ICCM 2007 · Vol. I · 348–358

# Modular Ranks of Geometric Inclusion Matrices

Qing Xiang*

**Abstract**

We survey recent results on $p$-ranks of certain inclusion matrices arising from a finite projective space or a finite symplectic space.

**2000 Mathematics Subject Classification:** 05E20, 20G05, 20C33.
**Keywords and Phrases:** Composition factor, Hamada's formula, Module, Partially ordered set, Rank, Smith normal form, Subset-inclusion matrix, Subspace-inclusion matrix.

## 1 Introduction

Let $X$ be a set of size $v$ and let $r$, $s$ be two integers such that $1 \leqslant s \leqslant r < v$. Let $W^v_{r,s}$ be the $(0,1)$-incidence matrix with rows indexed by the $r$-subsets $R$ of $X$ and columns indexed by the $s$-subsets $S$ of $X$, and with $(R,S)$-entry equal to one if and only if $S \subseteq R$. These subset-inclusion matrices $W^v_{r,s}$ play an important role in applications of linear algebra to combinatorics (see [1], [12]); and in many such applications, it is important to know the rank of $W^v_{r,s}$ over various fields. It is well known that $W^v_{r,s}$ has full rank over the field of rational numbers. That is, $\mathrm{rank}_{\mathbb{Q}}(W^v_{r,s}) = \min\{\binom{v}{r}, \binom{v}{s}\}$. This result goes back at least to Gottlieb [10]. More recently, Wilson [23] determined the rank of $W^v_{r,s}$ over all finite fields, and also found a diagonal form of $W^v_{r,s}$.

There is a $q$-analog of this problem. Let $V$ be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$, where $q = p^t$, $p$ is a prime. For any integer $r$, $1 \leqslant r \leqslant n$, we denote the set of $r$-dimensional subspaces of $V$ by $\mathcal{L}_r$. For integers $r, s$, $1 \leqslant s \leqslant r \leqslant n$, let $A^{n+1}_{r,s}(q)$ be the (0,1)-incidence matrix with rows indexed by elements $Y$ of $\mathcal{L}_r$ and columns indexed by elements $Z$ of $\mathcal{L}_s$, and with $(Y,Z)$-entry equal to 1 if and only if $Z \subseteq Y$. The matrices $A^{n+1}_{r,s}(q)$ are $q$-analogues of the matrices $W^v_{r,s}$ defined above. We consider the following problem.

**Problem 1.1.** *What is the rank of $A^{n+1}_{r,s}(q)$ over $\mathbb{F}_p$? Moreover, what is the Smith normal form of $A^{n+1}_{r,s}(q)$ as a matrix over $\mathbb{Z}$?*

*Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA. E-mail: xiang@math.udel.edu.

For convenience the rank of $A_{r,s}^{n+1}(q)$ over $\mathbb{F}_p$ will be called the *p-rank* of $A_{r,s}^{n+1}(q)$. The first part of Problem 1.1 appeared in [11] as Problem 10.1. The second part of Problem 1.1 is more general than the first part (hence much more difficult), and is motivated by our recent work on the Smith normal form of $A_{r,1}^{n+1}(q)$ ([5]).

The subspace-inclusion matrices $A_{r,s}^{n+1}(q)$ have been investigated at least since 1960s. Most of the investigations were on the rank of $A_{r,s}^{n+1}(q)$ over fields $K$ of various characteristics. When $K = \mathbb{Q}$, Kantor in [15] showed that the matrix $A_{r,s}^{n+1}(q)$ has full rank under certain natural conditions. When $\mathrm{char}(K) = \ell$, where $\ell$ does not divide $q$, the rank of $A_{r,s}^{n+1}(q)$ over $K$ was given by Frumkin and Yakir [9]. The most interesting case is when $\mathrm{char}(K) = p$. In this case, the problem of finding the rank of $A_{r,s}^{n+1}$ over $K$ is open in general (cf. [11]). However, in the special case where $s = 1$, Hamada [13] gave a complete solution (known as Hamada's formula) to the problem of finding the $p$-rank of $A_{r,1}^{n+1}(q)$. We note that the incidence structure behind $A_{r,1}^{n+1}(q)$ is a classical 2-design ([3, p. 13]). Motivated by the desire to understand the algebraic structure behind Hamada's formula, Bardoe and Sin [2] gave a modern treatment of Hamada's formula from representation-theoretic point of view. Their work paved the way for further research in this area. The Smith normal form of $A_{r,1}^{n+1}(q)$ (as a matrix over $\mathbb{Z}$) was obtained only recently in [5]. The result can be viewed as a partial $q$-analogue of Wilson's work on subset-inclusion matrices, and a full generalization of Hamada's $p$-rank formula to the integers.

