicated. So we start with a lemma, which says that the sum $S_{a,b}$ can be expressed in terms of Gauss sums of order $4N$ of E .

Lemma 4.3. Assume that $a, b \neq 0$. Then

$$S_{a,b} = \frac{(q-1)}{4(q^3-1)} \sum_{\ell \in I_X} \sum_{\substack{u,v=0 \\ u+v \equiv 0 \pmod{4} \\ u \equiv v \pmod{N}}}^{4N-1} G(\chi_{4N}^{-v})G(\chi_{4N}^{-u})\chi_{4N}^v(a)\chi_{4N}^u(b)\chi_{4N}^u(\omega^\ell), \quad (4.4)$$

where χ_{4N} is a multiplicative character of order $4N$ of E .

Proof. By orthogonality of characters, the quadruple sum $S_{a,b}$ is equal to

$$\frac{1}{4} \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{x,y \in F^*} \psi_E(ax\omega^{(q-1)i} + bx\omega^{-(q-1)i}y^4\omega^\ell)$$

$$\begin{aligned}
 &= \frac{1}{4(q^3 - 1)^2} \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{x, y \in F^*} \sum_{j, k=0}^{q^3-2} G(\chi_{q^3-1}^{-j}) \chi_{q^3-1}^j (ax\omega^{(q-1)i}) G(\chi_{q^3-1}^{-k}) \\
 &\quad \times \chi_{q^3-1}^k (bx\omega^{-(q-1)i} y^4 \omega^\ell) \\
 &= \frac{1}{4(q^3 - 1)^2} \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{x \in F^*} \sum_{j, k=0}^{q^3-2} G(\chi_{q^3-1}^{-j}) G(\chi_{q^3-1}^{-k}) \chi_{q^3-1}^j (a) \chi_{q^3-1}^k (b) \chi_{q^3-1}^{j+k} (x) \\
 &\quad \times \chi_{q^3-1}^{j-k} (\omega^{(q-1)i}) \chi_{q^3-1}^k (\omega^\ell) \left(\sum_{y \in F^*} \chi_{q^3-1}^k (y^4) \right). \tag{4.5}
 \end{aligned}$$

Let $\chi_{4N} = \chi_{q^3-1}^{\frac{q-1}{4}}$. Since $\sum_{y \in F^*} \chi_{q^3-1}^k (y^4) = q - 1$ or 0 according as $\frac{(q-1)}{4} | k$ or not, continuing from (4.5), we have

$$\begin{aligned}
 S_{a,b} &= \frac{q-1}{4(q^3 - 1)^2} \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{x \in F^*} \sum_{j=0}^{q^3-2} \sum_{u=0}^{4N-1} G(\chi_{q^3-1}^{-j}) G(\chi_{4N}^{-u}) \chi_{q^3-1}^j (a) \chi_{4N}^u (b) \chi_{q^3-1}^{j+\frac{q-1}{4}u} (x) \\
 &\quad \times \chi_{q^3-1}^{j-\frac{q-1}{4}u} (\omega^{(q-1)i}) \chi_{4N}^u (\omega^\ell) \\
 &= \frac{q-1}{4(q^3 - 1)^2} \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{j=0}^{q^3-2} \sum_{u=0}^{4N-1} G(\chi_{q^3-1}^{-j}) G(\chi_{4N}^{-u}) \chi_{q^3-1}^j (a) \chi_{4N}^u (b) \chi_{q^3-1}^{j-\frac{q-1}{4}u} (\omega^{(q-1)i}) \\
 &\quad \times \chi_{4N}^u (\omega^\ell) \left(\sum_{x \in F^*} \chi_{q^3-1}^{j+\frac{q-1}{4}u} (x) \right). \tag{4.6}
 \end{aligned}$$

Let $\chi_N = \chi_{q^3-1}^{\frac{q-1}{4}}$. Since $\sum_{x \in F^*} \chi_{q^3-1}^{j+\frac{q-1}{4}u} (x) = q - 1$ or 0 according as $j \equiv 0 \pmod{\frac{q-1}{4}}$ and $j + \frac{q-1}{4}u \equiv 0 \pmod{q-1}$ or not, continuing from (4.6), we have

$$\begin{aligned}
 S_{a,b} &= \frac{(q-1)^2}{4(q^3 - 1)^2} \sum_{\ell \in I_X} \sum_{i=0}^{N-1} \sum_{\substack{u,v=0 \\ u+v \equiv 0 \pmod{4}}}^{4N-1} G(\chi_{4N}^{-v}) G(\chi_{4N}^{-u}) \chi_{4N}^v (a) \chi_{4N}^u (b) \chi_{4N}^{v-u} (\omega^{(q-1)i}) \\
 &\quad \times \chi_{4N}^u (\omega^\ell) \\
 &= \frac{(q-1)^2}{4(q^3 - 1)^2} \sum_{\ell \in I_X} \sum_{\substack{u,v=0 \\ u+v \equiv 0 \pmod{4}}}^{4N-1} G(\chi_{4N}^{-v}) G(\chi_{4N}^{-u}) \chi_{4N}^v (a) \chi_{4N}^u (b) \\
 &\quad \times \chi_{4N}^u (\omega^\ell) \left(\sum_{i=0}^{N-1} \chi_N^{(v-u)\frac{q-1}{4}} (\omega^i) \right) \\
 &= \frac{(q-1)}{4(q^3 - 1)} \sum_{\ell \in I_X} \sum_{\substack{u,v=0 \\ u+v \equiv 0 \pmod{4} \\ u \equiv v \pmod{N}}}^{4N-1} G(\chi_{4N}^{-v}) G(\chi_{4N}^{-u}) \chi_{4N}^v (a) \chi_{4N}^u (b) \chi_{4N}^u (\omega^\ell). \tag{4.7}
 \end{aligned}$$

The proof of the lemma is now complete. \square

Remark 4.4. In (4.4) of Lemma 4.3, v is expressed as $v = -Nc + 4d$ if $u = Nc + 4d$ for $c = 0, 1, 2, 3$, and $d = 0, 1, \dots, N - 1$. Write $\chi_4 = \chi_{4N}^N$ and we can assume that $\chi_4(\omega^N) = i$, where $i = \sqrt{-1}$. Then, by the definition of I_X , the sum in (4.4) can be expanded as follows:

$$\begin{aligned} \frac{4(q^3 - 1)}{q - 1} S_{a,b} &= \sum_{s=0,1} \sum_{t \in A} \sum_{c=0,1,2,3} \sum_{d=0}^{N-1} G(\chi_{4N}^{Nc-4d}) G(\chi_{4N}^{-Nc-4d}) \chi_4^c(a^{-1}b) \chi_N^d(ab) \\ &\quad \times \chi_{4N}^{Nc+4d}(\omega^{4t+Ns}) \\ &\quad + \sum_{s=2,3} \sum_{t \in B} \sum_{c=0,1,2,3} \sum_{d=0}^{N-1} G(\chi_{4N}^{Nc-4d}) G(\chi_{4N}^{-Nc-4d}) \chi_4^c(a^{-1}b) \chi_N^d(ab) \\ &\quad \times \chi_{4N}^{Nc+4d}(\omega^{4t+Ns}) \\ &= \sum_{t \in A} \sum_{c=0,1,2,3} \sum_{d=0}^{N-1} G(\chi_{4N}^{Nc-4d}) G(\chi_{4N}^{-Nc-4d}) \chi_4^c(r') \chi_N^d(r) \chi_N^d(\omega^{4t})(1 + (i)^c) \\ &\quad + \sum_{t \in B} \sum_{c=0,1,2,3} \sum_{d=0}^{N-1} G(\chi_{4N}^{Nc-4d}) G(\chi_{4N}^{-Nc-4d}) \chi_4^c(r') \\ &\quad \times \chi_N^d(r) \chi_N^d(\omega^{4t})((-1)^c + (-i)^c) \end{aligned} \tag{4.8}$$

where $r' := a^{-1}b$ and $r := ab$.

