Wave Motion 47 (2010) 552–563

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

A Hamiltonian approach to nonlinear modulation of surface water waves

Walter Craig [a,*,1], Philippe Guyenne [b,2], Catherine Sulem [c,3]

[a] Department of Mathematics, McMaster University, Hamilton, ON, Canada L8S 4K1
[b] Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA
[c] Department of Mathematics, University of Toronto, Toronto, ON, Canada M5S 3G3

ARTICLE INFO

Article history:
Received 15 September 2009
Received in revised form 4 April 2010
Accepted 7 April 2010
Available online 13 April 2010

Keywords:
Nonlinear Schrödinger equation
Dispersive equations
Hamiltonian PDEs
Modulation theory

ABSTRACT

Many wave phenomena in physics are described by weakly nonlinear nearly monochromatic solutions in the form of modulated wave packets. The examples include ocean waves as well as waves in optics and in plasmas. There are a number of approaches to deriving the envelope equations for these theories of amplitude modulation. In this paper, we give a unified approach, based on the principles of a Hamiltonian formulation of the equations of motion. Our principal example is the system of equations of free surface water waves, for which we give a new derivation of the classical nonlinear Schrödinger and Davey–Stewartson equations, as well as the higher-order Dysthe system. One consequence of our analysis from this point of view is that the Dysthe equation can be posed as a Hamiltonian partial differential equation.

1. Introduction

Modulation theory is a well-established method in applied mathematics to study the long time evolution and stability of oscillatory solutions to partial differential equations (PDEs). Typical PDEs to which the theory is applied are nonlinear dispersive evolution equations describing wave phenomena that arise in physical applications. The usual modulational Ansatz is to anticipate a weakly nonlinear monochromatic form for solutions, and to derive equations describing the evolution of its envelope. For surface gravity water waves which are the focus of this paper, one typically finds the nonlinear Schrödinger (NLS) equation as a canonical equation for the first nontrivial term (or possibly the Davey–Stewartson system in the case of higher space dimensions). Well-known cases include descriptions of problems in optics [1,2], in plasma physics [3], and in the problem of water waves [4–8]. See also Ref. [9] for a review. For the next-order correction to this description of water waves, one finds the Dysthe equation ([10–12] and more recently [13]). Higher-order corrections have been understood to play an important role in the modulational approximation; indeed it has been observed that the addition of the higher-order terms provides improvements for stability properties of finite amplitude waves [10,14], as compared to the NLS description. Further numerical study of the Dysthe equation and comparison to experiments are presented in Refs. [15] and [16].

One of the usual approaches to modulation theory is a direct perturbation method involving multiple spatial and temporal scales. This approach is however known for often giving rise to long calculations. An alternative approach was codified in Whitham [17], who developed an elegant method of averaged Lagrangians and a transformation theory. A third approach to such derivations was given by Zakharov et al. [3], based on a Fourier mode coupling formalism and an expansion of the dispersion relation with respect to a small parameter.

In this paper, we present a systematic approach to the derivation of the equations of modulation theory, based on averaged Hamiltonians. It gives a simple and straightforward method for Hamiltonian PDEs. This is in the spirit of the paper of Craig and

* Corresponding author.
E-mail addresses: craig@math.mcmaster.ca (W. Craig), guyenne@math.udel.edu (P. Guyenne), sulem@math.toronto.edu (C. Sulem).
[1] Partially supported by the Canada Research Chairs Program and NSERC through grant number 238452-06.
[2] Partially supported by NSF through grants numbers DMS-0625931 and DMS-0920850.
[3] Partially supported by NSERC through grant number 46179-05.

0165-2125/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.wavemoti.2010.04.002

Groves [19] which gives a uniform approach to the water wave problem in the long wave scaling regime. The method involves a formal expansion in small parameter, and elements which enter our analysis include an expansion of Fourier multiplier operators, and in the case of water waves, a precise description of the Dirichlet–Neumann operator. Our approach is logically independent of the classical methods of multiple scales. It is closer to those of Stiassnie [11] and Zakharov et al. [3], however what differentiates it is that throughout the approach we are careful to retain a certain point of view. Namely, that our PDEs are considered to be Hamiltonian systems, that scaling transformations and Ansätze are considered to be canonical transformations under change of symplectic form, and the formal expansion is performed in the expression of the Hamiltonian.

Our point of view does have essential differences from the method used in [11,3] where the modulational Ansatz comes into play only after deriving the equations of motion from the truncated Hamiltonian, and, as a result, the original Hamiltonian structure is not systematically preserved throughout the derivation of the model equations, as presented e.g. in [11]. In the present formalism, the modulational Ansatz is introduced directly in the Hamiltonian and the symplectic structure of the system is transformed accordingly, thus leading to the derivation of Hamiltonian model equations in a consistent and systematic manner. Furthermore, several aspects of the formal expansion method as presented here are able to be placed on mathematically rigorous grounds, including two analytic results which concern the form of a Fourier multiplier operator expansion and a scale separation lemma for multiple scale functions.

We first illustrate the method in Section 2 on the nonlinear Klein–Gordon equation as a model problem. Subsequently we address the problem of the water wave equations, both in two and higher space dimensions. The approach uses only a few basic features of these equations, namely that they are dispersive Hamiltonian PDEs and that momentum and wave action are conserved quantities of motion. We distinguish two cases; in Section 3 we study the case of finite depth, and in Section 4 the case of infinite depth. In all cases we give our derivation of the NLS equation at the leading order, and our version of the Dysthe system at higher order, through which we exhibit the simplicity and straightforward nature of the approach. As a byproduct, we exhibit a Hamiltonian for this version of the Dysthe equation; as far as we know, the facts that it can be written in the form of a Hamiltonian PDE, and its subsequent properties of energy conservation, have not previously been noticed.

2. A model equation

This section illustrates the approach to modulation theory through the example of the nonlinear Klein–Gordon equation, which is the following dispersive equation

$$\partial_t^2 v = \Delta_x v - m^2 v - v^3, \quad x \in \mathbb{R}^d, t \in \mathbb{R}, \tag{2.1}$$

for real valued functions v and p. It can be written as a Hamiltonian PDE with the Hamiltonian

$$H = \int \left(\frac{1}{2} p^2 + \frac{1}{2} |\nabla v|^2 + \frac{m^2}{2} v^2 + \frac{1}{4} v^4 \right) dx. \tag{2.2}$$

so that Eq. (2.1) is rewritten in first-order form

$$\partial_t v = p$$
$$\partial_t p = \Delta_x v - m^2 v - v^3.$$

This exhibits the Hamiltonian character of the problem, namely

$$\partial_t \begin{pmatrix} v \\ p \end{pmatrix} = J \begin{pmatrix} \delta_v H \\ \delta_p H \end{pmatrix}, \tag{2.3}$$

with the symplectic structure expressed through

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix},$$

and with $\mathbb{I}$ denoting the identity operator. The linear dispersion relation is $\omega^2(k) = |k|^2 + m^2$ and it is thus natural to define the Fourier multiplier operator $\omega(D_x) = (|D_x|^2 + m^2)^{1/2}$ where $D_x = -i\partial_x$ is the usual choice of self-adjoint form for partial derivatives.