More recently, we [6] extended Hamada's work in another direction. Let $V$ be a vector space over $\mathbb{F}_q$ of dimension $2m \geqslant 4$, where $q = p^t$ is a prime power. We equip $V$ with a nonsingular alternating bilinear form $b(\cdot,\cdot)$. Let $\mathcal{I}_r$ denote the set of totally isotropic $r$-dimensional subspaces of $V$ with respect to $b(\cdot,\cdot)$, where $1 \leqslant r \leqslant m$. Let $B_{r,1}^{2m}(q)$ be the incidence matrix of the inclusion relation between $\mathcal{I}_r$ and $\mathcal{I}_1$. We remark that in the case where $m = 2$ and $r = 2$, the incidence structure behind $B_{2,1}^{4}(q)$ is a symplectic generalized quadrangle ([22],[18]). The following problem arises naturally.

**Problem 1.2.** *What is the rank of $B_{r,1}^{2m}(q)$ over $\mathbb{F}_p$? Moreover, what is the Smith normal form of $B_{r,1}^{2m}(q)$ as a matrix over $\mathbb{Z}$?*

For $1 \leqslant s \leqslant r \leqslant m$ we can also define $B_{r,s}^{2m}(q)$ in the same fashion as above. One could of course ask what the $p$-rank and the Smith normal form of $B_{r,s}^{2m}(q)$ are when $1 < s < r$. It seems that these questions are completely out of reach at present since the corresponding questions on $A_{r,s}^{n+1}(q)$ are not answered yet.

The first result on the $p$-rank of $B_{r,1}^{2m}(q)$ was obtained in the case where $m = 2$, $r = 2$ and $q = 2^t$ by Sastry and Sin [19]. These authors used very detailed information about the extensions of simple modules for the symplectic group $\mathrm{Sp}(4, q)$ to obtain the following result.

$$\mathrm{rank}_2(B_{2,1}^4(2^t)) = 1 + \left(\frac{9+\sqrt{17}}{2}\right)^t + \left(\frac{9-\sqrt{17}}{2}\right)^t \tag{1.1}$$

In the case where $q = p$ is an odd prime, de Caen and Moorhouse [4] determined the $p$-rank of $B^4_{2,1}(p)$, which was later generalized in [20], giving the $p$-ranks of $B^{2m}_{r,1}(p)$, where $1 \leqslant r \leqslant m$, $p$ is an odd prime, and $m$ is not necessarily 2. In [6], we obtained an additive formula for the $p$-rank of $B^{2m}_{r,1}(q)$, $p$ odd, which can be viewed as the symplectic analogue of Hamada's formula for the $p$-rank of $A^{n+1}_{r,1}(q)$. As a corollary, we obtain the following closed formula for the $p$-rank of $B^4_{2,1}(p^t)$, $p$ an odd prime.

**Theorem 1.1.** *Let $p$ be an odd prime and let $t \geqslant 1$ be an integer. Then the $p$-rank of $B^4_{2,1}(p^t)$ is equal to*

$$1 + \alpha_1^t + \alpha_2^t,$$

*where*

$$\alpha_1, \alpha_2 = \frac{p(p+1)^2}{4} \pm \frac{p(p+1)(p-1)}{12}\sqrt{17}. \tag{1.2}$$

We remark that in (1.2), if we simply set $p = 2$, then we actually obtain (1.1). This circumstance is somewhat coincidental, however, since the examples in [7] show that for $m = r = 3$, the ranks are not given by the same function of $p$ and $t$ for the even characteristic and odd characteristic cases. The problem of finding the 2-rank of $B^{2m}_{r,1}(2^t)$ is more difficult (see reasons in Section 3.2). Very recently, we [7] gave a solution to this problem, which generalizes the 2-rank formula for $B^4_{2,1}(2^t)$ in (1.1).

As for the second part of Problem 1.2, Lataille [17] found the Smith normal form of $B^{2m}_{r,1}(q)$ when $q = p$ is a prime. The general case of the second part of Problem 1.2 is still unsolved.

## 2 $\mathbb{F}_q^{\mathcal{L}_1}$ as a $\mathrm{GL}(V)$-module, Hamada's formula

Let $k = \mathbb{F}_q$, where $q = p^t$ is a prime power. Let $V$ be an $(n+1)$-dimensional vector space over $k$. For convenience we use $\mathcal{P}$ to denote $\mathcal{L}_1$, the set of 1-dimensional subspaces of $V$, and use $G$ to denote the general linear group $\mathrm{GL}(V)$. We consider $k[V]$ (the space of $k$-valued functions on $V$) and $k[\mathcal{P}]$ (the space of $k$-valued functions on $\mathcal{P}$) as $kG$-modules. Let $r$ be an integer such that $1 < r \leqslant n$. Then each row of $A^{n+1}_{r,1}(q)$ is the characteristic vector in $\mathcal{P}$ of the corresponding $r$-dimensional subspace. Let $\mathcal{C}_r$ be the $k$-span of the rows of $A^{n+1}_{r,1}(q)$. The group $G$ acts naturally on $\mathcal{L}_r$. So $\mathcal{C}_r$ is a $kG$-submodule of $k[\mathcal{P}]$. In [2], among other results, Bardoe and Sin described all $kG$-submodules of $k[\mathcal{P}]$ by using a partially ordered set (see below). Also for any $f \in k[\mathcal{P}]$, they gave a way to identify the submodule $kGf$ generated by $f$. From these general results, Bardoe and Sin [2] could identify the $kG$-submodule $\mathcal{C}_r$ and then deduce Hamada's formula. To explain these results in detail, we first need to give some necessary background information.