We now compute the right hand side of Eq. (4.8) by dividing it into three partial sums: P_1, P_2 and P_3 , where P_1 is the contribution of the summands with $c = 2$, P_2 is the contribution of the summands with $c = 0$, and P_3 is the contribution of the summands with $c = 1, 3$. That is, we have

$$\frac{4(q^3 - 1)}{q - 1} S_{a,b} = P_1 + P_2 + P_3.$$

It is obvious that P_1 is equal to 0. Next we evaluate P_2 .

Lemma 4.5. *We have*

$$P_2 = 2(q + 1)(1 - q^3) + 2q^3 \delta_r,$$

where

$$\delta_r := \begin{cases} N, & \text{if } \log_\omega(r) \pmod N \in T, \\ 0, & \text{if } \log_\omega(r) \pmod N \notin T, \end{cases}$$

and T is as defined in Theorem 3.7.

Proof. Recall that $T = \{\log_\omega(x) \pmod N : \text{Tr}(x) = 0\}$. Note that $4(A \cup B) \equiv T \pmod N$ by the definition of A, B and the fact that $G(\chi_N^d) = q \sum_{t \in A \cup B} \chi_N^d(\omega^{4t})$ which follows from [Theorem 2.3](#). We have

$$\begin{aligned} P_2 &= 2 \sum_{t \in A \cup B} \sum_{d=0}^{N-1} G(\chi_N^{-d}) G(\chi_N^{-d}) \chi_N^d(r) \chi_N^d(\omega^{4t}) \\ &= \frac{2}{q} \sum_{d=1}^{N-1} G(\chi_N^{-d}) G(\chi_N^{-d}) \chi_N^d(r) G(\chi_N^d) + 2(q+1). \end{aligned} \tag{4.9}$$

Since $G(\chi_N^{-d}) G(\chi_N^d) = q^3$ and $G(\chi_N^d) = q \sum_{x \in T} \chi_N^d(\omega^x)$, continuing from (4.9), we have

$$\begin{aligned} P_2 &= 2q^2 \sum_{d=1}^{N-1} G(\chi_N^{-d}) \chi_N^d(r) + 2(q+1) = 2q^3 \sum_{d=1}^{N-1} \sum_{x \in T} \chi_N^{-d}(\omega^x) \chi_N^d(r) + 2(q+1) \\ &= 2q^3 \sum_{d=0}^{N-1} \sum_{x \in T} \chi_N^{-d}(\omega^x) \chi_N^d(r) + 2(q+1) - 2(q+1)q^3 \\ &= 2(q+1)(1 - q^3) + 2q^3 \cdot \begin{cases} N, & \text{if } \log_\omega(r) \pmod N \in T \\ 0, & \text{if } \log_\omega(r) \pmod N \notin T \end{cases} \end{aligned}$$

The conclusion of the lemma now follows. \square

It remains to evaluate P_3 .

Lemma 4.6. *Let η be the quadratic character of E , and define $n_{r'} = \chi_4(r') + \chi_4^3(r') + i(\chi_4(r') - \chi_4^3(r'))$. Then*

$$P_3 = \eta(2)n_{r'} G(\eta) \left(2N \sum_{t \in \bar{X}} \psi_E(r^{\frac{N+1}{2}} \omega^t C_0^{(2N, q^3)}) + N - q\delta'_r \right), \tag{4.10}$$

where δ'_r is defined by

$$\delta'_r = \begin{cases} 0, & \text{if } \log_\omega(r) \pmod N \in T, \\ N, & \text{if } \log_\omega(r) \pmod N \notin T \text{ and } \log_\omega(r^{N+1}) \pmod{2N} \in T', \\ -N, & \text{if } \log_\omega(r) \pmod N \notin T \text{ and } \log_\omega(r^{N+1}) \pmod{2N} \notin T', \end{cases}$$

where $T' = \{\log_\omega(x) \pmod{2N} \mid \text{Tr}(x) = 1\}$.

Remark 4.7. Before going into the proof of the lemma, we make the following observations.

(i) The $n_{r'}$ in Lemma 4.6 can be explicitly evaluated:

$$n_{r'} = \begin{cases} 2 & \text{if } \chi_4(r') = 1 \text{ or } \chi_4(r') = -i, \\ -2 & \text{if } \chi_4(r') = -1 \text{ or } \chi_4(r') = i. \end{cases}$$

(ii) The character sum $\sum_{t \in \bar{X}} \psi_E(r^{\frac{N+1}{2}} \omega^t C_0^{(2N, q^3)}) = \sum_{t \in \bar{X}} \psi_F(\text{Tr}(r^{\frac{N+1}{2}} \omega^t) S)$ in Lemma 4.6 has been already evaluated in Theorem 3.7 (and Remark 3.8) as:

$$T_{\log_\omega(r^{\frac{N+1}{2}})} = \begin{cases} \frac{q-1}{2} + q^{\frac{-1+\eta'(2)G(\eta')}{2}}, & \text{if } \log_\omega(r^{\frac{N+1}{2}}) \pmod{2N} \in \bar{X}, \\ \frac{q-1}{2} + q^{\frac{-1-\eta'(2)G(\eta')}{2}}, & \text{if } \log_\omega(r^{\frac{N+1}{2}}) \pmod{2N} \in \bar{X} + N, \\ \frac{q-1}{2}, & \text{if } \log_\omega(r) \pmod{N} \notin T \\ & \text{and } \log_\omega(r^{N+1}) \pmod{2N} \in T', \\ -\frac{q+1}{2}, & \text{if } \log_\omega(r) \pmod{N} \notin T \\ & \text{and } \log_\omega(r^{N+1}) \pmod{2N} \notin T'. \end{cases}$$

Proof of Lemma 4.6. First we note that

$$P_3 = n_{r'} \sum_{d=0}^{N-1} G(\chi_{4N}^{N-4d}) G(\chi_{4N}^{-N-4d}) \chi_N^d(r) \left(\sum_{t \in A} \chi_N^d(\omega^{4t}) - \sum_{t \in B} \chi_N^d(\omega^{4t}) \right). \tag{4.11}$$

Applying the Hasse–Davenport product formula with $\ell = 2$, $\chi = \chi_N^{-d} \chi_4$, and $\theta = \eta$, we have

$$G(\chi_N^{-d} \chi_4) G(\chi_N^{-d} \chi_4^3) = \eta(2) G(\eta) G(\chi_N^{-2d} \eta). \tag{4.12}$$

Substituting Eq. (4.12) into Eq. (4.11), we obtain

$$\begin{aligned} \frac{\eta(2)}{n_{r'} G(\eta)} \cdot P_3 &= \sum_{d=0}^{N-1} G(\chi_N^{-2d} \eta) \chi_N^d(r) \left(\sum_{t \in A} \chi_N^{2d}(\omega^{2t}) - \sum_{t \in B} \chi_N^{2d}(\omega^{2t}) \right) \\ &= \sum_{d=0}^{N-1} G(\chi_N^{-2d} \eta) \chi_N^{2d}(r^{\frac{N+1}{2}}) \eta(r^{\frac{N+1}{2}}) \left(\sum_{t \in A} \chi_N^{2d} \eta(\omega^{2t}) + \sum_{t \in B} \chi_N^{2d} \eta(\omega^{2t+N}) \right) \\ &= \sum_{t \in \bar{X}} \sum_{d=0}^{N-1} G(\chi_N^{-2d} \eta) \chi_N^{2d} \eta(r^{\frac{N+1}{2}}) \chi_N^{2d} \eta(\omega^t) \end{aligned} \tag{4.13}$$

where we have used the definition of \bar{X} and the fact that $\eta(r^{\frac{N+1}{2}}) = 1$. By orthogonality of characters, we have

$$\frac{\eta(2)}{n_{r'} G(\eta)} \cdot P_3 = 2N \sum_{t \in \bar{X}} \psi_E(r^{\frac{N+1}{2}} \omega^t C_0^{(2N, q^3)}) - \sum_{t \in \bar{X}} \sum_{d=0}^{N-1} G(\chi_N^{-d}) \chi_N^d(r^{\frac{N+1}{2}} \omega^t). \tag{4.14}$$