Changing variables to complex symplectic coordinates, we define $a(D_x) = \omega(D_x)^{1/2}$ and then use it to define the complex symplectic coordinates[4]

$$z = \frac{1}{\sqrt{2}} \left(a(D_x) v + i a(D_x)^{-1} p \right), \tag{2.4}$$

[4] On the usual Sobolev space H^s the operator $a(D_x): H^s \to H^{s-1/2}$ is bounded and invertible.

so that

$$v=\frac{1}{\sqrt{2}}a(D_x)^{-1}(z+\overline{z}),\quad p=\frac{1}{\sqrt{2}i}a(D_x)(z-\overline{z}),\tag{2.5}$$

where the symbol $\bar{\cdot}$ denotes complex conjugation. This transformation changes the expression of the symplectic structure to standard complex form

$$J_1=\begin{pmatrix}0 & -i\mathbb{I}\\ i\mathbb{I} & 0\end{pmatrix},$$

and the Hamiltonian is rewritten as

$$\begin{aligned}H_1(z,\overline{z})&=\int\left(|a(D_x)z|^2+\frac{1}{16}\left|a(D_x)^{-1}(z+\overline{z})\right|^4\right)dx\\&=\int\overline{z}\omega(D_x)zdx+\frac{1}{16}\int\left|\omega^{-1/2}(D_x)(z+\overline{z})\right|^4dx.\end{aligned}\tag{2.6}$$

We now introduce the modulational Ansatz

$$z=\varepsilon u(X)e^{ik_0\cdot x},\quad X=\varepsilon x\in\mathbb{R}^d,\tag{2.7}$$

for a modulated plane wave solution with carrier wave number k_0. That is, $z=z(x,X)$ has the form of a multiple scale function and ε is a small parameter. However, maintaining the point of view that our scaling and Ansätze are transformations, the mapping from z to u results from a translation of Fourier space variables, followed by a spatial scaling [20], and it is invertible. In order to analyse the resulting expression in the Hamiltonian, the following result describes the action of Fourier multiplier operators on multiple scale functions.

Theorem 1. *Let $m(D_x)$ be a Fourier multiplier. For sufficiently smooth functions $f(X)$, we have*

$$\begin{aligned}m(D_x)\left(e^{ik_0\cdot x}f(X)\right)&=e^{ik_0\cdot x}m(k_0+\varepsilon D_X)f(X)\\&=e^{ik_0\cdot x}\Big(m(k_0)+\varepsilon\partial_{k_j}m(k_0)D_{X_j}f(X)+\frac{\varepsilon^2}{2}\partial^2_{k_jk_\ell}m(k_0)D^2_{X_jX_\ell}f(X)\\&\quad+\frac{\varepsilon^3}{3!}\partial^3_{k_jk_\ell k_n}m(k_0)D^3_{X_jX_\ell X_n}f(X)+\ldots+\frac{\varepsilon^{|j|}}{j!}\partial^{j_1}_{k_1}\ldots\partial^{j_d}_{k_d}m(k_0)D^{j_1}_{X_1}\ldots D^{j_d}_{X_d}f(X)+O\left(\varepsilon^{|j|+1}\right)\Big).\end{aligned}\tag{2.8}$$

In Eq. (2.8) and hereafter, the Einstein summation notation is used for repeated indices. The proof of this theorem appears as Theorem 4.1 and extended in A2.1 of [21] where it was used to give a rigorous setting to modulation theory in the context of water waves. The proof relies on the Fourier representation of Fourier multipliers and Taylor expansions. In formulation (2.8), the operator m acts on a multiple scale function in the simple monochromatic oscillatory form $e^{ik_0\cdot x}u(\varepsilon x)$, and the expansion (2.8) has the result of giving a Taylor expansion about the wave number k_0. In the X-variables, the truncation of this series at any finite order $|j|$ acts as a differential operator in the X-variables, of order $|j|$.

The change of variables expressed by the modulational Ansatz (2.4), (2.7) is symplectic, up to a scaling factor, and therefore changes the symplectic structure given by J_1 only up to this factor:

$$J_2=\varepsilon^d\begin{pmatrix}0 & -i\mathbb{I}\\ i\mathbb{I} & 0\end{pmatrix},$$

whose non-symplectic nature can be renormalized by simply rescaling time [20]. The transformed Hamiltonian is as follows

$$\varepsilon^dH_2(u,\overline{u})=\varepsilon^2\int\overline{u}\omega(k_0+\varepsilon D_X)udX+\frac{\varepsilon^4}{16}\int\left|e^{ik_0\cdot x}\omega^{-1/2}(k_0+\varepsilon D_X)u+e^{-ik_0\cdot x}\omega^{-1/2}(-k_0+\varepsilon D_X)\overline{u}\right|^4dX,\tag{2.9}$$

and the evolution equations are

$$\partial_tu=-i\delta_{\overline{u}}H_2,\ \partial_t\overline{u}=i\delta_uH_2.\tag{2.10}$$

The second Eq. (2.10) is of course the complex conjugate of the first because of the original reality conditions imposed on v and p. The quartic term in the Hamiltonian simplifies further because of the following scale separation lemma.

Theorem 2. *Let g be a function on $\mathbb{R}^d$ which is periodic over the fundamental domain $\mathbb{R}^d/\Gamma$, a torus, and denote $\mathsf{E}(g)$ its average value. For any function f(X) in the Schwartz space $\mathcal{S}\left(\mathbb{R}^d\right)$, we have*

$$\int g(X/\varepsilon)f(X)dX = \mathsf{E}(g)\int f(X)dX + O\left(\varepsilon^N\right), \tag{2.11}$$

for any N.

The proof of this result can be found in Ref. [22]. It expresses that on the LHS of Eq. (2.11), the contributions of the fast oscillations homogenize, and this approximation is true at order $O(\varepsilon^N)$ for $f(X)\in\mathcal{S}\left(\mathbb{R}^d\right)$. In general the order of approximation is related to the smoothness of the given test function f.

Using the scale separation result, the Hamiltonian is now expressed as

$$\begin{aligned}\varepsilon^{d-2}H_2(u,\overline{u}) = {} & \frac{1}{2}\int(\overline{u}\omega(k_0+\varepsilon D_X)u + u\omega(-k_0+\varepsilon D_X)\overline{u})dX \\ & + \frac{3\varepsilon^2}{8}\int\left(\omega^{-1/2}(k_0+\varepsilon D_X)u\right)^2\left(\omega^{-1/2}(-k_0+\varepsilon D_X)\overline{u}\right)^2 dX + O\left(\varepsilon^N\right).\end{aligned} \tag{2.12}$$

Following Theorem 1, the Taylor expansion of the linear dispersion relation gives

$$\omega(k_0+\varepsilon D_X)f(X) = \omega(k_0)f(X) + \varepsilon\partial_{k_j}\omega(k_0)D_{X_j}f(X) + \frac{1}{2}\varepsilon^2\partial^2_{k_jk_l}\omega(k_0)D^2_{X_jX_l}f(X) + \frac{1}{6}\varepsilon^3\partial^3_{k_jk_lk_m}\omega(k_0)D^3_{X_jX_lX_m}f(X) + \cdots. \tag{2.13}$$