Let $k[X_0, X_1, \ldots, X_n]$ denote the polynomial ring over $k$, in $n+1$ indeterminates. Since every function on $V$ is given by a polynomial in the $n+1$ coordinates $x_i$, the map $X_i \mapsto x_i$ defines a surjective $k$-algebra homomorphism $k[X_0, X_1, \ldots, X_n] \to k[V]$, with kernel generated by the elements $X_i^q - X_i$. Furthermore, this map is simply the coordinate description of the following canonical map. The polynomial ring is isomorphic to the symmetric algebra $S(V^*)$ of the dual space of $V$; so we have a natural evaluation map $S(V^*) \to k[V]$. This canonical description makes it clear that the map is equivariant with respect to the natural actions of $G$ on these spaces. A basis of $k[V]$ is obtained by taking monomials in $n+1$ coordinates $x_i$ such that the degree in each variable is at most $q-1$. We will call these the *basis monomials* of $k[V]$. Noting that the functions on $V \setminus \{0\}$ which descend to $\mathcal{P}$ are precisely those which are invariant under scalar multiplication by $k^*$, we obtain from the monomial basis of $k[V]$ a basis of $k[\mathcal{P}]$,

$$\mathcal{M} = \{\prod_{i=0}^{n} x_i^{b_i} \mid 0 \leqslant b_i \leqslant q-1, \sum_i b_i \equiv 0 \pmod{q-1},$$
$$(b_0, \ldots, b_n) \neq (q-1, \ldots, q-1)\}.$$

The elements of $\mathcal{M}$ are called *the basis monomials* of $k[\mathcal{P}]$.

## 2.1 Types and $\mathcal{H}$-types

We now recall the definitions of two $t$-tuples associated with each basis monomial of $k[\mathcal{P}]$. Let

$$f = \prod_{i=0}^{n} x_i^{b_i} = \prod_{j=0}^{t-1}\prod_{i=0}^{n}(x_i^{a_{ij}})^{p^j},$$

be a basis monomial of $k[\mathcal{P}]$, where $b_i = \sum_{j=0}^{t-1} a_{ij}p^j$ and $0 \leqslant a_{ij} \leqslant p-1$. Let $\lambda_j = \sum_{i=0}^{n} a_{ij}$. The $t$-tuple $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{t-1})$ is called the *type* of $f$. The set of all types of basis monomials of $k[\mathcal{P}]$ is denoted by $\boldsymbol{\Lambda}$.

In [2], there is another $t$-tuple associated with each basis monomial of $k[\mathcal{P}]$, which we will call its $\mathcal{H}$-*type.* This tuple will lie in the set $\mathcal{H}[0] = \mathcal{H} \cup \{(0,0,\ldots 0)\}$, where

$$\mathcal{H} = \{\mathbf{s} = (s_0, s_1, \ldots, s_{t-1}) \mid \forall j, 1 \leqslant s_j \leqslant n,\ 0 \leqslant ps_{j+1} - s_j \leqslant (n+1)(p-1)\}.$$

The $\mathcal{H}$-type $\mathbf{s}$ of $f$ is uniquely determined by the type via the equations

$$\lambda_j = ps_{j+1} - s_j, \quad 0 \leqslant j \leqslant t-1,$$

where the subscripts are taken modulo $t$. Moreover, these equations determine a bijection between the set $\boldsymbol{\Lambda}$ of types of elements of $\mathcal{M}$ and the set $\mathcal{H}[0]$. For $\mathbf{s} = (s_0, s_1, \ldots, s_{t-1}) \in \mathcal{H}$, $\mathbf{s}' = (s'_0, s'_1, \ldots, s'_{t-1}) \in \mathcal{H}$ we define $\mathbf{s} \leqslant \mathbf{s}'$ if $s_i \leqslant s'_i$ for all $i$. This relation "$\leqslant$" defines a partial order on $\mathcal{H}$.