Finally, we evaluate the second sum in Eq. (4.14). Using $G(\chi_N^{2^{-1}d}) = q \sum_{t \in \bar{X}} \chi_N^d(\omega^t)$ and applying the Hasse–Davenport product formula again with $\ell = 2$, $\chi = \chi_N^{-2^{-1}d}$, and $\theta = \eta$, we can rewrite the second sum in Eq. (4.14) as follows:

$$\begin{aligned} \sum_{t \in \bar{X}} \sum_{d=0}^{N-1} G(\chi_N^{-d}) \chi_N^d(r^{\frac{N+1}{2}} \omega^t) &= \frac{1}{q} \sum_{d=1}^{N-1} G(\chi_N^{-d}) G(\chi_N^{2^{-1}d}) \chi_N^d(r^{\frac{N+1}{2}}) - (q+1) \\ &= \frac{G(\eta)}{q} \sum_{d=1}^{N-1} G(\chi_N^{-2^{-1}d} \eta) \chi_N^d(r^{\frac{N+1}{2}}) - (q+1) \\ &= \frac{G(\eta)}{q} \sum_{d=0}^{N-1} G(\chi_N^{-d} \eta) \chi_N^d \eta(r^{N+1}) - N. \end{aligned} \tag{4.15}$$

Let χ_{2N} be a multiplicative character of order $2N$ of E and η' be the quadratic character of F . We now use the following formula:

$$G(\chi_N^{-d} \eta) = \left(\sum_{x \in E: \text{Tr}(x)=1} \chi_N^{-d} \eta(x) \right) G(\eta'),$$

which follows from Theorem 2.3. Noting that $G(\eta)G(\eta') = q^2$ (cf. [18, Theorem 5.18]), we have

$$\begin{aligned} &\sum_{t \in \bar{X}} \sum_{d=0}^{N-1} G(\chi_N^{-d}) \chi_N^d(r^{\frac{N+1}{2}} \omega^t) + N \\ &= \frac{G(\eta)}{q} G(\eta') \sum_{d=0}^{N-1} \left(\sum_{x \in E: \text{Tr}(x)=1} \chi_N^{-d} \eta(x) \right) \chi_N^d \eta(r^{N+1}) \\ &= q \sum_{d=0}^{2N-1} \left(\sum_{x \in E: \text{Tr}(x)=1} \chi_{2N}^{-d}(x) \right) \chi_{2N}^d(r^{N+1}) - q \sum_{d=0}^{N-1} \left(\sum_{x \in E: \text{Tr}(x)=1} \chi_N^{-d}(x) \right) \chi_N^d(r^{N+1}) \\ &= q \cdot \begin{cases} 0, & \text{if } \log_\omega(r) \pmod{N} \in T, \\ N, & \text{if } \log_\omega(r) \pmod{N} \notin T \text{ and } \log_\omega(r^{N+1}) \pmod{2N} \in T', \\ -N, & \text{if } \log_\omega(r) \pmod{N} \notin T \text{ and } \log_\omega(r^{N+1}) \pmod{2N} \notin T'. \end{cases} \end{aligned}$$

This completes the proof of the lemma. \square

We are now ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. If either $a = 0$ or $b = 0$, then $\psi_{a,b}(D) = -\frac{q^2-1}{2}$ by Lemma 4.2. If $a \neq 0$ and $b \neq 0$, by Lemmas 4.5, 4.6, and Remark 4.7, we have

$$\begin{aligned} \frac{4(q^3 - 1)}{q - 1} S_{a,b} &= 2(q + 1)(1 - q^3) + 2q^3\delta_r + \eta(2)n_{r'}G(\eta)(2NT_{\log_\omega(r^{\frac{N+1}{2}})} + N - q\delta'_r) \\ &= 2(q + 1)(1 - q^3) + \begin{cases} 4q^3N, & \text{if } \log_\omega(r^{\frac{N+1}{2}}) \pmod{2N} \in \bar{X}, \\ & \text{and } n_{r'} = 2, \\ \text{or } \log_\omega(r^{\frac{N+1}{2}}) \pmod{2N} \in \bar{X} + N, \\ & \text{and } n_{r'} = -2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus by Lemma 4.3 and Remark 4.4, the value of $\psi_{a,b}(D)$ in this case is given by $-\frac{q^2-1}{2}$ or $q^3 - \frac{q^2-1}{2}$.

Finally, we show that $\psi_{a,b}(D) = q^3 - \frac{q^2-1}{2}$ if and only if $(b, a) \in D$. If $(b, a) \in D$, then there are $x \in F^*$, $y \in C_0^{(q-1, q^3)}$, $z \in C_0^{(4N, q^3)}$, $u \in X$, and $v \in \{1, \omega^N\}$ such that $b = xy$ and $a = xy^{-1}zu^2v$. Then, $r := ab = x^2zu^2v$ and $r' := a^{-1}b = yz^{-1}u^{-2}v^{-1}$, and they satisfy

$$\log_\omega(r^{\frac{N+1}{2}}) \pmod{2N} \equiv \log_\omega(u^{N+1}) \pmod{2N} \in \bar{X} + \log_\omega(u^N)$$

and

$$\chi_4(r') = \chi_4(u^{-2}v^{-1}) = \begin{cases} 1, & \text{if } u \text{ is a square and } v = 1, \\ -i, & \text{if } u \text{ is a square and } v = \omega^N, \\ -1, & \text{if } u \text{ is a nonsquare and } v = 1, \\ i, & \text{if } u \text{ is a nonsquare and } v = \omega^N. \end{cases}$$

If u is a square, then $\log_\omega(r^{\frac{N+1}{2}}) \pmod{2N} \in \bar{X}$ and $n_{r'} = 2$. If u is a nonsquare, then $\log_\omega(r^{\frac{N+1}{2}}) \pmod{2N} \in \bar{X} + N$ and $n_{r'} = -2$. Thus, in both cases, we have $\psi_{a,b}(D) = q^3 - \frac{q^2-1}{2}$. The converse also holds since the size of the dual of D is $|D|$, cf. [19, Theorem 3.4]. The proof of the theorem is now complete. \square

Remark 4.8.

- (1) It is clear that the subset D defined in (4.2) is disjoint from both $\{(y, 0) \mid y \in E^*\}$ and $\{(0, y) \mid y \in E^*\}$. By adding one of these sets to D , we therefore obtain a Cameron–Liebler line class of parameter $\frac{q^2+1}{2}$.
- (2) We observe that $I_X = 2\bar{X} \cup (2\bar{X} + N)$. Consider the element $\sigma \in P\Gamma O^+(5, q)$ defined by $\sigma((x, y)) = (x^q, y^q), \forall (x, y) \in E \times E$. This automorphism σ belongs to the embedded image of $PGL(4, q)$ in $P\Gamma O^+(5, q)$ induced by the Klein correspondence. We claim that σ stabilizes our line set \mathcal{L} . This amounts to the fact that \bar{D} is invariant under the map $(x, y) \mapsto (x^q, y^q)$, which follows from the fact that \bar{X} is invariant under multiplication by q as proved in Remark 3.5. Summing up, our line set \mathcal{L} has an automorphism group isomorphic to $(\mathbb{Z}_{q^2+q+1} \times \mathbb{Z}_{q-1}) \rtimes \mathbb{Z}_3$.

