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Instability in supercritical nonlinear wave equations: Theoretical results and symplectic integration

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Abstract

Nonlinear wave evolutions involve a dynamical balance between linear dispersive spreading of the waves and nonlinear self-interaction of the waves. In sub-critical settings, the dispersive spreading is stronger and therefore solutions are expected to exist globally in time. We show that in the supercritical case, the nonlinear self-interaction of the waves is much stronger. This leads to some sort of instability of the waves. The proofs are based on the construction of high frequency approximate solutions. Preliminary numerical simulations that support these theoretical results are also reported.

Keywords: Nonlinear wave equations; Supercritical equations; Ill-posedness

1. Introduction

In this paper, we discuss some *ill-posedness* issues (mainly, established in [1,5]) for solutions of the *supercritical* non-linear wave equation

$$\partial_t^2 u - \Delta u + f(u) = 0 \ , \tag{1}$$

where $u = u(t, x) : \mathbb{R}^{1+d} \to \mathbb{R}$. We assume that the initial data

$$\gamma := \partial u_{|t=0} = (\nabla u, \partial_t u)_{|t=0} \tag{2}$$

is in the *homogeneous* Sobolev space $\dot{H}^{s-1}$ endowed with the norm

$$\|\gamma\|^2_{\dot{H}^{s-1}} := \int_{\mathbb{R}^d} |\xi|^{2(s-1)} |\hat{\gamma}(\xi)|^2 \ d\xi.$$

The nonlinear interaction f satisfies $f(0) = 0$ and it is supposed of the form $f = \partial V/\partial \bar{z}$ with a *potential* $V \in C^\infty(\mathbb{C}; \mathbb{R})$. This assumption on f formally guarantees the *conservation of the energy*

$$E(u(t)) := \int_{\mathbb{R}^d} |\partial u|^2 \, dx + \int_{\mathbb{R}^d} V(u) \, dx \ = E(u(0)). \tag{3}$$

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First, we recall our definition of *well-posedness* of the Cauchy problem.

Definition. We say that the Cauchy problem (1) and (2) is *locally well-posed* in $\dot{H}^s(\mathbb{R}^d)$ if for any bounded subset $\mathcal{B} \subset \dot{H}^{s-1}$, there exists a time $T = T(\mathcal{B}) > 0$ such that for every $\gamma \in \mathcal{B}$ there exists a (distributional) solution u to (1);

$$\partial u \in \mathcal{C}([-T, T]; \dot{H}^{s-1}),$$

and such that the solution map $\gamma \mapsto \partial u$ is uniformly continuous from $\mathcal{B}$ to $\mathcal{C}([-T, T]; \dot{H}^{s-1})$. Moreover, there is an additional space X in which u lies, such that u is the unique solution to the Cauchy problem in $\mathcal{C}([-T, T]; \dot{H}^s) \cap X$.

Second, we define the notion of *supercriticality*.

Definition. Let $s \in [0, d/2)$. The Cauchy problem (1) and (2) is said $\dot{H}^s$-*supercritical* if

$$(\mathbf{H}_{\mathrm{s,d}}) \qquad \frac{V(u)}{|u|^{(4/(d-2s))+2}} \uparrow +\infty \quad \text{as} \quad u \to +\infty.$$

To better illustrate the above condition let's first refer to the "model" case when the nonlinearity is given by

$$f(u) = |u|^{2\sigma} u. \tag{4}$$

In such a case, solutions to the nonlinear wave equation enjoy a scaling property. Indeed, if u solves (1) on $(-T^*, T^*)$ with initial data $\partial u_{|t=0} \in \dot{H}^{s-1}$, then for any $\lambda > 0$, the function $u^\lambda : (-T^*\lambda^2, T^*\lambda^2) \times \mathbb{R}^d$ defined by $u^\lambda(t, x) := \lambda^{-1/\sigma} u(\lambda^{-2}t, \lambda^{-2}x)$ also does. Moreover, the norm of the Sobolev space $\dot{H}^{s_c}$ with $s_c = (d/2) - (1/\sigma)$ is also invariant under the dilation $u \mapsto u^\lambda$. It turns out that this space is relevant in the theory of the initial value problem (1) and (2). Obviously, when f is given by (4), the *supercriticality* condition ($\mathbf{H}_{s,d}$) is equivalent to $s < s_c$.