## 2.2 Composition factors

The types, or equivalently the $\mathcal{H}$-types parametrize the composition factors of $k[\mathcal{P}]$ in the following sense. The $kG$-module $k[\mathcal{P}]$ is multiplicity-free. We can associate to each $\mathcal{H}$-type $\mathbf{s} \in \mathcal{H}[0]$ a composition factor, which we shall denote by $L(\mathbf{s})$, such that these simple modules are all nonisomorphic. The simple modules $L(\mathbf{s})$ occur as subquotients of $k[\mathcal{P}]$ in the following way. For each $\mathbf{s} \in \mathcal{H}$, we let $Y(\mathbf{s})$ be the subspace spanned by monomials of $\mathcal{H}$-types in $\mathcal{H}_{\mathbf{s}} = \{\mathbf{s}' \in \mathcal{H} \mid \mathbf{s}' \leqslant \mathbf{s}\}$. Then $Y(\mathbf{s})$ is a $kG$-submodule of $k[\mathcal{P}]$ and $Y(\mathbf{s}') \subseteq Y(\mathbf{s})$ if and only if $\mathbf{s}' \leqslant \mathbf{s}$. The $kG$-module $Y(\mathbf{s})$ has a unique maximal submodule $\sum_{\mathbf{s}' \lneqq \mathbf{s}} Y(\mathbf{s}')$, and

$$Y(\mathbf{s})/\sum_{\mathbf{s}' \lneqq \mathbf{s}} Y(\mathbf{s}') \cong L(\mathbf{s}).$$

The isomorphism type of the simple module $L(\mathbf{s})$ is most easily described in terms of the corresponding type $(\lambda_0, \dots, \lambda_{t-1}) \in \mathbf{\Lambda}$. Let $S^{\lambda}$ be the degree $\lambda$ component in the truncated polynomial ring $k[X_0, X_1, \dots, X_n]/(X_i^p; 0 \leqslant i \leqslant n)$. Here $\lambda$ ranges from 0 to $n+1$. Note that the dimension of $S^{\lambda}$ is

$$d_{\lambda} = \sum_{j=0}^{\lfloor \lambda/p \rfloor} (-1)^j \binom{n+1}{j} \binom{n+\lambda-pj}{n}.$$

The simple module $L(\mathbf{s})$ is isomorphic to the twisted tensor product

$$S^{\lambda_0} \otimes (S^{\lambda_1})^{(p)} \otimes \cdots \otimes (S^{\lambda_{t-1}})^{(p^{t-1})}.$$

## 2.3 Submodule structure

The reason for considering $\mathcal{H}$-types is that they allow a simple description of the submodule structure of the $kG$-module $k[\mathcal{P}]$. The space $k[\mathcal{P}]$ has a $kG$-decomposition

$$k[\mathcal{P}] = k1 \oplus Y_{\mathcal{P}},$$

where $Y_{\mathcal{P}}$ is the kernel of the map $k[\mathcal{P}] \to k$, $f \mapsto |\mathcal{P}|^{-1} \sum_{Q \in \mathcal{P}} f(Q)$. The $kG$-module $Y_{\mathcal{P}}$ is an indecomposable module whose composition factors are parametrized by $\mathcal{H}$. Theorem A in [2] states that given any $kG$-submodule of $Y_{\mathcal{P}}$, the set of $\mathcal{H}$-types of its composition factors is an ideal in the partially ordered set $\mathcal{H}$ and that this correspondence is an order isomorphism from the submodule lattice of $Y_{\mathcal{P}}$ to the lattice of ideals in $\mathcal{H}$.

The submodules of $Y_{\mathcal{P}}$ can also be described in terms of basis monomials [2, Theorem B]. Any submodule of of $Y_{\mathcal{P}}$ has a basis consisting of the basis monomials which it contains. Moreover, the $\mathcal{H}$-types of these basis monomials are precisely the $\mathcal{H}$-types of the composition factors of the submodule. Furthermore, in any composition series, the images of the monomials of a fixed $\mathcal{H}$-type form a basis of the composition factor of that $\mathcal{H}$-type. Theorem B in [2] gives us a way to identify the submodule $kGf$ generated by any function $f \in k[\mathcal{P}]$. For $f \in k[\mathcal{P}]$, let $\mathcal{H}_f \subseteq \mathcal{H}[0]$ denote the set of $\mathcal{H}$-types of the basis monomials appearing with nonzero coefficients in the expression for $f$. Then

$$kGf = \sum_{\mathbf{s} \in \mathcal{H}_f} Y(\mathbf{s}).$$

Now Hamada's result on the $p$-rank of $A_{r,1}^{n+1}(q)$ follows easily from the above general results. Here is the proof given in [2]. Let $\mathcal{C}_r$ be the $k$-span of the rows of