5. Affine two-intersection sets

A set \mathcal{K} of points of a projective or affine plane is called a *set of type (m, n)* if every line of the plane intersects \mathcal{K} in m or n points; we assume that $m < n$, and we require both values to occur. There are many known sets of type (m, n) in $\text{PG}(2, q)$ for both even and odd q , e.g., a maximal arc of degree n in $\text{PG}(2, 2^f)$ is a set of type $(0, n)$ with $n \mid 2^f$, and a unital in $\text{PG}(2, q^2)$ is a set of type $(1, q + 1)$. The situation is quite different for affine planes. When q is even, let \mathcal{K} be a maximal arc of degree n in $\text{PG}(2, q)$ and let ℓ be a line of $\text{PG}(2, q)$ such that $|\ell \cap \mathcal{K}| = 0$. Then \mathcal{K} is a set of type $(0, n)$ in $\text{AG}(2, q) = \text{PG}(2, q) \setminus \ell$. Since nontrivial maximal arcs do not exist in $\text{PG}(2, q)$ when q is odd, cf. [1], the construction just mentioned does not work in $\text{AG}(2, q)$, q odd. In fact, for affine planes of odd order, we only know examples of sets of type (m, n) in affine planes of order 9 and in $\text{AG}(2, 81)$. Penttila and Royle [24] classified sets of type $(3, 6)$ in all affine planes of order 9 by exhaustive computer search. In [26], Rodgers developed a method to obtain new affine two-intersection sets in $\text{AG}(2, 3^{2e})$ by establishing certain tactical decompositions of the points and lines of $\text{PG}(3, 3^{2e})$ induced by Cameron–Liebler line classes with some nice properties. He was thus able to rediscover an example of sets of type $(3, 6)$ in $\text{AG}(2, 9)$, and obtain a new example of affine two-intersection sets in $\text{AG}(2, 81)$. In his thesis [25], Rodgers made the following conjecture.

Conjecture 5.1. (See [25].) *For each integer $e \geq 1$, there exists a set of type $(\frac{1}{2}(3^{2e} - 3^e), \frac{1}{2}(3^{2e} + 3^e))$ in $\text{AG}(2, 3^{2e})$.*

Since we just established the existence of Cameron–Liebler line classes of parameters $\frac{3^{2e}-1}{2}$ in $\text{PG}(3, 3^{2e})$ for any $e \geq 1$, it is natural to ask whether Rodgers’ method can be applied to these new line classes to produce affine two-intersection sets in $\text{AG}(2, 3^{2e})$. We have not been able to check whether the tactical decompositions induced by the Cameron–Liebler line classes constructed in Section 4 have the properties predicted by Rodgers. On the other hand, we are able to establish Conjecture 5.1 by a direct and explicit algebraic construction. Before doing so, we describe the model of $\text{AG}(2, 3^{2e})$ we are going to use.

Notation 5.2. Let $q = 3^{2e}$ with $e \geq 1$. Write $N = q^2 + q + 1$, and let ω be a primitive element of $E = \mathbb{F}_{q^3}$. Let w_1 be an element of order N in E . We identify the points of $\text{PG}(2, q)$ with \mathbb{Z}_N as follows: View E as a 3-dimensional vector space over $F = \mathbb{F}_q$, and use E as the underlying vector space of $\text{PG}(2, q)$. We identify the projective point $\langle w_1^i \rangle$ with $i \in \mathbb{Z}_N$, $0 \leq i \leq N - 1$. Let $J = \{i : i \in \mathbb{Z}_N, \text{Tr}(w_1^i) = 0\}$. We obtain the affine plane $\text{AG}(2, q)$ from $\text{PG}(2, q)$ by deleting J . The points and lines of $\text{AG}(2, q)$ are given below:

- (1) the points: $\mathbb{Z}_N \setminus J$,
- (2) the lines: $l_i = \{j \in \mathbb{Z}_N \setminus J : \text{Tr}(w_1^{i+j}) = 0\}$, $1 \leq i \leq N - 1$.

The following is our main theorem of this section.

Theorem 5.3. *With the notation as above, let $w_0 = \omega^N \in F$ and $C_i^{(4,q)} = w_0^i \langle w_0^4 \rangle$, $0 \leq i \leq 3$. Then $\mathcal{K} := \{k \in \mathbb{Z}_N : \text{Tr}(w_1^k) \in C_0^{(4,q)} \cup C_1^{(4,q)}\} \subseteq \mathbb{Z}_N \setminus J$ is a set of type $(\frac{1}{2}(3^{2e} - 3^e), \frac{1}{2}(3^{2e} + 3^e))$ in $\text{AG}(2, 3^{2e})$.*

We divide the proof into a series of lemmas. The following lemma reduces the problem to the computation of the modulus of a certain exponential sum.

Lemma 5.4. *Let γ be any fixed element of $E \setminus F$, and let χ_4 be a multiplicative character of E^* of order 4 such that $\chi_4(\omega^N) = i$, where $i = \sqrt{-1}$. Define $a = \bar{c} = \frac{1-i}{4}$ and*

$$H_{\gamma,j} := \sum_{x \in F} \chi_4^j(1 + \gamma x), \quad j = 1, 3. \tag{5.1}$$

Then the size of the set $\{i \in \mathbb{Z}_N : \text{Tr}(w_1^i) \in C_0^{(4,q)} \cup C_1^{(4,q)}, \text{Tr}(\gamma w_1^i) = 0\}$ is equal to

$$\frac{q}{2} + aH_{\gamma,1} + cH_{\gamma,3}. \tag{5.2}$$

Proof. The size of the set $\{i \in \mathbb{Z}_N : \text{Tr}(w_1^i) \in C_0^{(4,q)} \cup C_1^{(4,q)}, \text{Tr}(\gamma w_1^i) = 0\}$ is given by

$$\begin{aligned} & \frac{1}{q^2} \sum_{e,f \in F} \sum_{y \in C_0^{(q-1,q^3)}} \sum_{x \in C_0^{(4,q)} \cup C_1^{(4,q)}} \psi_E(ey)\psi_F(-ex)\psi_E(f\gamma y) \\ &= \frac{1}{q^2} \sum_{e \in F^*} \sum_{\lambda \in F} \sum_{y \in C_0^{(q-1,q^3)}} \sum_{x \in C_0^{(4,q)} \cup C_1^{(4,q)}} \psi_E(ye(1 + \lambda\gamma))\psi_F(-ex) \end{aligned} \tag{5.3}$$

$$+ \frac{1}{q^2} \sum_{f \in F} \sum_{y \in C_0^{(q-1,q^3)}} \sum_{x \in C_0^{(4,q)} \cup C_1^{(4,q)}} \psi_E(yf\gamma). \tag{5.4}$$

The sum in (5.4) is equal to

$$S_2 := \frac{1}{q^2} \sum_{z \in E^*} \sum_{x \in C_0^{(4,q)} \cup C_1^{(4,q)}} \psi_E(z\gamma) + \frac{N(q-1)}{2q^2} = \frac{q^2 - 1}{2q}.$$