Finally, recall that in the power nonlinearity case, problem (1) and (2) is locally well-posed for $s > s_c$ with an existence time interval depending only upon $\|\gamma\|_{\dot{H}^{s-1}}$, see [3]. It is also locally well-posed for $s = s_c$ with an existence time interval depending upon the solution to the linear wave equation $U_0(t)\gamma$, and is ill-posed for $s < s_c$ (see the reference work of [2], and the thorough paper of [6] concerning the loss of regularity of $\dot{H}^s$- *supercritical* waves). Based on this complete trichotomy, it is natural to refer to $\dot{H}^{s_c}$ as the *critical* regularity for (1). For more details, we refer to Tao's book [8] and the references therein. The case of $\dot{H}^1$-*supercritical* problems is of particular interest since in such a case, in addition, solutions enjoy the conservation of the energy (3). Moreover in this case, the definition of $\dot{H}^1$-*supercritical* problems ($H_{s,d}$) has to be extended when the space dimension $d = 2$. Indeed, energy-critical nonlinearities seem to be of exponential type.[1]

$$(\mathbf{H}_{1,2}) \qquad \begin{cases} (a) & \dfrac{\log(V(u))}{u^2} \quad \text{increases to} + \infty \\ (b) & liminf \dfrac{\log(V(u))}{u^2} = 4\pi \ \text{ and } \ E > 1. \end{cases}$$

The goal of this paper is to present some extensions of the results in [2,6] to $\dot{H}^s$-*supercritical* problems with more general nonlinearities given by ($H_{s,d}$). Similar results are also proved for the energy-*supercritical* problems in two space dimension. The last section is devoted to numerical simulations that show the change of dynamics through the different regimes. We refer to [1,5] for analogous results for nonlinear *Schrödinger* equations.

2. Ill posedness of $\dot{H}^s$ supercritical waves

The main result in this case is given in [1].

[1] In fact, the critical nonlinearity is of exponential type as shown in [4,5].

Theorem 2.1. *Let* $1 < s < s_{sob}$.[2] *One can find a sequence of times* $0 < t_n \ll 1$ *and a sequence of initial data* $(u_{0,n}, 0)$ *with bounded energy, and* $\lim_{n\to\infty} \|u_{0,n}\|_{H^s} = 0$, *so that the solution to* (1) *with initial data* $u_{0,n}$ *satisfies*:

$$\lim_{n\to\infty} \|u_n(t_n, \cdot)\|_{H^s} \to \infty.$$

In particular, for any $t > 0$, *the solution map* $(u_0, u_1) \to u(t)$ *fails to be continuous at the initial data* (0,0) *in the* H^s*-topology.*

Remark. Since the total energy of the solutions constructed in the above Theorem is bounded (with respect to n), the above result also applies to finite energy weak solutions to (1)(which are known to exist). This extends the result established in [2] where the energy is unbounded.

Sketch of the proof. For the sake of simplicity, we shall give only the proof when $d = 3$ and $f(u) = u^7$. Following the same steps, the proof extends to the other cases.

