Numerical Homogenization of the Time – Harmonic Acoustics of Bone: The Monophasic Case

Ming Fang

Department of Mathematics, Norfolk State University, Norfolk, VA 23504, USA

Robert P. Gilbert, Philippe Guyenne & Ana Vasilic

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

ABSTRACT

In the predecessor to this work, we undertook a derivation of the time-harmonic, acoustic equations, idealizing the bone as a periodic arrangement of a Kelvin-Voigt viscoelastic porous matrix containing a viscous fluid, where we assumed that the fluid was slightly compressible. The effective equations for the monophasic vibrations were obtained, and existence and uniqueness was proved. In the current article, we perform numerical experiments, assuming that the trabeculae are isotropic.

KEYWORDS

two scale convergence, time harmonic waves, viscoelasticity of Kelvin-Voigt

*Address all correspondence to mfang@nsu.edu

1. INTRODUCTION

In [1] we developed a theory for the harmonic excitation of cancellous bone where the bone fluid and the elastic matrix move together: the monophasic case.[1]

Using the method of homogenization, we described the microstructure of the composite material, bone plus blood-marrow, in terms of a cell problem, where all ingredients exist in equilibrium. The two phases of material are assumed to have the following constitutive equations:

$$\sigma^{\varepsilon} = \theta^{\varepsilon}\sigma^{f,\varepsilon} + (1-\theta^{\varepsilon})\sigma^{s,\varepsilon} \tag{1.1}$$

The viscoelastic behavior of the trabeculae is modeled by a Kelvin-Voigt constitutive equation

$$\sigma_{ij}^{s,\varepsilon} = (A^s + i\omega B^s)_{ijkl} e(u^{\varepsilon})_{kl} \tag{1.2}$$

Here ω is the wave frequency and $e(u^{\varepsilon})$ is the strain tensor defined by

$$e(u^{\varepsilon})_{ij} = \frac{1}{2}(\partial_i u_j^{\varepsilon} + \partial_j u_i^{\varepsilon}) \quad i,j = 1,2,3$$

The constants A^s_{ijkl} are the elasticity coefficients of the solid and are assumed to have the classical symmetry and positivity conditions. The constants B^s_{ijkl} describe viscosity of the solid, with the classical symmetry and positivity conditions.

The marrow was modeled as a slightly compressible viscous barotropic fluid with the constitutive equations

$$\sigma_{ij}^{f,\varepsilon} = (A^f + i\omega B^f)_{ijkl} e(u^{\varepsilon})_{kl} \tag{1.3}$$

In (1.3),

$$\begin{aligned} A^f_{ijkl} &= c^2\rho_f\delta_{ij}\delta_{kl} \\ B^f_{ijkl} &= 2\eta\delta_{ik}\delta_{jl} + \xi\delta_{ij}\delta_{kl} \end{aligned} \tag{1.4}$$

Here, c is the sound speed, $\rho_f > 0$ is a constant density of the marrow at rest, and η, ξ are constant viscosities, which are subject to the following conditions:

$$\eta > 0, \qquad \frac{\xi}{\eta} > -\frac{2}{3}$$

whose physical justification is to be found in [2]. From (1.4), one can obtain more explicit constitutive equations:

$$\sigma^{f\varepsilon} := c^2\rho_f\nabla\cdot u^{\varepsilon} I + 2i\omega\eta e(u^{\varepsilon}) + i\omega\xi\nabla\cdot u^{\varepsilon} I \tag{1.5}$$

The equations of motion for the trabeculae (solid part) are given by

$$-\omega^2\rho_s u^{\varepsilon} - \mathrm{div}(\sigma^{s,\varepsilon}) = F\rho_s \quad \text{in} \quad \Omega_s^{\varepsilon} \tag{1.6}$$

Here the trabeculae stress is defined in (1.2), and $\rho_s > 0$ is the constant density of the trabeculae at rest.

In the marrow part,

$$-\omega^2\rho_f\, u^{\varepsilon} - \mathrm{div}(\sigma^{f,\varepsilon}) = F\rho_f \quad \text{in} \quad \Omega_f^{\varepsilon} \tag{1.7}$$

The transition conditions between fluid and solid parts are given by the continuity of displacement

$$[u^{\varepsilon}] = 0 \quad \text{on} \quad \Gamma_{\varepsilon} \tag{1.8}$$

where $[\cdot]$ indicates the jump across the boundary of $\Gamma_{\varepsilon} = \partial\Omega_s^{\varepsilon} \cap \partial\Omega_f^{\varepsilon}$, and the continuity of the traction

$$\sigma^{s,\varepsilon}\cdot\nu = \sigma^{f,\varepsilon}\cdot\nu \quad \text{on} \quad \Gamma_{\varepsilon} \tag{1.9}$$