Let χ_{q-1} be a multiplicative character of order $q - 1$ of E and χ'_{q-1} be its restriction to F . Write $\chi_4 = \chi_{q-1}^{\frac{q-1}{4}}$ and $\chi'_4 = \chi'_{q-1}{}^{\frac{q-1}{4}}$, which are multiplicative characters of order 4 of E and F respectively. Here, we can assume that $\chi_4(\omega^N) = i$. Then, by orthogonality of characters, the sum (5.3), denoted by S_1 , is computed as follows:

$$S_1 = \frac{1}{4q^2(q-1)} \sum_{i=0}^{q-2} \sum_{j=0}^3 G(\chi_{q-1}^{-i})G(\chi_4^j) \sum_{a \in F^*} \sum_{\lambda \in F} \sum_{h=0,1} \chi_{q-1}^i(a(1 + \lambda\gamma))\chi_4'^{-j}(-aw_0^h)$$

$$\begin{aligned}
 &= \frac{1}{4q^2(q-1)} \sum_{i=0}^{q-2} \sum_{j=0}^3 G(\chi_{q-1}^{-i})G(\chi_4^j) \sum_{\lambda \in F} \sum_{h=0,1} \chi_{q-1}^i(1+\lambda\gamma)\chi_4^{\prime-j}(-w_0^h) \\
 &\quad \times \sum_{a \in F^*} \chi_{q-1}^{i-\frac{q-1}{4}j}(a). \tag{5.5}
 \end{aligned}$$

Here $\sum_{a \in F^*} \chi_{q-1}^{i-\frac{q-1}{4}j}(a) = q-1$ or 0 according as $i \equiv \frac{q-1}{4}j \pmod{q-1}$ or not. Hence we have

$$S_1 = \frac{1}{4q^2} \sum_{j=0}^3 G(\chi_4^{-j})G(\chi_4^j) \sum_{\lambda \in F} \sum_{h=0,1} \chi_4^j(1+\lambda\gamma)\chi_4^{\prime-j}(-w_0^h).$$

Noting that $G(\chi_4^{-j})G(\chi_4^j) = q^2$ for $j = 1, 2, 3$, cf. [2, Theorem 11.6.3], and $\chi_4^{\prime}(-1) = 1$, we can rewrite the above as

$$\frac{1}{4} \sum_{j=0}^3 \sum_{\lambda \in F} \sum_{h=0,1} \chi_4^j(1+\lambda\gamma)\chi_4^{\prime-j}(w_0^h) + \frac{1-q^2}{2q}.$$

Therefore, the size of the set $\{i \in \mathbb{Z}_N : \text{Tr}(w_1^i) \in C_0^{(4,q)} \cup C_1^{(4,q)}, \text{Tr}(\gamma w_1^i) = 0\}$ is equal to

$$\begin{aligned}
 S_1 + S_2 &= \frac{1}{4} \sum_{j=0}^3 \sum_{\lambda \in F} \sum_{h=0,1} \chi_4^j(1+\lambda\gamma)\chi_4^{-j}(\omega^{Nh}) \\
 &= \frac{1}{4} \sum_{\lambda \in F} \sum_{h=0,1} 1 + \frac{1}{4} \left(\sum_{h=0,1} \chi_4^2(\omega^{Nh}) \right) \sum_{\lambda \in F} \chi_4^2(1+\lambda\gamma) \\
 &\quad + \frac{1}{4} \left(\sum_{h=0,1} \chi_4^3(\omega^{Nh}) \right) \sum_{\lambda \in F} \chi_4(1+\lambda\gamma) + \frac{1}{4} \left(\sum_{h=0,1} \chi_4(\omega^{Nh}) \right) \sum_{\lambda \in F} \chi_4^3(1+\lambda\gamma) \\
 &= \frac{q}{2} + aH_{\gamma,1} + cH_{\gamma,3}.
 \end{aligned}$$

This completes the proof of the lemma. \square

Now, the exponential sum $H_{\gamma,j}$ can be transformed to an exponential sum over F as follows: Since $\chi_4(z) = \chi_4^3(\text{Norm}(z))$ for $z \in E$, where Norm is the norm from E to F , we have

$$\begin{aligned}
 H_{\gamma,j} &= \chi_4^j(\gamma) \sum_{x \in F} \chi_4^j(\gamma^{-1} + x) \\
 &= \chi_4^j(\gamma) \sum_{x \in F} \chi_4^{-j}(\text{Norm}(\gamma^{-1} + x)) \\
 &= \chi_4^j(\gamma) \sum_{x \in F} \chi_4^{-j}(x^3 + \text{Tr}(\gamma^{-1})x^2 + \text{Tr}(\gamma^{-1-q})x + \text{Norm}(\gamma^{-1})). \tag{5.6}
 \end{aligned}$$

Write

$$f_\gamma(x) := x^3 + \text{Tr}(\gamma^{-1})x^2 + \text{Tr}(\gamma^{-1-q})x + \text{Norm}(\gamma^{-1}).$$

The rest of this section is devoted to showing that the exponential sum $\sum_{x \in F} \chi_4^{-j}(f_\gamma(x))$ has modulus 3^e . Note that if $|\sum_{x \in F} \chi_4^{-j}(f_\gamma(x))| = 3^e$, then $\sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) = \alpha 3^e$, for a fourth root of unity α , since $\mathbb{Z}[i]$ is a unique factorization domain and 3 is a prime in $\mathbb{Z}[i]$; it follows from Lemma 5.4 that the intersection sizes of \mathcal{K} with the lines of $\text{AG}(2, 3^{2e})$ are $\frac{3^{2e}+3^e}{2}$ or $\frac{3^{2e}-3^e}{2}$.

Therefore, we are interested in the modulus of the exponential sum $\sum_{x \in F} \chi_4^{-j}(f_\gamma(x))$. The well known theorem of Weil on multiplicative character sums implies that $|\sum_{x \in F} \chi_4^{-j}(f_\gamma(x))| \leq 2 \cdot 3^e$, cf. [18, Theorem 5.41], which is useful but not enough.

Lemma 5.5. *For $j = 1$ or 3 , it holds that*

$$\begin{aligned} \sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) &= \frac{-1}{3^e} \sum_{a \in F^*} \chi_4^j(a) \sum_{x \in F} \psi(a \text{Tr}(\gamma^{-1})x^2 + (a \text{Tr}(\gamma^{-1-q}) + a^{1/3})x \\ &\quad + a \text{Norm}(\gamma^{-1})). \end{aligned} \tag{5.7}$$

Proof. By orthogonality of characters,

$$\sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) = \frac{G(\chi_4^{-j})}{3^{2e}} \sum_{a \in F^*} \chi_4^j(a) \sum_{x \in F} \psi(a f_\gamma(x)). \tag{5.8}$$

Noting that $G(\chi_4^{-j}) = -3^e$, cf. [2, Theorem 11.6.3], we have

$$\begin{aligned} \sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) &= \frac{-1}{3^e} \sum_{a \in F^*} \chi_4^j(a) \sum_{x \in F} \psi(a(x^3 + \text{Tr}(\gamma^{-1})x^2 + \text{Tr}(\gamma^{-1-q})x + \text{Norm}(\gamma^{-1}))) \\ &= \frac{-1}{3^e} \sum_{a \in F^*} \chi_4^j(a) \sum_{x \in F} \psi(a \text{Tr}(\gamma^{-1})x^2 + (a \text{Tr}(\gamma^{-1-q}) + a^{1/3})x \\ &\quad + a \text{Norm}(\gamma^{-1}))). \end{aligned}$$

The proof of the lemma is complete. \square

First we consider the case where $\text{Tr}(\gamma^{-1}) = 0$ in Lemma 5.5.

Lemma 5.6. *Let $\gamma \in E \setminus F$ be such that $\text{Tr}(\gamma) = 0$. Then $\text{Tr}(\gamma^{1+q})$ is a nonzero square of F .*

Proof. By Hilbert’s Theorem 90 [18, Theorem 2.24], there exists $y \in E$ such that $\gamma = y - y^q$. We can directly compute that $\text{Tr}(\gamma^{1+q}) = \text{Tr}(y^{1+q}) - \text{Tr}(y^2)$ and $\text{Tr}(y)^2 =$

$\text{Tr}(y^2) + 2\text{Tr}(y^{1+q})$. It follows that $\text{Tr}(\gamma^{1+q}) = -\text{Tr}(y)^2$. Since $q \equiv 1 \pmod{4}$, -1 is a square, and hence $\text{Tr}(\gamma^{1+q})$ is a square of F .