Let Φ denote the (real valued), smooth and periodic solution to the ordinary differential equation

$$\Phi'' + \Phi^7 = 0, \qquad \Phi(0) = 1, \qquad \Phi'(0) = 0. \tag{5}$$

Let $v_{1,n}(x) = 0$ and $v_{0,n} = \kappa_n n^{s/3} \varphi(nx_1, n^\beta x')$ where φ is a non-negative smooth function supported in the unit ball, $\beta = \beta(s) := (4/3)s - (1/2)$ and $\kappa_n = \log^{-\delta}(n)$ with $\delta > 0$ to be fixed later. Observe that thanks to the anisotropic scaling in the initial data and the choice β, we have for $s < s_{sob}$

$$\|u_{n,0}\|_{H^s(\mathbb{R}^3)} + \|u_{n,0}\|_{L^8(\mathbb{R}^3)} \sim \kappa_n.$$

On the other hand, note that

$$v_n(t, x) = \kappa_n n^{s/3} \varphi(nx_1, n^\beta x') \Phi(t|\kappa_n n^{s/3} \varphi(nx_1, n^\beta x')|^3) \tag{6}$$

solves $\partial_t^2 v_n + v_n^7 = 0$ with initial data $(v_{0,n}, v_{1,n})$.

A straightforward but tedious computation shows that for all $t \neq 0$ and all integer $\sigma \geq 0$, we have

$$\|v_n(t)\|_{H^\sigma} \sim \kappa_n (\kappa_n^3 t n^s)^\sigma n^{\sigma - s}, \quad \text{as } n \to +\infty.$$

In particular, for all t such that $\kappa_n (\kappa_n^3 t n^s)^s \gg 1$ we have $\|v_n(t, \cdot)\|_{H^s} \gg 1$.

The next step is to show that v_n is a suitable Ansatz to approximate u_n for times much larger than $\kappa_n^{-1/s} \kappa_n^{-3} n^{-s}$. Set

$$E_n(f) := [n^{s-1} E_0^{1/2}(f) + n^{s-3} E_0^{1/2}(\partial_{x_1}^2 f) + n^{s-2\beta-1} E_0^{1/2}(\partial_{x'}^2 f)]. \tag{7}$$

Thanks to Poincaré inequality, one has $\|f\|_{H^s} \leq C E_n(f)$.

Lemma 2.2. *Let* $t_n = n^{-s} \log^{1/8}(n)$. *Then the solution* u_n *of* (1) *with the initial data* $(v_{0,n}, v_{1,n})$ *exists for* $0 \leq t \leq t_n$ *and there exists* $\varepsilon > 0$ *such that for* $t \in [0, t_n]$,

$$\|u_n(t) - v_n(t, \cdot)\|_{H^s} \leq C E_n(u_n(t) - v_n(t)) \leq C n^{-\varepsilon}.$$

The proof of the above lemma is based on the standard energy estimate on $w_n := u_n - v_n$ together with an anisotropic Gagliardo–Nirenberg inequality[3] to get

$$\frac{d}{dt} E_n(w) \leq C n^{2s-1} \log^{1/2}(n)(E_n(w) + E_n^7(w)) + Cn \log^2(n). \tag{8}$$

Assuming that $E_n(w) \ll 1$, leads to

$$E_n(w) \leq n^{2-2s} \log^{3/2}(n) e^{C t n^{2s-1} \log^{1/2}(n)}.$$

[2] $s < s_{sob} := \sigma d / 2(\sigma + 1)$.

[3] see [1] for the precise statement and more details.

For every $\gamma > 0$ there exists C_γ such that for $t \in [0, t_n]$,

$$C\, t\, n^{2s} \log^{1/2}(n) \le C \log^{3/4}(n) \le \gamma \log n + C_\gamma.$$

Since $s > 1$, by taking $\gamma > 0$ small enough, we deduce that there exists $\varepsilon > 0$ such that for $t \in [0, t_n]$, we have

$$E_n(w) \le C\, n^{-\varepsilon}. \tag{9}$$

Finally the usual bootstrap argument allows to drop the assumption $E_n(w) \le 1$. This completes the proof of Lemma 2.2, and therefore Theorem 2.1 follows.