$A_{r,1}^{n+1}(q)$. Since $G$ acts transitively on $\mathcal{L}_r$, we see that $\mathcal{C}_r = kG\chi_L$, where $L$ is the $r$-dimensional subspace of $V$ defined by the equations $x_i = 0$, $i = r, r+1, \dots, n$, and $\chi_L$ is the characteristic function of $L$. Note that

$$\begin{aligned}\chi_L &= (1 - x_r^{q-1})(1 - x_{r+1}^{q-1}) \cdots (1 - x_n^{q-1}) \\ &= 1 + f,\end{aligned}$$

where $f = \sum_{\emptyset \neq I \subseteq \{r, r+1, \dots, n\}} (-1)^{|I|} \mathbf{x}_I^{q-1}$, and $\mathbf{x}_I$ stands for $\prod_{i \in I} x_i$. Therefore, we have

$$\mathcal{C}_r = k1 \oplus kGf.$$

For any $I \subseteq \{r, r+1, \dots, n\}$, $0 < |I| < n+1-r$, the $\mathcal{H}$-type of $x_I^{q-1}$ is $(|I|, |I|, \dots, |I|)$, which lies below the $\mathcal{H}$-type $(n+1-r, \dots, n+1-r)$ of the monomial $x_r^{q-1} x_{r+1}^{q-1} \cdots x_n^{q-1}$ in the poset $\mathcal{H}$. Hence

$$kGf = \sum_{i=1}^{n+1-r} Y((i, i, \dots, i)) = Y((n+1-r, \dots, n+1-r)).$$

It follows that

$$\dim_k(\mathcal{C}_r) = 1 + \sum_{\substack{\mathbf{s}=(s_0, s_1, \dots, s_{t-1}) \in \mathcal{H} \\ \mathbf{s} \leqslant (n+1-r, \dots, n+1-r)}} \prod_{j=0}^{t-1} d_{\lambda_j},$$

where $\lambda_j = ps_{j+1} - s_j$ and $d_{\lambda_j}$ are defined in Section 2.2. The above formula is usually referred to as Hamada's formula. Note that the above proof not only gives the dimension of $\mathcal{C}_r$, but also shows that the monomials 1 and $\prod_{j=0}^n x_j^{b_j} \in k[\mathcal{P}]$ of $\mathcal{H}$-type less than or equal to $(n+1-r, \dots, n+1-r)$ together form a basis of $\mathcal{C}_r$.

# 3 $p$-rank of $B_{r,1}^{2m}(q)$

In this section, let $V$ be a finite-dimensional vector space over $k = \mathbb{F}_q$, $q = p^t$. We assume in addition that $V$ has a nonsingular alternating form $b(\cdot, \cdot)$. Then the dimension of $V$ must be an even number, say $2m$. We fix a basis $e_1, e_2, \dots, e_m, f_m, \dots, f_1$ of $V$, with corresponding coordinates $x_1, x_2, \dots, x_m, y_m, \dots, y_1$ so that $b(e_i, f_j) = \delta_{ij}$, $b(e_i, e_j) = 0$, and $b(f_i, f_j) = 0$. Now we have

$$k[V] \cong k[X_1, \dots, X_m, Y_m, \dots, Y_1]/(X_i^q - X_i, Y_i^q - Y_i)_{i=1}^m$$

and the monomial basis of $k[\mathcal{P}]$ is

$$\begin{aligned}\mathcal{M} = \{\prod_{i=1}^m x_i^{a_i} y_i^{b_i} \mid 0 \leqslant a_i, b_i \leqslant q-1, \sum_i (a_i + b_i) \equiv 0 \pmod{q-1}, \\ (a_1, \dots, a_m, b_1, \dots, b_m) \neq (q-1, \dots, q-1)\}.\end{aligned}$$

We are interested in the $p$-rank of the inclusion matrix $B_{r,1}^{2m}(q)$, $1 \leqslant r \leqslant m$, defined in Section 1. It turns out that the case where $p$ is odd and the case where $p = 2$ are very different. We consider the $p$ odd case first.

### 3.1 The Odd Characteristic Case

Throughout this subsection we assume that $p$ is an odd prime. As we saw before the monomial basis $\mathcal{M}$ of $k[\mathcal{P}]$ played a vital role in Section 2. Here we will need a special basis $\mathcal{B}$ of $k[\mathcal{P}]$ which was introduced in [6] and called *the symplectic basis*. (We will not give the detailed definition of $\mathcal{B}$ here since it requires a lot of preparation to give the precise definition. The interested reader should consult [6].) The definition of $\mathcal{B}$ is motivated by considering how the simple $k\,\mathrm{GL}(V)$-modules $S^\lambda$ behave upon restriction to $\mathrm{Sp}(V)$.