Similarly,

$$\omega^{-1/2}(\pm k_0+\varepsilon D_X) = \omega^{-1/2}(\pm k_0)\mp\frac{\varepsilon}{2}\omega^{-3/2}(\pm k_0)\partial_{k_j}\omega(\pm k_0)D_{X_j} + \cdots.$$

These expressions in the Hamiltonian provide the expansion of H_2 in powers of ε that we seek:

$$\begin{aligned}\varepsilon^{d-2}H_2(u,\overline{u}) = {} & \int\omega(k_0)|u|^2dX \\ & + \frac{\varepsilon}{2}\int\partial_{k_j}\omega(k_0)(\overline{u}D_{X_j}u + u\overline{D_{X_j}u})dX + \varepsilon^2\int\left(\frac{1}{4}\partial^2_{k_jk_\ell}\omega(k_0)\left(\overline{u}D^2_{X_jX_\ell}u + u\overline{D^2_{X_jX_\ell}u}\right) + \frac{3}{8}\omega^{-2}(k_0)|u|^4\right)dX \\ & + \varepsilon^3\int\left(\frac{1}{12}\partial^3_{k_jk_\ell k_m}\omega(k_0)\left(\overline{u}D^3_{k_jk_\ell k_m}u + u\overline{D^3_{k_jk_\ell k_m}u}\right) - \frac{3}{8}\omega^{-3}(k_0)\partial_{k_j}\omega(k_0)|u|^2\left(\overline{u}D_{X_j}u + u\overline{D_{X_j}u}\right)\right)dX \\ & + O\left(\varepsilon^4\right),\end{aligned} \tag{2.14}$$

where we have used the fact that $\omega(k_0)$ is an even function.

Of the four orders represented in perturbation theory in this Hamiltonian, the first two can be eliminated through elementary considerations. Firstly, one transforms the system into a reference coordinate frame moving with the group velocity $\partial_k\omega(k_0)$. This is accomplished by subtracting from Eq. (2.14) a multiple of the impulse (or momentum)

$$\begin{aligned} I &= \int v\partial_x p dx = \frac{1}{2}\int(\overline{z}D_xz + z\overline{D_xz})dx, \\ \varepsilon^{d-2}I &= k_0\int|u|^2dX + \frac{\varepsilon}{2}\int(\overline{u}D_Xu + u\overline{D_Xu})dX,\end{aligned} \tag{2.15}$$

which is a conserved integral of motion [18–20], yielding

$$\begin{aligned}\varepsilon^{d-2}(H_2-\partial_k\omega(k_0)\cdot I) = {} & (\omega(k_0)-k_0\cdot\partial_k\omega(k_0))\int|u|^2dX \\ & + \varepsilon^2\int\left(\frac{1}{4}\partial^2_{k_jk_\ell}\omega(k_0)\left(\overline{u}D^2_{X_jX_\ell}u + u\overline{D^2_{X_jX_\ell}u}\right) + \frac{3}{8}\omega^{-2}(k_0)|u|^4\right)dX \\ & + \varepsilon^3\int\left(\frac{1}{12}\partial^3_{k_jk_\ell k_m}\omega(k_0)\left(\overline{u}D^3_{X_jX_\ell X_m}u + u\overline{D^3_{X_jX_\ell X_m}u}\right) - \frac{3}{8}\omega^{-3}(k_0)\partial_{k_j}\omega(k_0)|u|^2\left(\overline{u}D_{X_j}u + u\overline{D_{X_j}u}\right)\right)dX + O\left(\varepsilon^4\right).\end{aligned} \tag{2.16}$$

Secondly one adjusts the phase of solutions, subtracting a multiple of the L^2 norm

$$\varepsilon^{d-2}M = \int|u|^2dX, \tag{2.17}$$

to obtain the reduced Hamiltonian

$$\begin{aligned}\varepsilon^{d-2}\,\hat{H}_2 := {} & \varepsilon^{d-2}(H_2-\partial_k\omega(k_0)\cdot I-(\omega(k_0)-k_0\cdot\partial_k\omega(k_0))M)\\ = {} & \varepsilon^2\int\left(\frac{1}{4}\partial^2_{k_jk_\ell}\omega(k_0)\left(\overline{u}D^2_{X_jX_\ell}u+u\overline{D^2_{X_jX_\ell}u}\right)+\frac{3}{8}\omega^{-2}(k_0)|u|^4\right)dX\\ & +\varepsilon^3\int\left(\frac{1}{12}\partial^3_{k_jk_\ell k_m}\omega(k_0)\left(\overline{u}D^3_{X_jX_\ell X_m}u+u\overline{D^3_{X_jX_\ell X_m}u}\right)-\frac{3}{8}\omega^{-3}(k_0)\partial_{k_j}\omega(k_0)|u|^2\left(\overline{u}D_{X_j}u+u\overline{D_{X_j}u}\right)\right)dX+O\left(\varepsilon^4\right).\end{aligned} \tag{2.18}$$

Posed as a system of equations for u, and dropping terms of order $O(\varepsilon^4)$, Eq. (2.3) becomes

$$\partial_t u = -i\delta_{\overline{u}}\,\hat{H}_2. \tag{2.19}$$

The subtraction of Eq. (2.17) from H_2 is justified by the fact that, although it is not by itself a constant of motion for the original equation (2.1), it is however a conserved quantity for the approximate system (2.19). Indeed, it is a straightforward calculation to show that $dM/dt=0$ using Eq. (2.19) or, equivalently, that $\hat{H}_2$ Poisson commutes with M, meaning that

$$\left\{\hat{H}_2, M\right\} = \int\left(\delta_{\overline{u}}\,\hat{H}_2\delta_u M-\delta_u\,\hat{H}_2\delta_{\overline{u}}M\right)dX = 0.$$

The conservation of Eq. (2.17) is equivalent to the phase invariance of solutions of Eq. (2.19).

Expressed in coordinates, the system Eq. (2.19) is

$$\partial_t u = -i\varepsilon^2\left[\frac{1}{2}\partial^2_{k_jk_l}\omega(k_0)D^2_{X_jX_l}u+\frac{3}{4}\omega^{-2}(k_0)|u|^2u+\varepsilon\left(\frac{1}{6}\partial^3_{k_jk_lk_m}\omega(k_0)D^3_{X_jX_lX_m}u-\frac{3}{2}\omega^{-3}(k_0)\partial_{k_j}\omega(k_0)|u|^2D_{X_j}u\right)\right].$$

Introducting the slow time scale $\tau=\varepsilon^2 t$ gives directly the cubic NLS equation at the lowest order of approximation, and an equation reminiscent of the Dysthe equation when the next order of approximation is taken into account. Explicitly, this is

$$i\partial_\tau u = -\frac{1}{2}\partial^2_{k_jk_l}\omega(k_0)\partial^2_{X_jX_l}u+\frac{3}{4\omega^2(k_0)}|u|^2u+\varepsilon\left[\frac{i}{6}\partial^3_{k_jk_lk_m}\omega(k_0)\partial^3_{X_jX_lX_m}u+\frac{3i}{2}\omega^{-3}(k_0)\partial_{k_j}\omega(k_0)|u|^2\partial_{X_j}u\right]. \tag{2.20}$$

We note that the NLS equation takes essentially a universal form, as seen by this derivation. However the nonlinear term of higher order does not have in general a universal form, as we will find in the next sections devoted to the water wave problem, since it depends on the detailed structure of the nonlinear terms.