Next we show that $\text{Tr}(\gamma^{1+q}) \neq 0$. The conic $\{\langle x \rangle : \text{Tr}(x^{1+q}) = 0\}$ in $\text{PG}(2, q)$ contains the point $\langle 1 \rangle$, and the tangent line through $\langle 1 \rangle$ is $\{\langle x \rangle : \text{Tr}(x) = 0\}$. Since $\text{Tr}(\gamma) = 0$, the point $\langle \gamma \rangle$ lies on this tangent line, and thus $\text{Tr}(\gamma^{1+q}) \neq 0$. \square

Proposition 5.7. *Let γ be an element of $E \setminus F$ such that $\text{Tr}(\gamma^{-1}) = 0$. Then, for $j = 1$ or 3 , we have*

$$\left| \sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) \right| = 3^e.$$

Proof. By Lemma 5.5, we need to show that

$$\left| \sum_{a \in F^*} \chi_4^j(a) \sum_{x \in F} \psi((a \text{Tr}(\gamma^{-1-q}) + a^{1/3})x + a \text{Norm}(\gamma^{-1})) \right| = 3^{2e}.$$

Since $\text{Tr}(\gamma^{-1-q})$ is a nonzero square of F by Lemma 5.6 and -1 is a square in F (in fact, -1 is a fourth power in F), there is an element $t \in F$ such that $t^2 = -\text{Tr}(\gamma^{-1-q})$. Then $a \text{Tr}(\gamma^{-1-q}) + a^{1/3} = 0$ if and only if $a = 0, \pm t^{-3}$. Hence

$$\begin{aligned} & \left| \sum_{a \in F^*} \chi_4^j(a) \sum_{x \in F} \psi((a \text{Tr}(\gamma^{-1-q}) + a^{1/3})x + a \text{Norm}(\gamma^{-1})) \right| \\ &= 3^{2e} |\psi(t^{-3} \text{Norm}(\gamma^{-1})) + \psi(-t^{-3} \text{Norm}(\gamma^{-1}))|. \end{aligned}$$

Noting that

$$\begin{aligned} & \left\{ t^3 \left(\frac{x^3}{t^3} - \frac{x}{t} + \frac{\text{Norm}(\gamma^{-1})}{t^3} \right) : x \in F \right\} \\ &= \left\{ y : \text{Tr}_{q/3} \left(\frac{y}{t^3} \right) = \text{Tr}_{q/3} \left(\frac{\text{Norm}(\gamma^{-1})}{t^3} \right) \right\}, \end{aligned} \tag{5.9}$$

we have $\text{Tr}_{q/3}(t^{-3} \text{Norm}(\gamma^{-1})) \neq 0$; otherwise the set (5.9) contains zero, and $f_\gamma(x) = 0$ for some $x \in F$, which is impossible since $f_\gamma(x) = \text{Norm}(\gamma^{-1} + x)$. Hence, we have $\psi(t^{-3} \text{Norm}(\gamma^{-1})) + \psi(-t^{-3} \text{Norm}(\gamma^{-1})) = -1$, which completes the proof. \square

Next, we consider the case where $\text{Tr}(\gamma^{-1}) \neq 0$ in Lemma 5.5.

Lemma 5.8. *Let γ be an element of $E \setminus F$ such that $\text{Tr}(\gamma^{-1}) \neq 0$. Then, for $j = 1$ or 3 , there exists an element $z \in F$ such that*

$$\left| \sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) \right| = \left| \sum_{a \in F^*} \chi_4^j(a) \psi(za + a^{-1}) \right|. \tag{5.10}$$

Proof. If $a \operatorname{Tr}(\gamma^{-1}) \neq 0$, by [18, Theorem 5.33], we have

$$\begin{aligned} & \sum_{x \in F} \psi(a \operatorname{Tr}(\gamma^{-1})x^2 + (a \operatorname{Tr}(\gamma^{-1-q}) + a^{1/3})x + a \operatorname{Norm}(\gamma^{-1})) \\ &= G(\eta')\psi(a \operatorname{Norm}(\gamma^{-1}) - (a \operatorname{Tr}(\gamma^{-1-q}) + a^{1/3})^2 a^{-1} \operatorname{Tr}(\gamma^{-1})^{-1})\eta'(a \operatorname{Tr}(\gamma^{-1})), \end{aligned}$$

where η' is the quadratic character of F . Write $a_0 = \operatorname{Norm}(\gamma^{-1})^{1/3}$, $a_1 = \operatorname{Tr}(\gamma^{-1-q})$, and $a_2 = \operatorname{Tr}(\gamma^{-1})^{-1}$. Then, by Lemma 5.5 and $G(\eta') = \pm 3^e$ (cf. [18, Theorem 5.15]), we obtain

$$\begin{aligned} & \left| \sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) \right| \\ &= \left| \sum_{a \in F^*} \chi_4^j(a)\psi(a \operatorname{Norm}(\gamma^{-1}) - (a \operatorname{Tr}(\gamma^{-1-q}) + a^{1/3})^2 a^{-1} \operatorname{Tr}(\gamma^{-1})^{-1})\eta'(a \operatorname{Tr}(\gamma^{-1})) \right| \\ &= \left| \sum_{a \in F^*} \chi_4^{3j}\eta'(a)\psi(a^3 \operatorname{Norm}(\gamma^{-1}) - (a^3 a_1 + a)^2 a^{-3} a_2) \right| \\ &= \left| \sum_{a \in F^*} \chi_4^j(a)\psi(a_0 a - a_2 a^{-3}(a^6 a_1^2 - a^4 a_1 + a^2)) \right| \\ &= \left| \sum_{a \in F^*} \chi_4^j(a)\psi(ya - a_2 a^{-1}) \right| = \left| \sum_{a \in F^*} \chi_4^j(a)\psi(za + a^{-1}) \right| \end{aligned}$$

where $y = a_0 - (a_1^2 a_2)^{1/3} + a_1 a_2$ and $z = -y a_2$. \square

The exponential sum $K_{j,z} := \sum_{a \in F^*} \chi_4^j(a)\psi(za + a^{-1})$ appearing in the right-hand side of Eq. (5.10) is a *generalized Kloosterman sum* [18, p. 265]. It is clear that $\sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) = \alpha K_{j,z}$ for a fourth root of unity α by the proof above. If $z = 0$ in Eq. (5.10), then the sum $K_{j,z}$ is just a Gauss sum, and hence $\sum_{x \in F} \chi_4^{-j}(f_\gamma(x))$ has modulus 3^e .

Now, we assume that $z \neq 0$. Again by orthogonality of characters, the sum $K_{j,z}$ can be expressed in terms of Gauss sums as follows:

$$K_{j,z} = \frac{1}{(q-1)^2} \sum_{h,i=0}^{q-1} G(\chi^{-h})G(\chi^{-i}) \sum_{a \in F^*} \chi_4^j(a)\chi^h(za)\chi^{-i}(a),$$

where χ is a multiplicative character of order $q-1$ of F . Here, we can assume that $\chi^{\frac{q-1}{4}} = \chi_4$. Since the inner sum $\sum_{a \in F^*} \chi_4^j(a)\chi^h(a)\chi^{-i}(a) = q-1$ or 0 according as $i \equiv h + \frac{q-1}{4}j \pmod{q-1}$ or not, we see that

$$K_{j,z} = \frac{1}{q-1} \sum_{h=0}^{q-1} G(\chi^{-h})G(\chi^{-h-\frac{q-1}{4}j})\chi^h(z).$$

In the following lemma, we show that $3^e \mid G(\chi^{-h})G(\chi^{-h-\frac{q-1}{4}j})$ for every $h = 0, 1, \dots, q - 2$, which implies that 3^e divides $K_{j,z}$.

