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Association schemes from ovoids in $PG(3, q)$

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Abstract. We discuss three proofs of a conjecture of De Caen and Van Dam on the existence of some four-class association scheme on the set of unordered pairs of distinct points of the projective line $PG(1, 4^f)$, where $f \geq 2$ is an integer. Our emphasis is on the proof using inversive planes and ovoids in $PG(3, q)$.

1. Introduction

Let Ω be a finite set. A (*symmetric*) *association scheme* with s classes on Ω is a partition of $\Omega \times \Omega$ into sets $R_0, R_1, \dots, R_s$ (relations, or associate classes) such that

1. $R_0 = \{(\omega, \omega) : \omega \in \Omega\}$ (the diagonal relation);
2. R_i is symmetric for $i = 1, 2, \dots, s$;
3. for all i, j, k in $\{0, 1, 2, \dots, s\}$ there is an integer p_{ij}^k such that, for all $(\alpha, \beta) \in R_k$,

$$|\{\gamma \in \Omega : (\alpha, \gamma) \in R_i \text{ and } (\gamma, \beta) \in R_j\}| = p_{ij}^k.$$

Elements α and β of Ω are called i-th associates if $(\alpha, \beta) \in R_i$. The numbers p_{ij}^k, $0 \leq k, i, j \leq s$, are called the *intersection parameters* of the scheme. That p_{ii}^0 exists means that there is a constant number of i-th associates of any element of Ω, which is usually denoted by n_i. We have

$$p_{ii}^0 = n_i, \text{ and } p_{ij}^0 = 0 \text{ if } i \neq j$$

and

$$n_0 = 1, \ n_0 + n_1 + \cdots + n_s = |\Omega|.$$

The numbers $n_0, n_1, \dots, n_s$ are called the *valencies* (*or degrees*) of the scheme. One of the classical association schemes is the triangular association scheme. Let X be an n-set, and let Ω be the set of all 2-subsets of X. We define the following relations on Ω.

$$R_1 = \{(\alpha, \beta) \in \Omega \times \Omega : |\alpha \cap \beta| = 1\},$$
$$R_2 = \{(\alpha, \beta) \in \Omega \times \Omega : \alpha \cap \beta = \emptyset\}.$$

Then it is easy to verify that R_1, R_2 together with the diagonal relation R_0 form a two-class association scheme on Ω. This scheme is called the triangular association scheme and is usually denoted by $\mathrm{T}(n)$. We remark that $\mathrm{T}(n)$ is a special case of the Johnson schemes.

De Caen and Van Dam [3] considered the following fission scheme of $\mathrm{T}(q+1)$ where q is a prime power. Let $X = PG(1, q)$, and let $\Omega = \binom{X}{2}$, the set of 2-subsets of X. Since the group $PGL(2, q)$ acts (as Möbius transformations) on the projective line $PG(1, q)$, there is a natural induced action of $PGL(2, q)$ on Ω. Since the action of $PGL(2, q)$ on $PG(1, q)$ is sharply triply transitive, one can show that $PGL(2, q)$ acts generously transitively on Ω (cf. [3]), that is, the orbits of $PGL(2, q)$ on $\Omega \times \Omega$ are symmetric, hence they may be taken as the relations of an association scheme on Ω. This association scheme will be denoted by $\mathrm{FT}(q+1)$, and it is a fission scheme of $\mathrm{T}(q+1)$. The relations of $\mathrm{FT}(q+1)$ can be described as follows.The *cross-ratio*

$$\rho(a, b; c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

is a complete invariant for ordered quadruples (a, b, c, d) of distinct points of $PG(1, q)$, that is, a quadruple can be mapped by an element of $PGL(2, q)$ to another quadruple if and only if they have the same cross-ratio. As a consequence, the scheme $\mathrm{FT}(q+1)$ has relations

1. $R_0 = \{(\omega, \omega) : \omega \in \Omega\}$ (the diagonal relation),
2. $R_1 = \{(\{a, b\}, \{c, d\}) : |\{a, b\} \cap \{c, d\}| = 1\}$, and
3. $R_{s,s^{-1}} = \{(\{a, b\}, \{c, d\}) : \rho(a, b; c, d) = s \text{ or } s^{-1}\}$ for each pair $\{s, s^{-1}\}$ in $\mathbb{F}_q \setminus \{0, 1\}$, where $\mathbb{F}_q$ is the finite field with q elements.