3. The water wave problem with finite depth

We now apply this analysis to the water wave problem that describes the motion of a free surface on top of an incompressible, ideal and irrotational fluid in a two- or three-dimensional channel. We consider the Euler equations in Hamiltonian form where the free surface $\eta(x)$ and the boundary values $\xi(x)$ of the velocity potential $\varphi(x,y)$ constitute the canonical variables for the system [5]

$$\partial_t\begin{pmatrix}\eta\\ \xi\end{pmatrix} = \begin{pmatrix}0 & \mathbb{I}\\ -\mathbb{I} & 0\end{pmatrix}\begin{pmatrix}\delta_\eta H\\ \delta_\xi H\end{pmatrix} = J\nabla H. \tag{3.1}$$

The Hamiltonian functional is the total energy

$$H = \frac{1}{2}\int\xi(x)G(\eta)\xi(x)+g\eta^2(x)dx. \tag{3.2}$$

In the above Hamiltonian, g is the acceleration of gravity, and $G(\eta)$ is the free-surface Dirichlet–Neumann operator associated with the Laplace equation for the velocity potential φ:

$$G(\eta)\xi(x) = \nabla\varphi(x,\eta(x))\cdot N(\eta)\left(1+|\partial_x\eta|^2\right)^{1/2}, \tag{3.3}$$

where $N(\eta)$ is the exterior unit normal on the free surface.

The time-dependent fluid domain consists of the region

$$S(\eta) = \left\{(x,y)\in\mathbb{R}^{n-1}\times\mathbb{R} : -h<y<\eta(x,t)\right\},$$

where h is the (constant) fluid depth, and $n=2$ or 3 is the space dimension. For a given function $f(x)$, let $\hat{f}(k)=\sqrt{2\pi}^{-(n-1)}\int_{\mathbb{R}^{n-1}}f(x)e^{-ik\cdot x}dx$ denote its Fourier transform, and P_0 is the projection that associates to f its zero frequency mode $\hat{f}(0)$.

3.1. Canonical transformations and Ansatz

Use the Dirichlet–Neumann operator $G_0(D_x)$ for a flat surface elevation and a finite-depth channel [23], given by

$$G_0(D_x) = |D_x| tanh(h|D_x|),$$

to define the Fourier multiplier

$$a(D_x) = \sqrt[4]{(g/G_0(D_x))},$$

and use this in turn for the canonical transformation $(\eta,\xi)\rightarrow(z,\bar{z},\tilde{\eta},\tilde{\xi})$ given by

$$\eta = \frac{1}{\sqrt{2}}a^{-1}(D_x)(z+\overline{z})+\tilde{\eta}, \quad \tilde{\eta} = P_0\eta, \tag{3.4}$$

$$\xi = \frac{1}{\sqrt{2}i}a(D_x)(z-\overline{z})+\tilde{\xi}, \quad \tilde{\xi} = P_0\xi. \tag{3.5}$$

As above, the notation is that $D_x=-i\partial_x$. This operator reduces to $G_0(D_x)=|D_x|$ if the depth is infinite. The presence of the mean fields $\tilde{\eta}$ and $\tilde{\xi}$ is due to the fact that $a^{-1}(0)=0$, so that the change of variables $(\eta,\xi)\rightarrow(z,\bar{z})$ is not invertible.[5] The new variables $(z,\bar{z},\tilde{\eta},\tilde{\xi})^\top$ are expressed in terms of $(\eta,\xi)^\top$ as follows

$$\left(z,\bar{z},\tilde{\eta},\tilde{\xi}\right)^\top = A_1(\eta,\xi)^\top,$$

where the 4×2 matrix A_1 is given by

$$A_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} a(D_x)(\mathbb{I}-P_0) & ia^{-1}(D_x)(\mathbb{I}-P_0) \\ a(D_x)(\mathbb{I}-P_0) & -ia^{-1}(D_x)(\mathbb{I}-P_0) \\ \sqrt{2}P_0 & 0 \\ 0 & \sqrt{2}P_0 \end{pmatrix}. \tag{3.6}$$

The equations of motion become

$$\partial_t\begin{pmatrix} z \\ \overline{z} \\ \tilde{\eta} \\ \tilde{\xi} \end{pmatrix} = J_1\begin{pmatrix} \delta_z H \\ \delta_{\bar{z}} H \\ \delta_{\tilde{\eta}} H \\ \delta_{\tilde{\xi}} H \end{pmatrix}, \tag{3.7}$$

with the symplectic form given by the matrix

$$J_1 = A_1 J A_1^\top = \begin{pmatrix} 0 & -i(\mathbb{I}-P_0) & 0 & 0 \\ i(\mathbb{I}-P_0) & 0 & 0 & 0 \\ 0 & 0 & 0 & P_0 \\ 0 & 0 & -P_0 & 0 \end{pmatrix}, \tag{3.8}$$

where we have used the fact that $P_0^2=P_0$.

We look for solutions in the form of monochromatic waves with a modulated complex amplitude depending upon the second variables $X=\varepsilon x$,

$$z = \varepsilon u(X,t)e^{ik_0\cdot x}, \quad \overline{z} = \varepsilon\overline{u}(X,t)e^{-ik_0\cdot x}, \tag{3.9}$$

$$\tilde{\eta} = \varepsilon^\alpha\tilde{\eta}_1(X,t), \quad \tilde{\xi} = \varepsilon^\beta\tilde{\xi}_1(X,t), \tag{3.10}$$

where the exponents $\alpha\geq1$ and $\beta\geq1$ are to be chosen.

[5] Actually the projection P_0 is not well defined on the classical Sobolev spaces H^s, but it is so on weighted Sobolev spaces.

Equivalently, the system has the form

$$\begin{pmatrix} u \\ \overline{u} \\ \tilde{\eta}_1 \\ \tilde{\xi}_1 \end{pmatrix} = A_2 \begin{pmatrix} z \\ \overline{z} \\ \tilde{\eta} \\ \tilde{\xi} \end{pmatrix}, \tag{3.11}$$

where A_2 is the 4×4 diagonal matrix with the entries on the diagonal being $(\varepsilon^{-1}e^{-ik_0\cdot x}, \varepsilon^{-1}e^{ik_0\cdot x}, \varepsilon^{-\alpha}, \varepsilon^{-\beta})$. The equations of motion are

$$\partial_t v = J_2 \nabla_v H, \tag{3.12}$$

with $v=(u,\bar{u},\tilde{\eta}_1,\tilde{\xi}_1)^\top$ and

$$J_2 = \varepsilon^{n-1}A_2 J_1 A_2^\top$$
$$= \begin{pmatrix} 0 & -i\varepsilon^{n-3}e^{-ik_0\cdot x}(\mathbb{I}-P_0)(e^{ik_0\cdot x}\cdot) & 0 & 0 \\ i\varepsilon^{n-3}e^{ik_0\cdot x}(\mathbb{I}-P_0)\left(e^{-ik_0\cdot x}\cdot\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{n-1-\alpha-\beta}P_0 \\ 0 & 0 & -\varepsilon^{n-1-\alpha-\beta}P_0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -i\varepsilon^{n-3}\mathbb{I}' & 0 & 0 \\ i\varepsilon^{n-3}\mathbb{I}' & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{n-1-\alpha-\beta} \\ 0 & 0 & -\varepsilon^{n-1-\alpha-\beta} & 0 \end{pmatrix},$$

where $\mathbb{I}'$ is the identity on the class of functions $\{u(X)\}$, and the final 2×2 block retains essentially the standard symplectic form on the two-dimensional space of constants $(\tilde{\eta}_1,\tilde{\xi}_1)$.