It is known from [16], [21] that the simple $k\,\mathrm{GL}(V)$-modules $S^\lambda$, $0 \leqslant \lambda \leqslant 2m(p-1)$, all remain simple as $k\,\mathrm{Sp}(V)$-modules except when $\lambda = m(p-1)$, in which case we have

$$S^{m(p-1)} = S^+ \oplus S^-.$$

Here, $S^+$ and $S^-$ are simple $k\,\mathrm{Sp}(V)$-modules, and

$$\dim_k(S^+) = (d_{(p-1)m} + p^m)/2, \quad \dim_k(S^-) = (d_{(p-1)m} - p^m)/2.$$

The splitting of $S^{m(p-1)}$ also motivated us to define the following new partially ordered set. For $(0,0,\dots,0) \neq \boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, let $\mathbf{s}$ be the corresponding $\mathcal{H}$-type in $\mathcal{H}$. Set $J(\mathbf{s}) = \{j \mid 0 \leqslant j \leqslant t-1,\ \lambda_j = m(p-1)\}$. For any $\mathbf{s}, \mathbf{s}' \in \mathcal{H}$, define $Z(\mathbf{s}, \mathbf{s}') = \{j \mid s'_j = s_j,\ s'_{j+1} = s_{j+1},\ \lambda_j = m(p-1)\}$. Let

$$\mathcal{S} = \{(\mathbf{s}, \varepsilon) \mid \mathbf{s} \in \mathcal{H}, \varepsilon \subseteq J(\mathbf{s})\}.$$

We define $(\mathbf{s}', \varepsilon') \leqslant (\mathbf{s}, \varepsilon)$ if and only if $\mathbf{s}' \leqslant \mathbf{s}$ and $\varepsilon \cap Z(\mathbf{s}', \mathbf{s}) = \varepsilon' \cap Z(\mathbf{s}', \mathbf{s})$. It is not difficult to check that this relation indeed defines a partial order on $\mathcal{S}$.

Since each $\mathrm{GL}(V)$-composition factor of $Y_{\mathcal{P}}$ has the form

$$S^{\lambda_0} \otimes (S^{\lambda_1})^{(p)} \otimes \cdots \otimes (S^{\lambda_{t-1}})^{(p^{t-1})},$$

from the splitting of $S^{m(p-1)}$ into simple $\mathrm{Sp}(V)$-modules, it is clear that the $k\,\mathrm{Sp}(V)$-composition factors of $Y_{\mathcal{P}}$ are given by their types, together with the additional choice of signs for each $j$ with $\lambda_j = m(p-1)$. In terms of $\mathcal{H}$-types, we see that each $\mathcal{H}$-type gives a $k\,\mathrm{GL}(V)$-composition factor of $Y_{\mathcal{P}}$ and then the choice of signs determines the simple $k\,\mathrm{Sp}(V)$-composition factors of this simple $k\,\mathrm{GL}(V)$-module. In this way, the elements of $\mathcal{S}$ label the $k\,\mathrm{Sp}(V)$-composition factors of $Y_{\mathcal{P}}$. It should be noted here that just as each basis monomial in $\mathcal{M}$ has an $\mathcal{H}$-type, each element of the symplectic basis $\mathcal{B}$ has a "signed" $\mathcal{H}$-type $(\mathbf{s}, \varepsilon) \in \mathcal{S}$. (This would be very clear if we had given the explicit definition of $\mathcal{B}$ here.)

For $(\mathbf{s}, \varepsilon) \in \mathcal{S}$, let $Y(\mathbf{s}, \varepsilon)$ be the $k$-subspace spanned by all the elements of $\mathcal{B}$ with signed $\mathcal{H}$-types $(\mathbf{s}', \varepsilon') \leqslant (\mathbf{s}, \varepsilon)$. We prove in [6] that
(i) $Y(\mathbf{s}, \varepsilon)$ is a $k\,\mathrm{Sp}(V)$-submodule of $Y_{\mathcal{P}}$,
(ii) $Y(\mathbf{s}, \varepsilon)$ has a unique maximal submodule $\sum_{(\mathbf{s}', \varepsilon') \lneq (\mathbf{s}, \varepsilon)} Y(\mathbf{s}', \varepsilon')$, and
(iii) $Y(\mathbf{s}, \varepsilon) = k\,\mathrm{Sp}(V) f$, where $f$ is any element of $Y(\mathbf{s}, \varepsilon)$ not in the unique maximal submodule of $Y(\mathbf{s}, \varepsilon)$.

For $1 \leqslant r \leqslant m$, we define $\mathcal{E}_r$ to be the $k$-span of the rows of $B^{2m}_{r,1}(q)$. Since $\mathrm{Sp}(V)$ acts transitively on the set of totally isotropic $r$-dimensional subspaces of

$V$, we again have $\mathcal{E}_r = k\,\mathrm{Sp}(V)\chi_L$, where $\chi_L$ is the characteristic function of a carefully chosen totally isotropic $r$-dimensional subspace of $V$. Using the above general results on $Y(\mathbf{s},\varepsilon)$, we [6] proved the following theorem.