In the future, we will simply write R_s instead of $R_{s,s^{-1}}$ to simplify notation. The scheme $\mathrm{FT}(q+1)$ has $(q+1)/2$ classes if q is odd and has $q/2$ classes if q is even. Its intersection parameters can be computed, see [3] for the computations in the case q is even. In [3], De Caen and Van Dam proceeded to consider possible fusion schemes of $\mathrm{FT}(q+1)$. Clearly the original triangular scheme $\mathrm{T}(q+1)$ is a 2-class fusion scheme of $\mathrm{FT}(q+1)$. Also since $P\Gamma L(2, q)$ is an overgroup of $PGL(2, q)$, the orbitals of the action $P\Gamma L(2, q)$ on Ω form a fusion scheme of $\mathrm{FT}(q+1)$. Besides these fusion schemes, De Caen and Van Dam [3] observed that when q is an even power of two, there seems exist an interesting fusion scheme of $\mathrm{FT}(q+1)$. Specifically, let $f \geq 2$ be an integer, let

$$X = PG(1, 4^f), \text{ and } \Omega = \binom{X}{2}.$$

Define the following relations on Ω:

1. $S_0 =$ the diagonal relation,
2. $S_1 = \{(\{a, b\}, \{c, d\}) : |\{a, b\} \cap \{c, d\}| = 1\}$,
3. $S_2 = \{(\{a, b\}, \{c, d\}) : \rho(a, b; c, d) \in \mathbb{F}_{2^f} \setminus \{0, 1\}\}$,
4. $S_3 = \{(\{a, b\}, \{c, d\}) : \rho(a, b; c, d) \neq 1 \text{ and } \rho(a, b; c, d)^{2^f+1} = 1\}$,
5. $S_4 = (\Omega \times \Omega) \setminus (S_0 \cup S_1 \cup S_2 \cup S_3)$.

CONJECTURE 1.1 (The De Caen-Van Dam Conjecture). The above relations S_i, $i = 0, 1, \ldots, 4$, form a four-class association scheme on Ω.

We will discuss three proofs of the above conjecture contained in [5], [11], [6]. Our discussion of the proofs in [5] and [11] will be sketchy, much more discussion will be devoted to the proof in [6], since the proof in [6] is related to finite geometry, and it actually proves a generalization of the De Caen-Van Dam conjecture.

2. Sketches of the first two proofs

We first sketch the proof in [5]. Let $f \geq 2$ be an integer. Let

$$B_0 = \mathbb{F}_{2^f} \setminus \{0, 1\}, \text{ and } B_1 = \{x \in \mathbb{F}_{4^f} \mid x \neq 1, x^{2^f+1} = 1\}.$$

Then $S_1 = R_1$ (of $\mathrm{FT}(4^f + 1)$), S_2 and S_3 are the union of all relations R_s with $s \in B_0$ and $s \in B_1$ respectively, and S_4 is the union of all remaining R_s. Recall that De Caen and Van Dam [3] computed the intersection parameters p_{st}^r of $\mathrm{FT}(4^f + 1)$, so for example, in order to show that the product of the adjacency matrices of S_2 and S_3 is a linear combination of those of the S_i, $i = 0, 1, 2, \ldots, 4$, it suffices to show that $\sum_{s \in B_0, t \in B_1} p_{st}^r$ does not depend on r itself, only depends on whether $r \in B_0$, $r \in B_1$, or $r \notin B_0 \cup B_1$. In general, from the computations of p_{st}^r in [3], it was clear that in order to prove the De Caen-Van Dam conjecture, it suffices to show that for all $i, j \in \{0, 1\}$, the numbers

$$\pi_{i,j}(r) := \frac{1}{2} \sum_{s \in B_i, t \in B_j} |\{x \in \mathbb{F}_{4^f} \mid x^2 + (r + s + t + rst)x + rs + rt + st = 0\}|$$