At the exterior boundary, we imposed the zero Dirichlet condition:

$$u^{\varepsilon} = 0 \quad \text{on} \quad \partial\Omega \tag{1.10}$$

This led to a weak formulation of the slightly compressible problem as

$$-\omega^2\int_{\Omega}\rho^{\varepsilon}u^{\varepsilon}(x)\bar{\phi}(x) + \int_{\Omega}\theta^{\varepsilon}(\Lambda^{\mathcal{E}} + i\omega B^{\mathcal{E}})e(u^{\varepsilon}) : e(\bar{\phi}) + \int_{\Omega}(1-\theta^{\varepsilon})(A^s + i\omega B^s)e(u^{\varepsilon}) : e(\bar{\phi}) = \int_{\Omega}F\rho^{\varepsilon}\bar{\phi}$$
$$\forall\phi \in H_0^1(\Omega)^n \tag{1.11}$$

where an over bar denotes the complex conjugate.

[1] We are currently studying the biphasic case.

2. TWO-SCALE CONVERGENCE

The main convergence results of [1] were obtained by using the method of *two-scale convergence*[2]. We passed to the limit in the two-scale sense using the weak formulation (1.11), first using a test function $\varphi(x)$, and then a test function $\psi\left(\frac{\varepsilon}{x}\right)$, where $\psi(y)$ was assumed to be $\mathcal{Y}$-periodic with zero average. This yielded the following equations for u^0, u^1, the first two terms in the asymptotic expansion of the displacement:

$$u(x,y,\epsilon) = u^0(x,u) + \epsilon u^1(x,y) + \epsilon^2 u^2(x,y) + \cdots$$

namely,

$$\begin{aligned}&\int_\Omega \rho F\cdot\bar{\varphi} = -\omega^2\int_\Omega \rho u^0\cdot\bar{\varphi}\\&+\int_\Omega\int_{\mathcal{Y}} \theta(y)(A^f+i\omega B^f)\left(e_x\left(u^0\right)+e_y\left(u^1\right)\right):e_x(\bar{\varphi})\\&+\int_\Omega\int_{\mathcal{Y}}(1-\theta(y))(A^s+i\omega B^s)\left(e_x\left(u^0\right)+e_y\left(u^1\right)\right):e_x(\bar{\varphi})\\&\forall\varphi\in H_0^1(\Omega)^n \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned}0=&\int_\Omega\int_{\mathcal{Y}}\theta(y)(A^f+i\omega B^f)\left(e_x\left(u^0\right)+e_y\left(u^1\right)\right):e_y(\bar{\psi})\\&+\int_\Omega\int_{\mathcal{Y}}(1-\theta(y))(A^s+i\omega B^s)\left(e_x\left(u^0\right)+e_y\left(u^1\right)\right):e_y(\bar{\psi})\\&\forall\psi\in H^1_{\text{per}}(\mathcal{Y})^n)/\mathbb{C} \qquad (2.2)\end{aligned}$$

In [1] we constructed the cell problem by substitution of u^1 given in the special form

$$\begin{aligned}u^1(x,y) =& \mathbf{N}^{pq}(y)e_x(u^0)_{pq}(x)\\&+\mathbf{M}^{pq}(y,\omega)e_x(u^0)_{pq}(x) \qquad (2.3)\end{aligned}$$

into the equation (2.2). The above summation is over p and q; moreover, as $u^1(x,y)$ is a vector, the matrices $\mathbf{N}^{pq}$ and $\mathbf{M}^{pq}$ have vector components, i.e., the right–hand side is a linear combination of these vectors, with scalar coefficients $\left(e_x(u^0)\right)_{pq}$.

[2] Two-scaled convergence was first introduced in [3] and developed further in [4,5].

Integration by parts of the variational formulation showed that the strong form of the variation formulation for $\mathbf{N}^{pq}$ seeks a solution such that

$$\begin{aligned}\text{div}\,(K_N(\mathcal{E}^{pq}+e_y(\mathbf{N}^{pq})))=0 \quad &\text{in} \quad \mathcal{Y} \qquad (2.4)\\ B^f(\mathcal{E}^{pq}+e_y(\mathbf{N}^{pq}))\nu = A^s(\mathcal{E}^{pq}+e_y(&\mathbf{N}^{pq}))\nu\\ &\text{on} \quad \partial\mathcal{Y}_f\cap\partial\mathcal{Y}_s\\ [\mathbf{N}^{pq}]=0, \quad &\text{on} \quad \partial\mathcal{Y}_f\cap\partial\mathcal{Y}_s\end{aligned}$$

where

$$K_N = i\omega\theta B^f + (1-\theta)A^s \qquad (2.5)$$

Similarly, the strong form of the variation equation for $\mathbf{M}^{pq}$ is to find a solution of