Lemma 5.9. *For any $h = 0, 1, \dots, q - 2$ and $j = 1$ or 3 , we have*

$$3^e \mid G(\chi^{-h})G(\chi^{-h-\frac{q-1}{4}j}),$$

where χ is a multiplicative character of order $q - 1$ of F .

Proof. We will only prove the lemma in the case where $j = 1$. The case where $j = 3$ is similar. If $h = 0$, $G(\chi^{-h})G(\chi^{-h-\frac{q-1}{4}}) = G(\chi^{-\frac{q-1}{4}}) = -3^e$ by [2, Theorem 11.6.3]. Similarly, if $h = \frac{3(q-1)}{4}$, then $G(\chi^{-h})G(\chi^{-h-\frac{q-1}{4}}) = G(\chi^{-\frac{3(q-1)}{4}}) = -3^e$. Thus, we will assume that $h \neq 0$ or $\frac{3(q-1)}{4}$ below.

By Theorem 2.5, it is enough to show that

$$s(h) + s(h + \frac{q-1}{4}) \geq 2e$$

for all $h = 1, \dots, q - 2$, where $s(x)$ is the sum of the 3-adic digits of the reduction of x modulo $q - 1$.

Define $a \equiv h + \frac{q-1}{4} \pmod{q-1}$. For any $x \in \mathbb{Z}_{q-1}$, $x \neq 0$, write $x = \sum_{i=0}^{2e-1} x_i 3^i = x_{2e-1}x_{2e-2} \cdots x_0$ with $x_i \in \{0, 1, 2\}$, where the subscripts are taken modulo $2e$. Note that $\frac{q-1}{4} = \sum_{i=0}^{e-1} 2 \cdot 3^{2i} = 0202 \cdots 02$. We now use the modular p -ary add-with-carry algorithm described in [17, Theorem 4.1], which says that there is a unique carry sequence $c = c_{2e-1}c_{2e-2} \cdots c_0$ with $c_i \in \{0, 1\}$ such that for all $0 \leq i \leq 2e - 1$

$$a_i + 3c_i = c_{i-1} + h_i + (1 + (-1)^i). \tag{5.11}$$

It follows that

$$s(h) + s(h + \frac{q-1}{4}) = \sum_{i=0}^{2e-1} h_i + \sum_{i=0}^{2e-1} a_i = 2 \sum_{i=0}^{2e-1} h_i - 2 \sum_{i=0}^{2e-1} c_i + 2e.$$

Thus, if $\sum_{i=0}^{2e-1} h_i \geq \sum_{i=0}^{2e-1} c_i$ is shown, then we obtain the assertion of this lemma. We now prove a stronger inequality, namely, $h_{2j+1} + h_{2j+2} \geq c_{2j+1} + c_{2j+2}$ for $0 \leq j \leq e - 1$, from which it follows that $\sum_{i=0}^{2e-1} h_i \geq \sum_{i=0}^{2e-1} c_i$.

If either of h_{2j+1} or h_{2j+2} is greater than or equal to 2, then the stronger inequality clearly holds since $c_i \in \{0, 1\}$. So we assume that both h_{2j+1} and h_{2j+2} are less than 2.

By Eq. (5.11), we have

$$a_{2j+1} + 3c_{2j+1} = c_{2j} + h_{2j+1} \quad \text{and} \quad a_{2j+2} + 3c_{2j+2} = c_{2j+1} + h_{2j+2} + 2.$$

If $c_{2j+1} = 1$, then $a_{2j+1} = 0$, $c_{2j} = 1$, and $h_{2j+1} = 2$, which implies that $h_{2j+1} + h_{2j+2} \geq c_{2j+1} + c_{2j+2}$. If $c_{2j+1} = 0$, then $a_{2j+2} + 3c_{2j+2} = h_{2j+2} + 2$. In the case where $h_{2j+2} = 0$,

we have $a_{2j+2} = 2$ and $c_{2j+2} = 0$. In the case where $h_{2j+2} = 1$, we have $a_{2j+2} = 0$ and $c_{2j+2} = 1$. In both cases, we have $h_{2j+1} + h_{2j+2} \geq c_{2j+1} + c_{2j+2}$. The proof of the lemma is complete. \square

Now recall that $\sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) = \alpha K_{j,z}$ for a fourth root unity α . Since now we have shown that 3^e divides $K_{j,z}$, we can write $\sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) = 3^e(a + bi)$ for some $a, b \in \mathbb{Z}$. Since $|\sum_{x \in F} \chi_4^{-j}(f_\gamma(x))| \leq 2 \cdot 3^e$, we see that (a, b) is equal to one of

$$(a, b) = (0, 0), (0, \pm 1), (0, \pm 2), (\pm 1, \pm 1), (\pm 1, 0), (\pm 2, 0).$$

If $(a, b) = (0, 0), (0, \pm 2), (\pm 1, \pm 1)$, or $(\pm 2, 0)$, then the intersection size stated in (5.2) is not integral, a contradiction. Therefore, it must be that $(a, b) = (0, \pm 1)$ or $(\pm 1, 0)$, i.e., $|\sum_{x \in F} \chi_4^{-j}(f_\gamma(x))| = 3^e$. Summing up, we have proved the following.

Proposition 5.10. *Let γ be an element of $E \setminus F$ such that $\text{Tr}(\gamma^{-1}) \neq 0$. Then, for $j = 1$ or 3 , it holds that*

$$\left| \sum_{x \in F} \chi_4^{-j}(f_\gamma(x)) \right| = 3^e.$$

By Lemma 5.4 together with Propositions 5.7 and 5.10, we obtain the assertion of Theorem 5.3.

Remark 5.11. It is well known (cf. [5]) that projective two-intersection sets are equivalent to certain strongly regular Cayley graphs (hence certain two-class association schemes). It is natural to ask what combinatorial objects are behind affine two-intersection sets. We give an answer in this remark. The notation here is almost the same as that in Notation 5.2. The only difference is that we simply identify the points of $\text{PG}(2, q)$ with $i \in \mathbb{Z}_N$, and define $J = \{j : j \in \mathbb{Z}_N, \text{Tr}(\omega^j) = 0\}$, and $l_i = \{j \in \mathbb{Z}_N \setminus J : \text{Tr}(\omega^{i+j}) = 0\}$, $1 \leq i \leq N - 1$, where Tr is the relative trace from E to F . Assume that $X \subseteq \mathbb{Z}_N \setminus J$ is a set of type (m, n) in $\text{AG}(2, q)$. Then there exists a set $Y \subseteq \mathbb{Z}_N \setminus \{0\}$ such that

$$|X \cap l_i| = \begin{cases} m, & \text{if } i \in Y, \\ n, & \text{if } i \in \mathbb{Z}_N \setminus (Y \cup \{0\}). \end{cases} \tag{5.12}$$

Define

$$D_0 := \{0\}, D_1 := \bigcup_{i \in J} C_i^{(N, q^3)}, D_2 := \bigcup_{i \in X} C_i^{(N, q^3)}, D_3 := \bigcup_{i \in \mathbb{Z}_N \setminus (J \cup X)} C_i^{(N, q^3)}.$$

Let ψ be the canonical additive character of E . Then, by (5.12), the values of $\psi(\omega^a D_2) := \sum_{x \in D_2} \psi(\omega^a x)$, $a = 0, 1, \dots, N - 1$, can be computed as follows:

$$\begin{aligned} \sum_{x \in D_2} \psi(\omega^a x) &= \sum_{i \in X} \sum_{\lambda \in \mathbb{F}_q^*} \psi(\omega^{a+i} \lambda) = -|X| + q(|X \cap (J - a)|) \\ &= -|X| + \begin{cases} qm, & \text{if } a \in Y, \\ qn, & \text{if } a \in \mathbb{Z}_N \setminus (Y \cup \{0\}), \\ 0, & \text{if } a = 0. \end{cases} \end{aligned}$$