are constant for $r \in B_0$, $r \in B_1$ and $r \in \mathbb{F}_{4^f} \setminus (\{0, 1\} \cup B_0 \cup B_1)$, respectively. This was done in [5] by some elementary but tricky substitutions. We refer the reader to [5] for details. Now let us turn to the proof in [11]. De Caen and Van Dam commented in [3] that if one can compute explicitly the first eigenmatrix (the character table) of $\mathrm{FT}(q + 1)$, then one can use the Bannai-Muzychuk criterion (cf. [1], [9, Lemma 1]) to check whether S_i, $i = 0, 1, \ldots, 4$, constitute an association scheme on Ω. Tanaka [11] did exactly this. He first used a method of Kwok [8] to compute the character table of $\mathrm{FT}(q + 1)$, where q is a power of 2, then based on the character table, he used the Bannai-Muzychuk criterion to verify the De Caen-Van Dam conjecture. For detailed computations of the character table, we refer the reader to [11].

3. Inversive planes, ovoids and association schemes

In this section, we discuss the proof in [6]. In this proof, we first give geometric interpretations of the relations S_i, $i = 1, 2, 3, 4$, by using the classical inversive plane. Then we realize that one can even define these relations in a more general setting (see details below), and we prove that the relations so defined constitute an association scheme. Therefore the De Caen-Van Dam conjecture follows as a corollary of our main theorem (Theorem 3.4).

Let $n \geq 2$ be an integer. Any $3-(n^2+1, n+1, 1)$ design is called an *inversive plane of order n*, and the blocks of this design are often referred to as its *circles*. All known finite inversive planes are "egglike" in the following sense. An *ovoid* $\mathcal{O}$ of $PG(3, q)$, where $q > 2$ is a prime power, is a set of q^2+1 points with no three collinear. The classical example of an ovoid in $PG(3, q)$ is an elliptic quadric. At each point P of $\mathcal{O}$ there is a unique tangent plane. All other planes in $PG(3, q)$ meet $\mathcal{O}$ in an *oval*; that is, all such planes meet $\mathcal{O}$ in a set of $q+1$ points, no three collinear. In the classical case where $\mathcal{O}$ is an elliptic quadric, these ovals are conics. If one takes as varieties the points of an ovoid $\mathcal{O}$, takes as blocks the nontangent planar intersections of $\mathcal{O}$, and defines incidence by inclusion, the resulting structure $I(\mathcal{O})$ is easily seen to be an inversive plane of order q. When $\mathcal{O}$ is an elliptic quadric in $PG(3, q)$, $I(\mathcal{O})$ is the *classical* (or Miquelian) inversive plane $M(q)$. When $q \geq 8$ is an odd power of 2 and $\mathcal{O}$ is the Tits ovoid of $PG(3, q)$, $I(\mathcal{O})$ is the Suzuki-Tits inversive plane $S(q)$. These are the only known finite inversive planes (see Chapter 6 of [4] for a generaldiscussion of inversive planes).

It should be noted that there are many other models of the classical inversive plane $M(q)$. In particular, the points of the projective line $PG(1, q^2)$ together with its Baer sublines (isomorphic copies of $PG(1, q)$) as "circles" form a model for $M(q)$. Thus one frequently identifies the points of $M(q)$ with $\mathbb{F}_{q^2} \cup \{\infty\}$, using parametric coordinates for $PG(1, q^2)$. In this model one particular circle is represented by $\mathbb{F}_q \cup \{\infty\}$, and all other circles are obtained as images of this base circle under the linear fractional mappings $x \mapsto \frac{ax+b}{cx+d}$, where $a, b, c, d \in \mathbb{F}_{q^2}$ with $ad - bc \neq 0$, and the usual conventions on the symbol ∞ are adopted. Using this model we see that $P\Gamma L(2, q^2)$ acts on $M(q)$ as an automorphism group.

Given two distinct points P and Q of $M(q)$, we use $J(P, Q)$ to denote the *bundle* of $q+1$ circles passing through P and Q. Next we define the notion of flock. In order to do this, we need some preparation. For each circle C in $M(q)$ there is a unique automorphism ϕ_C of $M(q)$ which has order 2 and whose fixed points are precisely the points of C (see [4]). This involution is called the *inversion with respect to C*, and distinct points P and Q in $M(q)$ are called *conjugate* with respect to C if $\phi_C(P) = Q$. Given any two distinct points P and Q of $M(q)$, we use $K(P, Q)$ to denote the *(linear) flock* of $q-1$ circles with P and Q as a conjugate pair. Thus the circles in $K(P, Q)$ partition the q^2-1 points of $M(q)\backslash\{P, Q\}$, and through any point $T \in M(q)\backslash\{P, Q\}$ there passes a unique circle of $J(P, Q)$.