3. Energy-supercritical equations in dimension 3 or higher

In this section, we assume that $d \ge 3$ and the nonlinearity is energy-*supercritical*:

$$(\mathbf{H}_{1,\mathrm{d}}) \qquad \frac{V(u)}{u^{(4/(d-2))+2}} \uparrow +\infty \quad \text{as} \quad u \to +\infty, \quad d \ge 3.$$

Then we have

Theorem 3.1. *A sequence (φ_k) in $\dot{H}^1$ and a sequence (t_k) satisfying*

$$\|\nabla \varphi_k\|_{L^2_x} \to 0, \quad \sup_k\ E(\varphi_k) < \infty, \quad t_k \to 0,$$

exist and such that any weak solution u_k of (1) with initial data $(\varphi_k, 0)$ satisfies

$$liminf_{k\to+\infty} \|\partial_t u_k(t_k)\|_{L^2_x} \gtrsim 1.$$

Sketch of the proof. The proof follows almost the same strategy as for the previous result. The main difference here is that we will be more specific in our choice of the initial data, and therefore one can get more quantitative information on the solutions to the corresponding ODE.

Define a sequence of continuous functions φ_k, supported in the unit ball such that $\varphi_k(x) = k^{(d-2)/2}$ if $|x| \le \varepsilon/k$, and $\varphi_k(x) = \varepsilon^{d-2} k^{(d-2)/2}/(k^{d-2} - \varepsilon^{d-2})(|x|^{2-d} - 1)$ if $\varepsilon/k \le |x| \le 1$.

An easy computation (using assumption $(\mathbf{H}_{1,\mathrm{d}})$) yields $\|\nabla \varphi_k\|^2_{L^2} \lesssim (\varepsilon^{d-2} k^{d-2})/(k^{d-2} - \varepsilon^{d-2})$, and

$$\int_{\mathbb{R}^d} V(\varphi_k(x))\ dx \lesssim V(k^{(d-2)/2}) \left(\frac{\varepsilon}{k}\right)^d \left(1 + \frac{1 - (\varepsilon/k)^d}{(1 - (\varepsilon/k)^{d-2})^{2d/(d-2)}}\right),$$

respectively. The choice $\varepsilon = \varepsilon_k \overset{def}{=} k(V(k^{(d-2)/2}))^{-1/d}$ together with the fact that $k(V(k^{(d-2)/2}))^{-1/d} \to 0$, guarantees that $\|\nabla \varphi_k\|_{L^2} \to 0$ and $\sup_k\ E(\varphi_k) < \infty$.

Next, we consider the solution to the ordinary differential equation

$$\Phi''(t) + V'(\Phi(t)) = 0, \qquad (\Phi, \Phi')(0) = (k^{\frac{d-2}{2}}, 0). \tag{10}$$

It is well known that the period T_k of Φ_k is given by

$$T_k = 2\sqrt{2} \int_0^{k^{(d-2)/2}} \frac{d\tau}{\sqrt{V(k^{(d-2)/2}) - V(\tau)}} = 2\sqrt{2} \frac{k^{(d-2)/2}}{\sqrt{V(k^{(d-2)/2})}} \int_0^1 \left(1 - \frac{V(\tau k^{(N-2)/2})}{V(k^{(N-2)/2})}\right)^{-1/2} d\tau.$$

Using the *supercriticality* assumption $(\mathbf{H}_{1,\mathrm{d}})$, we obtain $T_k \lesssim k^{(d-2)/2} (V(k^{(V-2)/2}))^{-1/2}$. It follows that $T_k \ll \varepsilon_k/k$.

Recall that by finite speed of propagation, any weak solution u_k of (1) with data $(\varphi_k, 0)$ satisfies $u_k(t, x) = \Phi_k(t)$ if $0 < t < \varepsilon_k/k$ and $|x| < (\varepsilon_k/k) - t$. Now let t_k such that $\Phi_k(t_k) = k^{(1-d)/2}$, i.e. $t_k = 1/\sqrt{2} \int_{k^{1-(d/2)}}^{k^{(d/2)-1}} (V(k^{(d-2)/2}) - V(\tau))^{-1/2}\ d\tau$. It is clear that $t_k \ll \varepsilon_k/k$ and, for $|x| < (\varepsilon_k/k) - t_k$,

$$|\partial_t u_k(t_k, x)| = \sqrt{2} \sqrt{V(k^{(d-2)/2}) - V(\Phi_k(t_k))} \gtrsim \sqrt{V(k^{(d-2)/2})}.$$

Hence,

$$\|\partial_t u_k(t_k)\|_{L^2}^2 \gtrsim V(k^{(d-2)/2})\left(\frac{\varepsilon_k}{k} - t_k\right)^d = \left(\frac{\varepsilon_k}{k}\right)^d V(k^{(d-2)/2})\left(1 - t_k\frac{k}{\varepsilon_k}\right)^d,$$

and the conclusion follows.