**Theorem 3.1.** *Let $r$ be an integer such that $1 \leqslant r \leqslant m$. Assume that $p$ is odd. We have*:
(i) *If $1 \leqslant r < m$, then* $\dim_k(\mathcal{E}_r) = \dim_k(\mathcal{C}_r)$.
(ii) *If $r = m$, then*

$$\dim_k(\mathcal{E}_r) = 1 + \sum_{\substack{(s_0,\dots,s_{t-1})\in\mathcal{H}\\ \forall j, 1\leqslant s_j\leqslant m}} \prod_{j=0}^{t-1} d_{(s_j,s_{j+1})},$$

*where*

$$d_{(s_j,s_{j+1})} = \begin{cases} \dim_k(S^+) = (d_{m(p-1)} + p^m)/2, & \text{if } s_j = s_{j+1} = m,\\ d_{\lambda_j}, \text{ where } \lambda_j = ps_{j+1} - s_j, & \text{otherwise.}\end{cases}$$

We view the above formula for $\dim_k(\mathcal{E}_r)$ as the symplectic analogue of Hamada's formula. As a corollary, we have

**Corollary 3.2.** *The $p$-rank of $B^{2m}_{m,1}(p^t)$, when $p$ is an odd prime, is given by*

$$\mathrm{rank}_p(B^{2m}_{m,1}(p^t)) = 1 + \mathrm{Trace}(D^t) = 1 + \alpha_1^t + \cdots + \alpha_m^t,$$

*where*

$$D = \begin{pmatrix} d_{(1,1)} & d_{(1,2)} & \cdots & d_{(1,m)}\\ d_{(2,1)} & d_{(2,2)} & \cdots & d_{(2,m)}\\ \vdots & \vdots & \ddots & \vdots\\ d_{(m,1)} & d_{(m,2)} & \cdots & d_{(m,m)}\end{pmatrix},$$

*and $\alpha_1, \alpha_2, \dots, \alpha_m$ are the eigenvalues of $D$.*

Specializing the above corollary to the case where $m = 2$, we obtain Theorem 1.1.

## 3.2 The Even Characteristic Case

As we saw in Section 3.1, when $p$ is odd, each $S^\lambda$ is either simple or semisimple as a $k\,\mathrm{Sp}(V)$-module. In the $p = 2$ case, the $k\,\mathrm{Sp}(V)$-submodule structure of $S^\lambda$ is much richer. The abundance of submodules of $S^\lambda$ in this case causes some difficulty in finding the 2-rank of $B^{2m}_{r,1}(2^t)$.

In characteristic 2 the truncated polynomial algebra

$$k[X_1,\dots,X_m,Y_m,\dots,Y_1]/(X_i^2, Y_i^2; 1 \leqslant i \leqslant m)$$

is an exterior algebra. The scalar extensions of the exterior powers to the algebraic closure $\overline{k}$ are examples of *tilting modules* for the algebraic group $\mathrm{Sp}(V \otimes \overline{k})$, as described in [8]; these have filtrations by *Weyl modules* and their duals. (See [14] for definitions.) For finite field $k$, we define in [7] a filtration

$$\{0\} \subset S_0^\lambda \subseteq \cdots \subseteq S_{\lfloor \frac{\lambda}{2} \rfloor}^\lambda = S^\lambda.$$

of $S^\lambda$ in an elementary way. Using this filtration, we similarly define a new basis of $k[\mathcal{P}]$, again called *the symplectic basis* in [7]. The symplectic basis allows us to construct a basis of $\mathcal{E}_r$, the $k$-span of the rows of $B_{r,1}^{2m}(2^t)$; hence we obtain the 2-rank of $B_{r,1}^{2m}(2^t)$.

**Theorem 3.3.** *Let $r$ be an integer such that $1 \leqslant r \leqslant m$. Let $E$ be the $(2m-r) \times (2m-r)$-matrix whose $(i,j)$-entry is*

$$e_{i,j} = \binom{2m}{2j-i} - \binom{2m}{2j+i+4r-6m-2}.$$

*Then*

$$\mathrm{rank}_2(B_{r,1}^{2m}(2^t)) = 1 + \mathrm{Trace}(E^t).$$

When $m = r = 2$, the matrix $E$ defined in Theorem 3.3 is

$$\begin{pmatrix} 4 & 4 \\ 1 & 5 \end{pmatrix},$$

whose eigenvalues are $\frac{9 \pm \sqrt{17}}{2}$. So from Theorem 3.2, we immediately obtain the result (1.1) of Sastry and Sin [19].