3.2. Expansion of Hamiltonian

We expand the Hamiltonian in powers of ε,

$$H = H^{(2)} + H^{(3)} + H^{(4)} + \cdots, \tag{3.13}$$

using the Taylor series expansion of the Dirichlet–Neumann operator [23],

$$G = G_0 + G_1 + G_2 + \cdots, \tag{3.14}$$

with the result that

$$H^{(2)} = \frac{1}{2}\int\left(\xi G_0 \xi + g\eta^2\right)dx, \tag{3.15}$$

$$H^{(3)} = \frac{1}{2}\int \xi G_1(\eta)\xi dx, \tag{3.16}$$

$$H^{(4)} = \frac{1}{2}\int \xi G_2(\eta)\xi dx, \tag{3.17}$$

where

$$G_1(\eta) = D_x\eta\cdot D_x - G_0\eta G_0,$$
$$G_2(\eta) = -\frac{1}{2}\left(|D_x|^2\eta^2 G_0 + G_0\eta^2|D_x|^2 - 2G_0\eta G_0\eta G_0\right).$$

Retaining terms of up to second order in η is sufficient for the purposes of the present paper, as this will include all of the contributions to the NLS and Dysthe systems. Denoting by $\omega(D_x) = (gG_0)^{1/2} = \sqrt{g|D_x|tanh(h|D_x|)}$ the linear dispersion relation, we get that

$$H^{(2)} = \int\left(\overline{z}\omega(D_x)z + \frac{i}{\sqrt{2}}a(D_x)(z-\overline{z})G_0\tilde{\xi} + \frac{1}{2}\tilde{\xi}G_0\tilde{\xi} + \frac{1}{2}g\tilde{\eta}^2 + \frac{g}{\sqrt{2}}\tilde{\eta}a^{-1}(D_x)(z+\overline{z})\right)dx. \tag{3.18}$$

In Eq. (3.18), the first term is calculated as before,

$$\int \bar{z}\omega(D_x)z\,dx = \varepsilon^{3-n}\int \bar{u}(X)\omega(k_0 + \varepsilon D_X)u(X)dX. \tag{3.19}$$

To calculate the second and last terms of Eq. (3.18), we apply Theorem 2 and find that all terms are nonresonant, and therefore they do not contribute to the Hamiltonian. The remaining terms in $H^{(2)}$ are

$$\begin{aligned}\varepsilon^{n-3}H^{(2)} &= \int \bar{u}(X)\omega(k_0 + \varepsilon D_X)u(X)dX \\ &\quad + \frac{1}{2}\varepsilon^{2\beta-2}\int \tilde{\xi}_1 G_0 \tilde{\xi}_1 dX + \frac{g}{2}\varepsilon^{2\alpha-2}\int \tilde{\eta}_1^2 dX.\end{aligned} \tag{3.20}$$

In the above expression, we furthermore have

$$\int \tilde{\xi}_1 G_0 \tilde{\xi}_1 dX = \varepsilon^2 h \int \tilde{\xi}_1 |D_X|^2 \tilde{\xi}_1 dX + O\left(\varepsilon^4\right). \tag{3.21}$$

We now compute $H^{(3)}$

$$H^{(3)} = \frac{1}{2}\int \eta\left(|D_x\xi|^2 - (G_0\xi)^2\right)dx. \tag{3.22}$$

From the terms that come out of the expansion, it is natural to choose the exponents α and β such that $\alpha = \beta + 1$. We find

$$\begin{aligned}\varepsilon^{n-3}H^{(3)} &= \varepsilon^{\beta+1}\int\left(\alpha_1(k_0)|u|^2\tilde{\eta}_1 - i|u|^2 k_0\cdot D_X\tilde{\xi}_1\right)dX \\ &\quad + \varepsilon^{\beta+2}\alpha_2^j(k_0)\int\left(\bar{u}D_{X_j}u + u\overline{D_{X_j}u}\right)\tilde{\eta}_1 dX \\ &\quad - \varepsilon^{\beta+2}\frac{i}{2}\int\left(\bar{u}D_{X_j}u + u\overline{D_{X_j}u}\right)D_{X_j}\tilde{\xi}_1 dX,\end{aligned}$$

where the coefficients are

$$\begin{aligned}\alpha_1(k_0) &= \frac{a^2(k_0)}{2}\left(|k_0|^2 - G_0^2(k_0)\right), \\ \alpha_2^j(k_0) &= \frac{a^2(k_0)}{2}\left(k_{0j} + |k_0|^2\frac{\partial_{k_j}a}{a(k_0)} - G_0(k_0)\partial_{k_j}G_0(k_0) - G_0^2(k_0)\frac{\partial_{k_j}a}{a(k_0)}\right).\end{aligned} \tag{3.23}$$

The contributions from $H^{(4)}$ are

$$\varepsilon^{n-3}H^{(4)} = \varepsilon^2\alpha_3(k_0)\int|u|^4 dX + \varepsilon^3\alpha_4^j(k_0)\int|u|^2\left(\bar{u}D_{X_j}u + u\overline{D_{X_j}u}\right)dX, \tag{2.24}$$

with coefficients

$$\begin{aligned}\alpha_3(k_0) &= \frac{1}{4}\left(G_0(k_0)^2 G_0(2k_0) - |k_0|^2 G_0(k_0)\right), \\ \alpha_4^j(k_0) &= \frac{3}{8}G_0(k_0)|k_0|^{-2}\left(G_0(2k_0)G_0(k_0) - |k_0|^2\right)k_{0_j} \\ &\quad + \frac{h}{8}\left(1 - tanh^2(h|k_0|)\right)\left(2G_0(2k_0)G_0(k_0) - |k_0|^2\right)k_{0_j} + \frac{h}{2}G_0(k_0)^2\left(1 - tanh^2(2h|k_0|)\right)k_{0_j}.\end{aligned}$$

The conserved impulse vector valued integral is given by

$$I = \int \eta\partial_x\xi\,dx, \tag{3.25}$$

which in the present coordinates is given by

$$\varepsilon^{n-3}I = \int\left(k_0|u|^2 + \varepsilon\bar{u}D_X u + i\varepsilon^{2\beta}\tilde{\eta}_1 D_X\tilde{\xi}_1\right)dX. \tag{3.26}$$