On the other hand, it is clear that

$$\sum_{x \in D_1} \psi(\omega^a x) = \begin{cases} q^2 - 1, & \text{if } a = 0, \\ -1, & \text{otherwise.} \end{cases}$$

Furthermore, $\sum_{x \in D_3} \psi(\omega^a x)$, $a = 0, 1, \dots, N - 1$, can be computed as $-1 - \sum_{x \in D_1 \cup D_2} \psi(\omega^a x)$. Thus, for each $i = 1, 2, 3$, the character values $\psi(\omega^a D_i)$, $a = 0, 1, \dots, N - 1$, are constant according as $a = 0$, $a \in Y$, or $a \in \mathbb{Z}_N \setminus (Y \cup \{0\})$. (In the language of association schemes, the Cayley graphs $\text{Cay}(E, D_i)$, $i = 0, 1, 2, 3$, form a three-class association scheme on E . See, e.g., [12, Theorem 10.1].) Thus, as an immediate consequence of our result on affine two-intersection sets, we obtain a three-class association scheme on \mathbb{F}_{36e} . Conversely, starting from a three-class association scheme defined by the three Cayley graphs $\text{Cay}(E, D_i)$, $1 \leq i \leq 3$, we obtain an affine two-intersection set in $\text{AG}(2, q)$. Summing up, we have the following proposition.

Proposition 5.12. *With the notation as above, a subset $X \subseteq \mathbb{Z}_N \setminus J$ is an affine two-intersection set in $\text{AG}(2, q)$ if and only if there exists a subset $Y \subseteq \mathbb{Z}_N \setminus \{0\}$ such that for each $i = 1, 2, 3$, the character values $\psi(\omega^a D_i)$, $a = 0, 1, \dots, N - 1$, are constant according as $a = 0$, $a \in Y$, or $a \in \mathbb{Z}_N \setminus (Y \cup \{0\})$ (or, equivalently, the Cayley graphs $\text{Cay}(E, D_i)$, $i = 0, 1, 2, 3$, form a three-class association scheme on E).*

Note that a result similar to Proposition 5.12 holds for affine two-intersection sets in $\text{AG}(s, q)$ for $s \geq 3$. We omit the detailed statement.

6. Concluding remarks

In this paper, we constructed an infinite family of Cameron–Liebler line classes in $\text{PG}(3, q)$ with parameter $x = \frac{q^2-1}{2}$, where $q \equiv 5$ or $9 \pmod{12}$. Furthermore we constructed the first infinite family of sets of type (m, n) in the affine plane $\text{AG}(2, q)$, where q is an even power of 3. It would be interesting to come up with a general construction of Cameron–Liebler line classes in $\text{PG}(3, q)$ when q is even since there are some known examples in this case in the thesis [25] of Rodgers.

We close this paper by referring the reader to a paper [8] by De Beule, Demeyer, Metsch, and Rodgers. Immediately after we finished a draft of this manuscript, we became aware that De Beule, Demeyer, Metsch and Rodgers [8] also obtained the same result on Cameron–Liebler line classes with parameter $x = \frac{q^2-1}{2}$ at almost the same

time. The approaches for proving the main result are comparable but different enough to justify that we write two separate papers; our approach is more algebraic and the approach taken by De Beule, Demeyer, Metsch and Rodgers is more geometric. The two teams of authors discussed this matter with each other, and decided to submit their papers separately.

References

- [1] S. Ball, A. Blokhuis, F. Mazzocca, Maximal arcs in Desarguesian planes of odd order do not exist, *Combinatorica* 17 (1997) 31–41.
- [2] B. Berndt, R. Evans, K.S. Williams, *Gauss and Jacobi Sums*, Wiley, 1997.
- [3] A.E. Brouwer, W.H. Haemers, *Spectra of Graphs*, Springer, New York, 2012.
- [4] A.A. Bruen, K. Drudge, The construction of Cameron–Liebler line classes in $PG(3, q)$, *Finite Fields Appl.* 5 (1999) 35–45.
- [5] R. Calderbank, W.M. Kantor, The geometry of two-weight codes, *Bull. Lond. Math. Soc.* 18 (1986) 97–122.
- [6] P.J. Cameron, R.A. Liebler, Tactical decompositions and orbits of projective groups, *Linear Algebra Appl.* 46 (1982) 91–102.
- [7] J. De Beule, A. Hallel, L. Storme, A non-existence result on Cameron–Liebler line classes, *J. Combin. Des.* 16 (2008) 342–349.
- [8] J. De Beule, J. Demeyer, K. Metsch, M. Rodgers, A new family of tight sets in $\mathcal{Q}^+(5, q)$, *Des. Codes Cryptogr.* (2015), <http://dx.doi.org/10.1007/s10623-014-0023-9>, in press.
- [9] K. Drudge, On a conjecture of Cameron and Liebler, *European J. Combin.* 20 (1999) 263–269.
- [10] A.L. Gavriluyk, K. Metsch, A modular equality for Cameron–Liebler line classes, *J. Combin. Theory Ser. A* 127 (2014) 224–242.
- [11] A.L. Gavriluyk, I.Y. Mogilyukh, Cameron–Liebler line classes in $PG(n, 4)$, *Des. Codes Cryptogr.* 73 (2014) 969–982.
- [12] C.D. Godsil, *Algebraic Combinatorics*, Chapman & Hall, New York, 1993.
- [13] P. Govaerts, T. Penttila, Cameron–Liebler line classes in $PG(3, 4)$, *Bull. Belg. Math. Soc. Simon Stevin* 12 (2005) 793–804.
- [14] P. Govaerts, L. Storme, On Cameron–Liebler line classes, *Adv. Geom.* 4 (2004) 279–286.
- [15] J.W.P. Hirschfeld, *Projective Geometries over Finite Fields*, Clarendon Press, Oxford, 1979.
- [16] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.
- [17] H. Hollmann, T. Helleseth, A. Kholosha, Z. Wang, Q. Xiang, Proofs of two conjectures on ternary weakly regular bent functions, *IEEE Trans. Inform. Theory* 55 (2009) 5272–5283.
- [18] R. Lidl, H. Niederreiter, *Finite Fields*, Cambridge University Press, 1997.
- [19] S.L. Ma, A survey of partial difference sets, *Des. Codes Cryptogr.* 4 (1994) 221–261.
- [20] K. Metsch, The non-existence of Cameron–Liebler line classes with parameter $2 < x \leq q$, *Bull. Lond. Math. Soc.* 42 (2010) 991–996.
- [21] K. Metsch, An improved bound on the existence of Cameron–Liebler line classes, *J. Combin. Theory Ser. A* 121 (2014) 89–93.
- [22] S.E. Payne, Topics in finite geometry: ovals, ovoids and generalized quadrangles, <http://math.ucdenver.edu/~spayne/classnotes/topics.pdf>, edition of 16 May 2007.
- [23] T. Penttila, Cameron–Liebler line classes in $PG(3, q)$, *Geom. Dedicata* 37 (1991) 245–252.
- [24] T. Penttila, G.F. Royle, Sets of type (m, n) in the affine and projective planes of order nine, *Des. Codes Cryptogr.* 6 (1995) 229–245.
- [25] M. Rodgers, On some new examples of Cameron–Liebler line classes, PhD thesis, University of Colorado, 2012.
- [26] M. Rodgers, Cameron–Liebler line classes, *Des. Codes Cryptogr.* 68 (2013) 33–37.
- [27] K. Yamamoto, M. Yamada, Williamson Hadamard matrices and Gauss sums, *J. Math. Soc. Japan* 37 (1985) 703–717.