Now we are in a position to give geometric interpretations for the relations S_i, $i = 2, 3$, by using $M(q)$. We will identify X with the point set of $M(q)$, $q = 2^f$. Thus Ω is the set of unordered pairs of distinct points of $M(q)$. (We remind the reader that the definitions of X and Ω are the same as that given in the De Caen-Van Dam conjecture.)

PROPOSITION 3.1. *Two unordered pairs* $\{a, b\}$ *and* $\{c, d\}$ *of distinct points of* $M(q)$ *are in relation* S_2 *if and only if the four points* a, b, c, d *are distinct and concircular.*

Proof. We will use the model for $M(q)$ arising from $PG(1, q^2)$. In particular, we identify the points of $M(q)$ with $\mathbb{F}_{q^2} \cup \{\infty\}$. Assume that $\{a, b\}$ and $\{c, d\}$ are in relation S_2. Then a, b, c, d are four distinct points of $M(q)$. Since $\mathrm{Aut}(M(q)) \cong P\Gamma L(2, q^2)$ contains $PGL(2, q^2)$, which is triply transitive on the points of $M(q)$ and preserves cross-ratio, we may assume $a = 0, b = \infty$, and $c = 1$. Thus $\rho = \rho(a, b; c, d) = 1/d$, which according to relation S_2 implies that $d \in \mathbb{F}_q^*$. However the unique circle in $M(q)$ containing 0, 1 and ∞ is $\mathbb{F}_q \cup \{\infty\}$, and thus a, b, c, d are four distinct concircular points.

Conversely, suppose a, b, c, d are four distinct concircular points of $M(q)$. Again we may assume without loss of generality that $a = 0$, $b = \infty$, and $c = 1$. Then as above, necessarily, $d \in \mathbb{F}_q^*$ and thus $\rho^{q-1} = 1$. That is $\{a, b\}$ and $\{c, d\}$ are in relation S_2. □

PROPOSITION 3.2. *Two unordered pairs* $\{a, b\}$ *and* $\{c, d\}$ *of distinct points of* $M(q)$ *are in relation* S_3 *if and only if* $K(a, b) \cap J(c, d) \neq \emptyset$.

Proof. Assume first that $\{a, b\}$ and $\{c, d\}$ are in relation S_3. Then, as in the proof of the previous proposition, we may assume that $a = 0$, $b = \infty$, and $c = 1$. Hence $\rho = \rho(a, b; c, d) = 1/d \in \mathbb{F}_{q^2} \setminus \{0, 1\}$, with $d^{q+1} = 1$. One easily checks that the mapping $\phi : z \mapsto \frac{1}{z^q}$ is an inversion interchanging a and b whose circle of fixed points contains c and d.

Conversely, assume that C is some circle containing c and d such that $\phi_C(a) = b$. In particular, a, b, c and d are four distinct points. Without loss of generality, we may assume $a = 0, b = \infty, c = 1$ and $\rho = \rho(a, b; c, d) = 1/d \in \mathbb{F}_{q^2} \setminus \{0, 1\}$. Using the transitive action of $PGL(2, q^2)$ on the circles of $M(q)$, we see that the unique circle through 1 whose inversion interchanges 0 and ∞ is $C = \{x \in \mathbb{F}_{q^2} : x^{q+1} = 1\}$. Since $d \in C$, this implies that $d^{q+1} = 1$, and therefore$\{a, b\}$ and $\{c, d\}$ are in relation S_3. □

REMARK. Since there is a circle containing $\{c, d\}$ with $\{a, b\}$ as a conjugate pair if and only if there is a circle containing $\{a, b\}$ with $\{c, d\}$ as a conjugate pair, it is clear that S_3 is indeed a symmetric relation.

In the following we will define relations S_i in a more general setting by using an arbitrary ovoid in $PG(3, q)$, q even. So we first gather a few well-known facts concerning ovoids in $PG(3, q)$. One of the fundamental results on ovoids in $PG(3, q)$, q even, is the construction of a symplectic polarity from an ovoid. A *polarity* of $PG(3, q)$ is a map from the set of subspaces of $PG(3, q)$ onto itself that maps points to planes, lines to lines, planes to points, preserves incidence and has order 2. A point, line or plane of $PG(3, q)$ is *absolute* with respect to a polarity if it is incident with its own image under the polarity. A *symplectic* *(or null) polarity* of $PG(3, q)$ is a polarity with the property that every point (hence every plane) is absolute. We now state a theorem of Segre.