Remark. In a recent paper [9], Tao has proved that a logarithmically *supercritical* quintic wave equation (in three space dimensions) is well-posed in $H^{1+\varepsilon}$ for all $\varepsilon > 0$. The above result shows that $\varepsilon = 0$ is not allowed.

4. Energy-supercritical nonlinear equations in 2D

First we start with the first type of *supercritical* nonlinearity given by $(\mathbf{H}_{1,2})$(a), i.e. $(\log(V(u))/u^2)$ increases to infinity as $u \to +\infty$. Examples of such nonlinearities include equations of the type

$$\partial_t^2 u - \Delta u + u^{2m-1} e^{\lambda u^{2m}} = 0, \quad \text{for any} \quad m \geq 2 \quad \text{and}\, \lambda > 0.$$

We have the same result as in higher dimension.

Theorem 4.1. *There exist a sequence* (φ_k) *in* $\dot{H}^1$ *and a sequence* (t_k) *satisfying*

$$||\nabla\varphi_k||_{L^2_x} \to 0, \quad \sup_k E(\varphi_k) < \infty, \quad t_k \to 0,$$

and such that any weak solution u_k *of* (1) *with initial data* $(\varphi_k, 0)$ *satisfies*

$$liminf_{k\to+\infty} ||\partial_t u_k(t_k)||_{L^2_x} \gtrsim 1.$$

Sketch of the proof. The proof goes along the same lines as the previous proof. We omit the details here.

Define a sequence of continuous functions supported in the unit ball φ_k which is the constant $\sqrt{k}$ if $|x| \leq \varepsilon_k\ e^{-k/2}$ and, $(-2\sqrt{k}/\log(V(\sqrt{k})))\log|x|$ if $\varepsilon_k\, e^{-k/2} \leq |x| \leq 1$. The scale ε_k is chosen $\varepsilon_k = e^{k/2}\,(V(\sqrt{k}))^{-1/2}$. Observe that by assumption $(\mathbf{H}_{1,2})$(a), we have $\varepsilon_k \to 0$ and

$$||\nabla\varphi_k||_{L^2}^2 \lesssim \frac{-1}{\log\varepsilon_k}.$$

Moreover, using the following lemma, one obtains the uniform bound on the potential term and therefore a bound on the energy.

Lemma 4.2. *Let* $0 < a < 1$, *then*

$$\int_a^1 r\, e^{4a^2\log^2 r}\, \mathrm{d}r \leq 2.$$

Next, let Φ_k solve the ODE (10) with initial data $(\sqrt{k}, 0)$. Then the following lemma

Lemma 4.3. *For any* $A > 1$, *we have*

$$\int_0^A \frac{du}{\sqrt{e^{A^2} - e^{u^2}}} \leq 2e^2\frac{A}{A^2-1}\, e^{-A^2/2}, \tag{11}$$

together with the *supercriticality* assumption, imply that the period T_k of Φ_k satisfies $T_k \ll \varepsilon_k\, e^{-k/2}$. Choosing t_k such that $\Phi_k(t_k) = (1/\sqrt{k})$, and arguing exactly in the same manner as in the previous case we finish the proof. We refer to [5] for all the details and proofs.

The second result concerns the case $(\mathbf{H}_2)$(b).