In analogy with the Klein–Gordon equation, we also make use of the quantity

$$\varepsilon^{n-3}M = \int |u|^2 dX,$$

which is known as the *wave action* integral. After subtracting terms proportional to I and M,

$$\hat{H} = H - \partial_k\omega(k_0)\cdot I - (\omega(k_0) - k_0\cdot\partial_k\omega(k_0))M, \tag{3.27}$$

we obtain the renormalized Hamiltonian

$$\begin{aligned}\varepsilon^{n-3}\hat{H} = \varepsilon^2\int\Big(&\frac{1}{2}\partial^2_{k_jk_l}\omega(k_0)\overline{u}D^2_{X_jX_l}u + \alpha_3(k_0)|u|^4 + \varepsilon^{2\beta-2}\left(\frac{h}{2}\tilde{\xi}_1|D_X|^2\tilde{\xi}_1 + \frac{g}{2}\tilde{\eta}_1^2 - i\partial_{k_j}\omega(k_0)\tilde{\eta}_1 D_{X_j}\tilde{\xi}_1\right)\\ &+ \alpha_1(k_0)\varepsilon^{\beta-1}\tilde{\eta}_1|u|^2 - i\varepsilon^{\beta-1}|u|^2k_0\cdot D_X\tilde{\xi}_1 + \frac{\varepsilon}{6}\partial^3_{k_jk_lk_m}\omega(k_0)\overline{u}D^3_{X_jX_lX_m}u\\ &+ \varepsilon\left(\alpha_2^j(k_0)\varepsilon^{\beta-1}\tilde{\eta}_1 - \frac{i}{2}\varepsilon^{\beta-1}D_{X_j}\tilde{\xi}_1 + \alpha_4^j(k_0)|u|^2\right)\left(\overline{u}D_{X_j}u + u\overline{D_{X_j}u}\right) + O\left(\varepsilon^2\right)\Big)dX.\end{aligned}$$

Here, as in the case of the Klein–Gordon equation, it can be shown that M is a conserved quantity of this system at the order of approximation that is being considered.

3.3. Hamilton equations

It is now natural to choose $\beta=1$ (and thus $\alpha=2$) for the exponents of the mean fields. Introducing the slow time variables $\tau=\varepsilon^2 t$, the corresponding equations of motion are

$$\begin{aligned}i\partial_\tau u =& -\frac{1}{2}\partial^2_{k_jk_l}\omega\partial^2_{X_jX_l}u + 2\alpha_3|u|^2u + \left(\alpha_1\tilde{\eta}_1 - ik_0\cdot D_X\tilde{\xi}_1\right)u\\ &+ i\varepsilon\left(\frac{1}{6}\partial^3_{k_jk_lk_m}\omega\partial^3_{X_jX_lX_m}u - \alpha_2^j\left(\tilde{\eta}_1\partial_{X_j}u + \partial_{X_j}(u\tilde{\eta}_1)\right) + \frac{1}{2}\left(\partial_{X_j}\tilde{\xi}_1\partial_{X_j}u + \partial_{X_j}\left(u\partial_{X_j}\tilde{\xi}_1\right)\right) - 4\alpha_4^j|u|^2\partial_{X_j}u\right) + O\left(\varepsilon^2\right),\\ \varepsilon\partial_\tau\tilde{\eta}_1 =& h|D_X|^2\tilde{\xi}_1 + k_0\cdot\partial_X|u|^2 + i\partial_{k_j}\omega D_{X_j}\tilde{\eta}_1 + \frac{\varepsilon}{2}\partial_{X_j}\left(\overline{u}D_{X_j}u + u\overline{D_{X_j}u}\right) + O\left(\varepsilon^2\right),\\ \varepsilon\partial_\tau\tilde{\xi}_1 =& -\left(g\tilde{\eta}_1 + \alpha_1|u|^2 - i\partial_{k_j}\omega D_{X_j}\tilde{\xi}_1\right) - \varepsilon\alpha_2^j\left(\overline{u}D_{X_j}u + u\overline{D_{X_j}u}\right) + O\left(\varepsilon^2\right).\end{aligned}$$

For simplicity of notation, we have dropped the explicit dependency on k_0 of the coefficients α_i and ω. This is the NLS equation in the case $n=2$ and the Davey–Stewartson system in a certain form, in the case $n=3$.

When $n=3$, in order to recover the more usual form of the Davey–Stewartson system, we neglect all the Dysthe terms, obtaining

$$i\partial_\tau u = -\frac{1}{2}\partial^2_{k_jk_l}\omega\partial^2_{X_jX_l}u + 2\alpha_3|u|^2u + \left(\alpha_1\tilde{\eta}_1 - ik_0\cdot D_X\tilde{\xi}_1\right)u, \tag{3.28}$$

$$h|D_X|^2\tilde{\xi}_1 + k_0\cdot\partial_X|u|^2 + \partial_{k_j}\omega\partial_{X_j}\tilde{\eta}_1 = 0, \tag{3.29}$$

$$g\tilde{\eta}_1 + \alpha_1|u|^2 - \partial_{k_j}\omega\partial_{X_j}\tilde{\xi}_1 = 0. \tag{3.30}$$

From Eq. (3.30) we can solve $\tilde{\eta}_1$. After substitution in Eqs. (3.28), (3.29), the system reads

$$\begin{aligned}&i\partial_\tau u = -\frac{1}{2}\partial^2_{k_jk_l}\omega\partial^2_{X_jX_l}u + \alpha_5|u|^2u - \alpha_6^j u\partial_{X_j}\tilde{\xi}_1,\\ &\mathcal{L}\tilde{\xi}_1 + \alpha_6^j\partial_{X_j}|u|^2 = 0,\end{aligned} \tag{3.31}$$

which is a standard form of the Davey–Stewartson system, where

$$\mathcal{L} = h|D_X|^2 + \frac{1}{g}\partial_{k_j}\omega\partial_{k_l}\omega\partial^2_{X_jX_l}$$

is a second-order differential operator with constant coefficients, and

$$\alpha_5 = 2\alpha_3 - \frac{\alpha_1^2}{g}, \quad \alpha_6^j = k_{0j} - \frac{\alpha_1}{g}\partial_{k_j}\omega.$$

When one retains the Dysthe terms, there is no further apparent simplification. However, by construction, the present method provides a conserved Hamiltonian for the above Dysthe system, which takes the form

$$H = \int\Big(\tfrac{1}{2}\partial^2_{k_jk_l}\omega(k_0)\partial_{X_j}\overline{u}\partial_{X_l}u + \alpha_3(k_0)|u|^4 + \frac{h}{2}\tilde{\xi}_1|D_X|^2\tilde{\xi}_1 + \frac{g}{2}\tilde{\eta}_1^2 - i\partial_{k_j}\omega(k_0)\tilde{\eta}_1 D_{X_j}\tilde{\xi}_1 + \alpha_1(k_0)\tilde{\eta}_1|u|^2 - |u|^2 k_{0j}\partial_{X_j}\tilde{\xi}_1 + \frac{\varepsilon}{6}\partial^3_{k_jk_lk_m}\omega(k_0)\Im\Big(\partial_{X_j}\overline{u}\partial^2_{X_lX_m}u\Big) + \varepsilon\Big(\alpha_2^j(k_0)\tilde{\eta}_1 - \tfrac{1}{2}\partial_{X_j}\tilde{\xi}_1 + \alpha_4^j(k_0)|u|^2\Big)\Big(\overline{u}D_{X_j}u + u\overline{D_{X_j}u}\Big)\Big)dX$$

where $\Im$ stands for the imaginary part.