THEOREM 3.3 (Segre [10]). *Let $\mathcal{O}$ be an ovoid of $PG(3, q)$ where $q > 2$ is even. Then $\mathcal{O}$ determines a symplectic polarity of $PG(3, q)$ which interchanges each tangent plane of $\mathcal{O}$ with its point of tangency and interchange each secant plane π with the nucleus of the oval $\pi \cap \mathcal{O}$.*

We will use $\perp$ to denote the symplectic polarity associated with $\mathcal{O}$ in the above theorem. Let us elaborate on $\perp$ a little bit. We will omit the proofs since they can be found in Chapters 15 and 16 of [7]. If a point $P \in PG(3, q)$ is on $\mathcal{O}$, then $P^{\perp}$ is the unique tangent plane to $\mathcal{O}$ at P, and all $q + 1$ tangent lines to $\mathcal{O}$ through P lie in $P^{\perp}$. If $P \in PG(3, q)$ is not on $\mathcal{O}$, then there are also $q + 1$ tangent lines to $\mathcal{O}$ through P. These $q + 1$ tangent lines are coplanar if and only if q is even. That is, for even q, the plane through P containing these $q + 1$ tangent lines meet $\mathcal{O}$ in an oval, and P is the nucleus of that oval. So $P^{\perp}$ is exactly the plane containing the $q + 1$ tangent lines to $\mathcal{O}$ through P. Now let us look at the effect of $\perp$ on lines. All tangent lines to $\mathcal{O}$ are absolute with respect to $\perp$. The secant and exterior lines to $\mathcal{O}$ get interchanged by $\perp$. Let ℓ be a secant line to $\mathcal{O}$ with $\ell \cap \mathcal{O} = \{P, Q\}$. Then $\ell^{\perp} = P^{\perp} \cap Q^{\perp}$. Among the $q + 1$ planes through $\ell^{\perp}$, two are tangent to $\mathcal{O}$, namely, $P^{\perp}$ and $Q^{\perp}$; each of the rest $q - 1$ planes meets $\mathcal{O}$ in an oval. This set of $q - 1$ ovals is a (linear) *flock*. When $\mathcal{O}$ is an elliptic quadric, this flock is exactly $K(P, Q)$.

We are now ready to define the relations S_i in a more general setting. To motivate this, we represent $M(q)$ as the egglike inversive plane $I(\mathcal{O})$, where $\mathcal{O}$ is an elliptic quadric in $PG(3, q)$. Thus the points on $\mathcal{O}$ are the points of the inversive plane $M(q)$, and the underlying set Ω on which the relations S_i are defined is now identified with the set of secant lines of $\mathcal{O}$. Given two pairs $\{a, b\}$ and $\{c, d\}$ of distinct points of $M(q)$, let $\ell(a, b)$ and $m(c, d)$ be the secant lines to $\mathcal{O}$ through a, b and c, d respectively. Then the four distinct points a, b, c and d being concircular is equivalent to $\ell(a, b)$ meeting $m(c, d)$ at a point off $\mathcal{O}$, and $K(a, b) \cap J(c, d) \neq \emptyset$ is equivalent to $\ell(a, b)^{\perp} \cap m(c, d) \neq \emptyset$. We thus reformulate the relations S_0, S_1, S_2, S_3 and S_4 on $\Omega = \{\ell : |\ell \cap \mathcal{O}| = 2\}$ as follows.

1. $S_0 = \{(\ell, \ell) : \ell \in \Omega\}$,
2. $S_1 = \{(\ell, m) : \ell \neq m,\ \ell$ and m meet at a point on $\mathcal{O}\}$,
3. $S_2 = \{(\ell, m) : \ell \neq m,\ \ell$ and m meet at a point off $\mathcal{O}\}$,
4. $S_3 = \{(\ell, m) : \ell \neq m,\ \ell^{\perp} \cap m \neq \emptyset\}$, and
5. $S_4 = (\Omega \times \Omega) \setminus (S_0 \cup S_1 \cup S_2 \cup S_3)$.