Theorem 4.4. *There exists a sequence of positive real numbers* (t_k) *tending to zero and two sequences* (U_k) *and* (V_k) *of solutions of the nonlinear Klein–Gordon equation*

$$\Box u + ue^{4\pi u^2} = 0 \tag{12}$$

satisfying the following:

$$||(U_k - V_k)(t=0,\cdot)||^2_{H^1} + ||\partial_t(U_k - V_k)(t=0,\cdot)||^2_{L^2} = \circ(1) \text{ as } k \to +\infty.$$

For any $\nu > 0$,

$$0 < E(U^k, 0) - 1 \leq e^3\nu^2 \text{ and } 0 < E(V^k, 0) - 1 \leq \nu^2,$$

and

$$liminf_{k\to\infty}||\partial_t(U_k - V_k)(t_k,\cdot)||^2_{L^2} \geq \frac{\pi}{4}(e^2 + e^{3-8\pi})\nu^2.$$

Remark. This result shows that the solution map (for *supercritical* problems) is not uniformly continuous. This result is weaker than the first two results. One cannot expect showing the discontinuity of the solution map at the zero initial data, simply because the zero data is not *supercritical*.

5. Numerical methods

We solve the Klein–Gordon equation in $1 + d = 1 + 2$ dimensions using the fourth-order symplectic "Position Extended Forest-Ruth Like (PEFRL)" algorithm of Omelyan et al. [7]. This is accomplished in the following way. First, the equation is recast into the Hamiltonian form

$$\partial_t u = \delta_v \mathcal{H} = v \qquad \partial_t v = -\delta_u \mathcal{H} = f(u), \tag{13}$$

where $\mathcal{H}(u, v) = (1/2)E(u, v)$ is the Hamiltonian of the system, and

$$f(u) = \Delta u - u^3 e^{u^4} \quad \text{or} \quad f(u) = \Delta u - ue^{4\pi u^2}. \tag{14}$$

Because of the "position-velocity" formulation of (13) and the fact that the RHS of the equation for $\partial_t v$ (i.e. $f(u)$) is a function of u alone, Eq. (13) is well suited to integration by the PEFRL scheme which has been successfully used in molecular dynamics simulations.

The PEFRL algorithm has the advantage of being both high-order and explicit. Omelyan et al. [7] showed that this algorithm is much more accurate than the standard fourth-order Forest-Ruth algorithm (although they have the same truncation error), and that it performs very well for long-time simulations. Although the latter aspect is not of central importance in this study, we found it natural to use a symplectic integrator, motivated by the fact that the Klein–Gordon equation is a Hamiltonian system and that energy conservation is an important issue here.

Spatial derivatives and nonlinear terms in $f(u)$ are computed using a Fourier pseudospectral method with the fast Fourier transform, thus assuming periodic boundary conditions in the x-direction. More precisely, spatial differentiation is performed in Fourier space while nonlinear products are evaluated in physical space. The Fourier pseudospectral method in space combined with the PEFRL algorithm in time makes the overall numerical solution very efficient and accurate.

6. Numerical results

We performed computations, typically with domain size $L_x = L_y = 5$ and resolution $N_x = N_y = 256$ points in the two horizontal directions (denoted by x and y here). Because we are interested in wave dynamics over very short times, the time step was chosen in the range $\Delta t = 10^{-3}$– 10^{-5} in order to properly resolve short-scale variations. In all of our experiments, the initial condition was chosen to be

$$u_0(x, y) = A\, e^{-30((x-L_x/2)^2+(y-L_y/2)^2)}, \qquad v_0(x, y) = 0, \tag{15}$$