4. The water wave problem with infinite depth

It is of interest to extend the analysis to the water wave problem with infinite depth as it gives a simplified Dysthe system. The only difference in the initial Hamiltonian lies in the form of the Dirichlet–Neumann operator $G_0(D_x)$ for the unperturbed fluid domain given by a flat surface elevation, which now has the form $G_0(D_x) = |D_x|$. As a consequence, it modifies the ordering in $H^{(2)}$ as Eq. (3.21) is now replaced by

$$\int\tilde{\xi}_1 G_0\tilde{\xi}_1 dX = \varepsilon\int\tilde{\xi}_1|D_X|\tilde{\xi}_1 dX. \tag{4.1}$$

This results in a modification of the values of the coefficients $\alpha_i(k_0)$; namely, $\alpha_1(k_0)$ as defined in Eq. (3.23) vanishes, $\alpha_3(k_0) = \frac{|k_0|^3}{4}$ and $\alpha_4^j(k_0) = \frac{3}{8}|k_0|k_{0j}$. The resulting modified Hamiltonian is rewritten as

$$\varepsilon^{n-3}\hat{H} = \varepsilon^2\int\Big(\frac{1}{2}\partial^2_{k_jk_l}\omega(k_0)\overline{u}D^2_{X_jX_l}u + \alpha_3(k_0)|u|^4 + \frac{1}{2}\varepsilon^{2\beta-3}\tilde{\xi}_1|D_X|\tilde{\xi}_1 + \varepsilon^{2\beta-2}\Big(\frac{g}{2}\tilde{\eta}_1^2 - i\partial_{k_j}\omega(k_0)\tilde{\eta}_1 D_{X_j}\tilde{\xi}_1\Big) - i\varepsilon^{\beta-1}|u|^2 k_0\cdot D_X\tilde{\xi}_1 + \frac{\varepsilon}{6}\partial^3_{k_jk_lk_m}\omega(k_0)\overline{u}D^3_{X_jX_lX_m}u + \varepsilon\Big(\alpha_2^j(k_0)\varepsilon^{\beta-1}\tilde{\eta}_1 - \frac{i}{2}\varepsilon^{\beta-1}D_{X_j}\tilde{\xi}_1 + \alpha_4^j(k_0)|u|^2\Big)\Big(\overline{u}D_{X_j}u + u\overline{D_{X_j}u}\Big) + O\Big(\varepsilon^2\Big)\Big)dX. \tag{4.2}$$

The presence of the term $\varepsilon^{2\beta-3}\tilde{\xi}_1|D_X|\tilde{\xi}_1$ in the second line of the RHS of Eq. (4.2) suggests that β should be equal to 2. Under this condition, Eq. (4.2) takes the form

$$\varepsilon^{n-3}\hat{H} = \varepsilon^2\int\Big(\frac{1}{2}\partial^2_{k_jk_l}\omega(k_0)\overline{u}D^2_{X_jX_l}u + \frac{|k_0|^3}{4}|u|^4 + \varepsilon\Big(\frac{1}{6}\partial^3_{k_jk_lk_m}\omega(k_0)\overline{u}D^3_{X_jX_lX_m}u + \frac{1}{2}\tilde{\xi}_1|D_X|\tilde{\xi}_1 - i|u|^2 k_0\cdot D_X\tilde{\xi}_1 + \frac{3}{8}|k_0|k_{0j}|u|^2\Big(\overline{u}D_{X_j}u + u\overline{D_{X_j}u}\Big)\Big) + O\Big(\varepsilon^2\Big)\Big)dX, \tag{4.3}$$

where we have written terms up to $O(\varepsilon)$ inside the integral. The conserved impulse vector valued integral I reduces to

$$I = \int\eta\partial_x\xi dx = \varepsilon^{3-n}\int\Big(k_0|u|^2 + \varepsilon\overline{u}D_X u\Big)dX + O\Big(\varepsilon^{7-n}\Big). \tag{4.4}$$

Proceeding as before, Hamilton's canonical equations are

$$\begin{aligned}\partial_t u &= -i\varepsilon^2\Big[\frac{1}{2}\partial^2_{k_jk_l}\omega(k_0)D^2_{X_jX_l}u + \frac{|k_0|^3}{2}|u|^2u + \varepsilon\Big(\frac{1}{6}\partial^3_{k_jk_lk_m}\omega(k_0)D^3_{X_jX_lX_m}u - ik_0\cdot uD_X\tilde{\xi}_1 + \frac{3|k_0|}{2}|u|^2k_0\cdot D_Xu\Big)\Big] + O\Big(\varepsilon^4\Big),\\ \partial_t\tilde{\eta}_1 &= |D_X|\tilde{\xi}_1 + ik_0\cdot D_X|u|^2 + O(\varepsilon),\\ \partial_t\tilde{\xi}_1 &= O(\varepsilon).\end{aligned} \tag{4.5}$$

Introducing the slow time $\tau = \varepsilon^2 t$ and solving for $\tilde{\xi}_1$ at lowest order in terms of u yield

$$\tilde{\xi}_1 = -i|D_X|^{-1}k_0\cdot D_X|u|^2 + O(\varepsilon), \tag{4.6}$$

which leads to the NLS equation at lowest order, and to a Dysthe-type equation [10] when one considers the next-order corrections. This latter takes the form

$$\begin{aligned}2i\partial_\tau u &= -\partial^2_{k_jk_l}\omega(k_0)\partial^2_{X_jX_l}u + |k_0|^3|u|^2u\\ &\quad + \varepsilon\Big(\frac{i}{3}\partial^3_{k_jk_lk_m}\omega(k_0)\partial^3_{X_jX_lX_m}u - 3i|k_0||u|^2k_0\cdot\partial_X u + 2uk_{0j}k_{0l}|D_X|^{-1}\partial^2_{X_jX_l}|u|^2\Big).\end{aligned} \tag{4.7}$$

We note the presence of the nonlocal operator $|D_X|^{-1} = (-\Delta)^{-1/2}$ in the equation, which is a consequence of the mean flow terms, and which can be expressed using the Hilbert transform.