Now we note that if we drop the condition that $\mathcal{O}$ is an elliptic quadric, but simply require that $\mathcal{O}$ is an ovoid in $PG(3, q)$, q even, these relations are still well-defined and symmetric. Thus from now on we no longer assume that $\mathcal{O}$ is an elliptic quadric, but only assume that $\mathcal{O}$ is an ovoid of $PG(3, q)$, q even. If we can show that the above relations S_i constitute an association scheme on the set of secant lines to an arbitrary ovoid in $PG(3,q)$, then we not only prove the De Caen-Van Dam conjecture, but also obtain another association scheme (with the same parameters) from the Tits ovoid when $q \geq 8$ is an odd power of 2.

In [6], using the above modified definitions of S_i on Ω, where Ω is the set of secant lines to an arbitrary ovoid $\mathcal{O}$ in $PG(3,q)$, we showed the existence of the intersection parameters p_{ij}^k, $i, j, k \in \{0,1,2,3,4\}$, and computed them. Thus we obtained the following theorem.

THEOREM 3.4. *The modified relations S_0, S_1, S_2, S_3, S_4 define an association scheme on the set of secant lines to any ovoid $\mathcal{O}$ in $PG(3,q)$, where $q = 2^f$ with $f \geq 2$. The first eigenmatrix of this scheme is*

$$P = \begin{pmatrix} 1 & 2(q^2-1) & (q/2-1)(q^2-1) & q(q^2-1)/2 & q(q/2-1)(q^2-1) \\ 1 & q^2-3 & 2-q & -q & -q(q-2) \\ 1 & -2 & 1-q & 0 & q \\ 1 & -2 & (q/2-1)(q-1) & q(q-1)/2 & -q(q-2) \\ 1 & -2 & q(q-1)/2+1 & -q(q+1)/2 & q \end{pmatrix}.$$

COROLLARY 3.5. *The original relations S_0, S_1, S_2, S_3, S_4 define an association scheme on the set of* 2*-subsets of $PG(1,q^2)$, where $q = 2^f$ with $f \geq 2$. The first eigenmatrix of this scheme is the same as that given in Theorem* 3.4.

REMARK. When $f \geq 3$ is odd and $\mathcal{O}$ is the Tits ovoid, the association scheme obtained from Theorem 3.4 is nonisomorphic to the one obtained in Corollary 3.5. To see that these schemes are indeed nonisomorphic, note that the subgroup of the automorphism group of the scheme fixing class S_1 is essentially the stabilizer of $\mathcal{O}$ in $P\Gamma L(4,q)$, which is not the same for the Tits ovoid as for an elliptic quadric (the former is the Suzuki group $Sz(q)$, having size $(q^2+1)q^2(q-1)$, the latter is the orthogonal group $PGO_-(4,q)$, having size $2(q^2+1)q^2(q^2-1)$).

From the character table P of the four-class scheme in Theorem 3.4, using the Bannai-Muzychuk criterion, we see that one gets a two-class association scheme by merging S_1, S_2, S_3, hence a strongly regular graph (srg). This srg will be denoted by $G(\mathcal{O})$, and can be defined as follows: The vertices of $G(\mathcal{O})$ are the set of secant lines to $\mathcal{O}$, two vertices are joined by an edge of $G(\mathcal{O})$ iff they are related by S_1 or S_2 or S_3. The existence of such an srg was pointed out in [3], modulo the De Caen-Van Dam conjecture. The parameters of $G(\mathcal{O})$ are

$$v = q^2(q^2+1)/2, \quad k = (q+1)(q^2-1)$$
$$\lambda = (q-1)(3q+2), \quad \mu = 2q(q+1)$$

where $q = 2^f$, $f \geq 2$.

In [6], we proved that $G(\mathcal{O})$ is isomorphic to the Brouwer-Wilbrink graph (see [2, Sect. 7B] for the definition of this graph) whenever $q \geq 4$ is a power of 2, independent of the ovoid $\mathcal{O}$ used in the construction of four-class association scheme in Theorem 3.4. The proof uses the Klein correspondence between the lines of $PG(3, q)$ and the points of the hyperbolic quadric in $PG(5, q)$. We refer the reader to [6] for details.

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