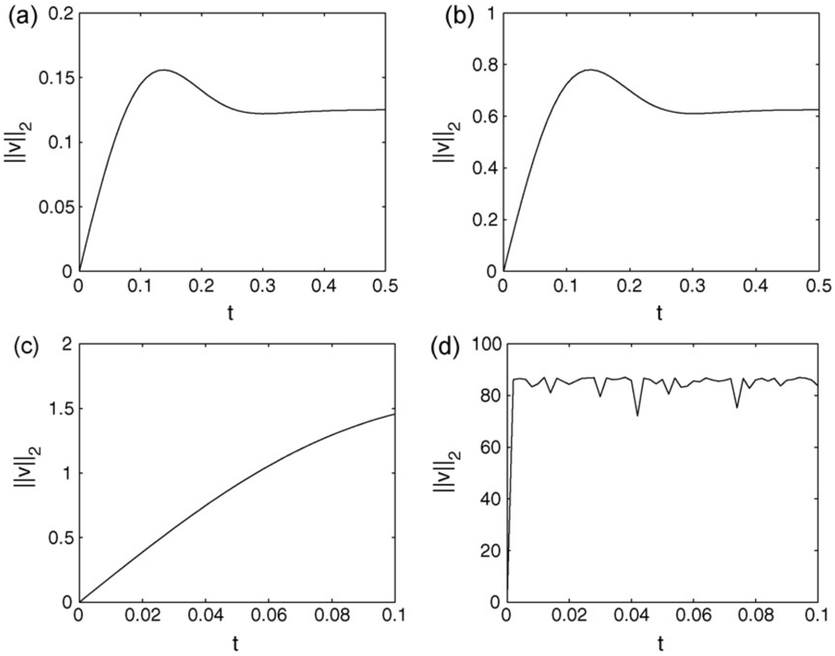

Fig. 1. Norm $||\partial_t u||_{L^2_x}$ vs. time for (a) $A = 0.1$ ($E = 0.0314$), (b) $A = 0.5$ ($E = 0.7862$), (c) $A = 1$ ($E = 3.1588$) and (d) $A = 2$ ($E = 7808.7309$).

i.e. a localized (symmetric) profile located in the center of the domain. We checked that energy is well conserved in all the cases we considered, and that convergence is achieved with respect to the numerical parameters.

The goal of this section is to test numerically the results of Theorems 4.1 and 4.4. In the case $f(u) = \Delta u - u^3 e^{u^4}$ (Theorem 4.1), Fig. 1 shows the time evolution of norm $||v||_{L^2_x} = ||\partial_t u||_{L^2_x}$ for $A = 0.1, 0.5, 1, 2$ (corresponding to

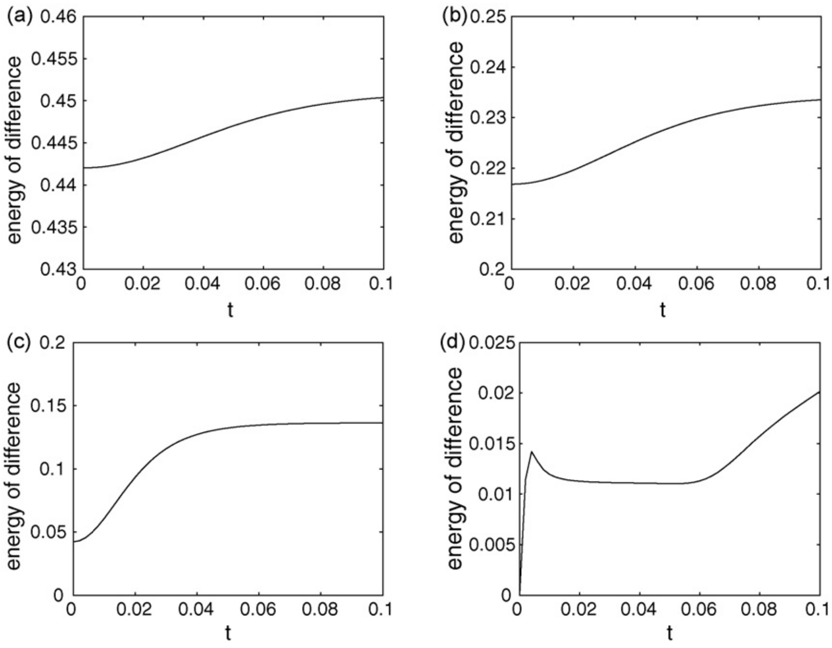

Fig. 2. Energy of difference of solutions $E(U - V)$ vs. time between (a) $E = 0.0319$ and $E = 0.7229$, (b) $E = 0.2339$ and $E = 0.9340$, (c) $E = 1.0991$ and $E = 1.7948$ and (d) $E = 13.9715$ and $E = 14.6969$.