Again, there is a conserved Hamiltonian for the Dysthe equation that we have derived, both for dimensions $n = 2$ or 3, given by

$$H = \frac{1}{2}\int\left[\partial^2_{k_jk_l}\omega(k_0)\partial_{X_j}u\partial_{X_l}\overline{u} + \frac{|k_0|^3}{2}|u|^4 + \varepsilon\left(\frac{1}{3}\partial^3_{k_jk_lk_m}\omega(k_0)\Im\left(\partial_{X_j}\overline{u}\partial^2_{X_lX_m}u\right) - k_{0j}k_{0l}\left(\partial_{X_j}|u|^2\right)|D_X|^{-1}\partial_{X_l}|u|^2 + \frac{3}{2}|k_0|k_{0j}|u|^2\Im\left(\overline{u}\partial_{X_j}u\right)\right)\right]dX. \tag{4.8}$$

In the above formulas,

$$\partial^2_{k_jk_l}\omega(k_0) = |k_0|^{-1}\left(\delta_{jl} - \frac{k_{0j}k_{0l}}{|k_0|^2}\right),$$

and

$$\partial^3_{k_jk_lk_m}\omega(k_0) = -|k_0|^{-3}\left(\delta_{jl}k_{0m} + \delta_{lm}k_{0j} + \delta_{mj}k_{0l} + \frac{k_{0j}k_{0l}k_{0m}}{|k_0|^2}\right).$$

It can be checked that evaluating

$$\partial_\tau u = -i\delta_{\overline{u}}H,$$

using the closed form (4.8) of the Hamiltonian yields Eq. (4.7).

In order to compare our Eq. (4.7) in Hamiltonian form with the Dysthe-type equations that have previously appeared in the literature, introduce the new variables

$$\psi = \frac{1}{\sqrt{2}}\left(\frac{|k_0|}{g}\right)^{1/4}\left(1 + \frac{\varepsilon}{4|k_0|^2}k_0\cdot D_X\right)u, \tag{4.9}$$

which is similar to Stiassnie [11]. This variable can be thought of as a first-order approximation of η as given by Eq. (3.4), which is to say that $\psi(X)$ is directly related to the free-surface elevation. Inverting Eq. (4.9), inserting it in Eq. (4.7) and retaining only $O(\varepsilon)$ terms, leads to the following equation

$$\begin{aligned} 2i\partial_\tau\psi = & -\partial^2_{k_jk_l}\omega(k_0)\partial^2_{X_jX_l}\psi + 2g^{1/2}|k_0|^{5/2}|\psi|^2\psi \\ & + \varepsilon\left(\frac{i}{3}\partial^3_{k_jk_lk_m}\omega(k_0)\partial^3_{X_jX_lX_m}\psi - 6ig^{1/2}|k_0|^{1/2}|\psi|^2k_0\cdot\partial_X\psi - ig^{1/2}|k_0|^{1/2}\psi^2k_0\cdot\partial_X\overline{\psi} + 4g^{1/2}|k_0|^{-1/2}\psi k_{0j}k_{0l}|D_X|^{-1}\partial^2_{X_jX_l}|\psi|^2\right). \end{aligned} \tag{4.10}$$

This equation contains all the usual Dysthe terms including the high-order nonlinear term in $\psi^2k_0\cdot\partial_X\overline{\psi}$ as can be seen in [10,11,14–16]. The Hamiltonian structure is however lost in the transformation to the ψ-variables. We also note that the numerical coefficients in Eq. (4.10) are very similar to those in the equation derived by Stiassnie [11]. However, in general, we should not expect to obtain precisely the same coefficients as in previous work, since the various approaches do not use precisely the same physical variables (in particular the choice of the velocity potential). Our own choice of physical variables is quite natural, consisting of the canonically conjugate variables η and ξ as related to $(z,\bar{z},\tilde{\eta},\xi)$ through Eqs. (3.4) and (3.5).

Therefore, our versions of the Dysthe equation can be viewed as new model equations in the sense that, although they agree with existing versions on their general form, details such as their numerical coefficients and the relation of their dependent variables to the original physical variables are different, and more importantly they possess a well-defined Hamiltonian structure which is consistent with the Hamiltonian formulation of the Euler equations for water waves.

References

[1] M.J. Ablowitz, H. Segur, Solitons and the inverse scattering transform, SIAM Studies in Applied Mathematics, Philadelphia, 1981.
[2] A.C. Newell, J.V. Moloney, Nonlinear Optics, Addison-Wesley, 1992.
[3] V.E. Zakharov, S.L. Musher, A.M. Rubenchik, Hamiltonian approach to the description of non-linear plasma phenomena, Phys. Rep. 129 (1985) 285–366.
[4] D.J. Benney, A.C. Newell, The propagation of nonlinear wave envelopes, J. Math. Phys. 46 (1967) 133–139.
[5] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Appl. Mech. Tech. Phys. 9 (1968) 190–194.
[6] D.J. Benney, G.J. Roskes, Wave instabilities, Stud. Appl. Math. 48 (1969) 377–385.
[7] H. Hasimoto, H. Ono, Nonlinear modulation of gravity waves, J. Phys. Soc. Jpn 33 (1972) 805–811.
[8] M.J. Ablowitz, H. Segur, On the evolution of packets of water waves, J. Fluid Mech. 92 (1979) 691–715.
[9] C. Sulem, P.-L. Sulem, The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse, Series in Mathematical Sciences, vol. 139, Springer-Verlag, 1999.

[10] K.B. Dysthe, Note on a modification of the nonlinear Schrödinger equation for application to deep water waves, Proc. R. Soc. Lond. Ser. A 369 (1979) 105–114.
[11] M. Stiassnie, Note on the modified nonlinear Schrödinger equation for deep water waves, Wave Motion 6 (4) (1984) 431–433.
[12] U. Brinch-Nielsen, I.G. Jonsson, Fourth-order evolution equations and stability analysis for Stokes waves on arbitrary water depth, Wave Motion 8 (1986) 455–472.
[13] D.S. Agafontsev, F. Dias, E.A. Kuznetsov, Deep-water internal solitary waves near critical density ratio, Physica D 225 (2) (2007) 153–168.
[14] P.A.E.M. Janssen, On a fourth-order envelope equation for deep-water waves, J. Fluid Mech. 126 (1983) 1–11.
[15] E. Lo, C.C. Mei, Numerical study of water-wave modulation based on a higher-order nonlinear Schrödinger equation, J. Fluid Mech. 150 (1985) 395–416.
[16] M.J. Ablowitz, J. Hammack, D. Henderson, C.M. Schober, Long-time dynamics of the modulational instability of deep water waves, Physica D 152–153 (2001) 416–433.
[17] G.B. Whitham, Linear and nonlinear waves, Wiley-Interscience, 1974.
[18] T.B. Benjamin, Impulse, flow-force and variational principles, IMA J. Appl. Math. 32 (1984) 2–68.
[19] W. Craig, M.D. Groves, Hamiltonian long-wave approximation to the water-wave problem, Wave Motion 19 (1994) 367–389.
[20] W. Craig, P. Guyenne, H. Kalisch, Hamiltonian long-wave expansions for free surfaces and interfaces, Commun. Pure Appl. Math. 58 (2005) 1587–1641.
[21] W. Craig, C. Sulem, P.-L. Sulem, Nonlinear modulation of gravity waves: a rigorous approach, Nonlinearity 5 (1992) 497–522.
[22] W. Craig, P. Guyenne, D.P. Nicholls, C. Sulem, Hamiltonian long-wave expansions for water waves over a rough bottom, Proc. R. Soc. Lond. Ser. A 461 (2005) 839–873.
[23] W. Craig, C. Sulem, Numerical simulation of gravity waves, J. Comput. Phys. 108 (1993) 73–83.