When $m = r = 3$, the matrix $E$ defined in Theorem 3.3 is

$$\begin{pmatrix} 6 & 20 & 6 \\ 1 & 15 & 14 \\ 0 & 6 & 14 \end{pmatrix},$$

whose eigenvalues are $\alpha_1 = 8$, $\alpha_2 = \frac{27}{2} + \frac{\sqrt{473}}{2}$, and $\alpha_3 = \frac{27}{2} - \frac{\sqrt{473}}{2}$. So by Theorem 3.2 the rank formula may be given as

$$\mathrm{rank}_2(B_{3,1}^6(2^t)) = 1 + \mathrm{Trace}(E^t) = 1 + \alpha_1^t + \alpha_2^t + \alpha_3^t.$$

Comparing the above formula with the one obtained from Corollary 3.2 by setting $m = r = 3$, we find that the two formulae are not given by the same function of $p$ and $t$.

**Acknowledgements.** The author would like to thank David Chandler and Peter Sin for numerous discussions on the topics of this survey. This work is partially supported by NSF grant DMS 0701049.

## References

[1] L. Babai, P. Frankl, *Linear algebra methods in combinatorics*, September 1992.

[2] M. Bardoe, P. Sin, The permutation modules for $\mathrm{GL}(n+1, \mathbb{F}_q)$ acting on $\mathrm{P}^n(\mathbb{F}_q)$ and ${\mathbb{F}_q}^{n+1}$, *J. Lond. Math. Soc.* **61** (2000), 58–80.

[3] T. Beth, D. Jungnickel, H. Lenz, *Design Theory.* Vol. I. Second edition. Encyclopedia of Mathematics and its Applications, 78. Cambridge University Press, Cambridge, 1999.

[4] D. de Caen and E. Moorhouse, The $p$-rank of the $\mathrm{Sp}(4, p)$ generalized quadrangle, unpublished notes (1998).

[5] D. B. Chandler, P. Sin, and Q. Xiang, The invariant factors of the incidence matrices of points and linear subspaces in $\mathrm{PG}(n, q)$, *Trans. Amer. Math. Soc.* **358** (2006), 4935–495

[6] D. B. Chandler, P. Sin, and Q. Xiang, The permutation action of finite symplectic groups of odd characteristic on their standard modules, *J. Algebra*, in press.

[7] D. B. Chandler, P. Sin, Q. Xiang, Incidence modules for a symplectic space in characteristic 2, *preprint*.

[8] S. Donkin, On tilting modules and invariants for algebraic groups, in *Finite Dimensional Algebras and related topics* (V. Dlab and L. L. Scott, Eds.), Kluwer Academic Publishers, (1994) 59–77.

[9] A. Frumkin, A. Yakir, Rank of inclusion matrices and modular representation theory, *Israel J. Math.* **71** (1990), 309–320.

[10] D. H. Gottlieb, A certain class of incidence matrices, *Proc. Amer. Math. Soc.* **17** (1966), 1233–1237.

[11] C. D. Godsil, Problems in algebraic combinatorics, *Electon. J. Combin.* 2 (1995), Feature 1, approx. 20 pp. (electronic).

[12] C. D. Godsil, *Tools from linear algebra*. With an appendix by L. Lovász. Handbook of Combinatorics, Vol. 1,2, 1705–1748, Elsevier, Amsterdam, 1995.

[13] N. Hamada, The rank of the incidence matrix of points and $d$-flats in finite geometries, *J. Sci. Hiroshima Univ. Ser. A-I* **32** (1968), 381-396.

[14] J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, London, 1987.

[15] W. M. Kantor, On incidence matrices of finite projective and affine spaces, *Math. Z.* **124** (1972), 315–318.

[16] J. Lahtonen, On the submodules and composition factors of certain induced modules for groups of type $C_n$, *J. Algebra* **140** (1991), 415–425.

[17] J. M. Lataille, The elementary divisors of incidence matrices between certain subspaces of a finite symplectic space, *J. Algebra* **268** (2003), 444–462.

[18] S. E. Payne, J. A. Thas, *Finite Generalized Quadrangles,* Research Notes in Mathematics, 110, Pitman (Advanced Publishing Program), Boston, MA, 1984.

[19] N. S. N. Sastry and P. Sin, The code of a regular generalized quadrangle of even order, *Proc. Symposia in Pure Mathematics* **63** (1998), 485–496.

[20] P. Sin, The permutation representation of $\mathrm{Sp}(2m, \mathbb{F}_p)$ acting on the vectors

of its standard module, *J. Algebra* **241** (2001), 578–591.

[21] I. D. Suprunenko and A. E. Zalesskii, Reduced symmetric powers of natural realizations of the groups $\mathrm{SL}_m(P)$ and $\mathrm{Sp}_m(P)$ and their restrictions to subgroups, *Siberian Mathematical Journal* (4) **31** (1990), 33–46.

[22] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, $P$ubl. Math. I. H. E. S. **2** (1959), 14 – 60.

[23] R. M. Wilson, A diagonal form for the incidence matrices of $t$-subsets vs. $k$-subsets, *Europ. J. Combin.* **11** (1990), 609–615.