$E = 0.0314, 0.7862, 3.1588, 7808.7309$ respectively). We see that for $A \geq 1$ this quantity increases from 0 to reach values higher than 1 over a very short time interval, which agrees well with Theorem 4.1. For lower amplitudes ($A = 0.1, 0.5$), $||v||_{L_x^2}$ still increases but then stabilizes around a value less than 1 (the order of magnitude of this value is nonetheless fairly close to 1). This may be explained by the fact that, for low amplitudes, $||v||_{L_x^2}$ is smaller and increases more slowly, and thus numerical diffusion (even small) in our experiments can play against it.

In the case $f(u) = \Delta u - ue^{4\pi u^2}$ (Theorem 4.4), we were unable to confirm to the full extent the result of Theorem 4.4. We observed numerically that two close solutions U and V with individual energies $E > 1$ can yield a norm $||\partial_t(U - V)||_{L^2}^2$ which does not satisfy the estimate of Theorem 4.4 for supercritical waves. The reasons for this are still unknown to us and work in this direction is in progress. Still, in an attempt to characterize the change of dynamics between sub and supercritical solutions (i.e. between $E < 1$ and $E > 1$), we plot in Fig. 2 the time evolution of energy $E(U - V)$ between two relatively close solutions U and V within the same regime. Cases (a) and (b) correspond to subcritical solutions ($E < 1$) while cases (c) and (d) correspond to supercritical solutions ($E > 1$). In these four cases, the two solutions differ by the same amount of energy $\Delta E \sim 0.7$, which was chosen not too small otherwise the time scale of the dynamics would be prohibitively too small to be examined numerically (see Theorem 4.4).

We see in Fig. 2(a) and (b) that $E(U - V)$ increases only by a few percent (2% and 8% respectively) on the interval $t \in [0, 0.1]$, while it varies much faster by 222% and 69937% in Fig. 2(c) and (d) respectively. These numerical results clearly support the idea that the regime $E > 1$ is highly nonlinear and differs in substance from that for $E < 1$. Further investigation is needed in order to better characterize the sub/supercritical transition and this is envisioned for a future work.

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References

[1] N. Burq, P. Gérard, S. Ibrahim, Instability results for nonlinear Schrödinger and wave equations, preprint.
[2] M. Christ, J. Colliander, T. Tao, Ill-posedness for nonlinear Schrödinger and wave equations, preprint 2003.
[3] J. Ginibre, G. Velo, The global Cauchy problem for nonlinear Klein–Gordon equation, Math. Z. 189 (1985) 487–505.
[4] S. Ibrahim, M. Majdoub, N. Masmoudi, Global solutions for a semilinear 2D Klein–Gordon equation with exponential type nonlinearity, Commun. Pure Appl. Math. 59 (2006) 1639–1658.
[5] S. Ibrahim, M. Majdoub, N. Masmoudi, Ill-posedness of H^1 supercritical waves, C. R. Math. Acad. Sci. Paris 345 (2007) 133–138.
[6] G. Lebeau, Perte de régularité pour les équations d'ondes sur-critiques, Bull. Soc. Math. France 133 (2005) 145–157.
[7] I.P. Omelyan, I.M. Mryglod, R. Folk, Optimized Forest-Ruth- and Suzuki-like algorithms for integration of motion in many-body systems, Comput. Phys. Commun. 146 (2002) 188–202.
[8] T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis, CBMS Regional Conference Series in Mathematics, AMS, Providence, RI, 2006. xvi + 373 pp, (ISBN: 0-8218-4143-2).
[9] T. Tao, Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data, J. Hyperbolic Differ. Eq. 4 (2007) 259–265.