$$\begin{aligned}&\text{div}(K_M(\mathcal{E}^{pq}+e_y(\mathbf{M}^{pq})))=-\text{div}(K_M e_y(\mathbf{N}^{pq})) \quad \text{in } \mathcal{Y}\\&(A^f+i\omega B^f)(\mathcal{E}^{pq}+e_y(\mathbf{M}^{pq})+e_y(\mathbf{N}^{pq}))\nu \qquad (2.6)\\&=(A^s+i\omega B^s)(\mathcal{E}^{pq}+e_y(\mathbf{M}^{pq})+e_y(\mathbf{N}^{pq}))\nu\\&\qquad \text{on} \quad \partial\mathcal{Y}_f\cap\partial\mathcal{Y}_s\\&[\mathbf{M}^{pq}]=0, \quad \text{on} \quad \partial\mathcal{Y}_f\cap\partial\mathcal{Y}_s\end{aligned}$$

where

$$K_M = \theta(A^f+i\omega B^f)+(1-\theta)(A^s+i\omega B^s) \qquad (2.7)$$

The problems (2.5) and (2.7) are uniquely solvable for each $\omega > 0$. To show this, observe that A^s, B^s, and B^f (but not A^f) are strongly elliptic. Therefore K_N and K_M given by, respectively, (2.5), (2.7) are strongly elliptic for each $\omega > 0$. Now apply the complex version of the Lax-Milgram theorem to conclude.

3. ISOTROPIC CASE

In this section we write out explicit forms of the solutions assuming that the trabeculae are isotropic, i.e., we assume that in $\mathcal{Y}_s$ (see Fig. 1),

$$\begin{aligned}(A^s e_y)_{ij} &= a^s_{ijkl}e_{kl} = (\lambda\delta_{ij}\delta_{kl}+2\mu\delta_{ik}\delta_{jl})\,e_{kl}\\&=\lambda\delta_{ij}e_{kk}+2\mu e_{ij} \qquad (3.1)\end{aligned}$$

$$(B^s e_y)_{ij} = b^s_{ijkl}e_{kl} = \bar{\lambda}\delta_{ij}e_{kk}+2\bar{\mu}e_{ij} \qquad (3.2)$$

$$(A^f e_y)_{ij} = c^2\rho_f\delta_{ij}e_{kk} \qquad (3.3)$$

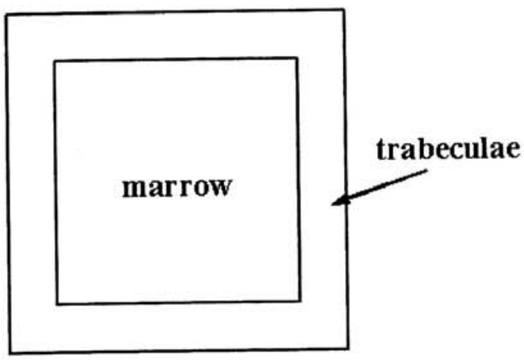

FIGURE 1. Sketch of the unit cell consisting of the marrow ($\mathcal{Y}_f$) and the trabeculae ($\mathcal{Y}_s$)

$$\begin{aligned}(B^f e_y)_{ij} &= (\xi\delta_{ij}\delta_{kl} + 2\eta\delta_{ik}\delta_{jl})\, e_{kl} \\ &= \xi\delta_{ij}e_{kk} + 2\eta e_{ij}\end{aligned} \tag{3.4}$$

Therefore, from (2.5), in $\mathcal{Y}_s$, we obtain the differential equation for the matrices $\mathbf{N}^{pq}$ ($p, q = 1, 2$):

$$\frac{\partial}{\partial y_j}(\lambda\delta_{ij}e_{kk}(\mathbf{N}^{pq}) + 2\mu e_{ij}(\mathbf{N}^{pq})) = 0 \quad \text{in} \quad \mathcal{Y}_s \tag{3.5}$$

$$\frac{\partial}{\partial y_j}(\xi\delta_{ij}e_{kk}(\mathbf{N}^{pq}) + 2\eta e_{ij}(\mathbf{N}^{pq})) = 0 \quad \text{in} \quad \mathcal{Y}_f \tag{3.6}$$

where on the boundary $\mathcal{Y}_s \bigcap \mathcal{Y}_f$ we have

$$\begin{aligned}&\nu_j^s\left[\lambda\left(e_{kk}\left(\mathbf{N}^{pq}\right) + \delta_{pq}\right)\delta_{ij} + 2\mu\left(e_{ij}(\mathbf{N}^{pq}) + \mathcal{E}_{ij}^{pq}\right)\right] \\ &= \nu_j^f\left[\xi\left(e_{kk}\left(\mathbf{N}^{pq}\right) + \delta_{pq}\right)\delta_{ij} + 2\eta\left(e_{ij}(\mathbf{N}^{pq}) + \mathcal{E}_{ij}^{pq}\right)\right]\end{aligned} \tag{3.7}$$

where $-\nu_k^s = \nu_k^f$ on $\mathcal{Y}_s \bigcap \mathcal{Y}_f$. Since another transmission condition is needed, we set $[\mathbf{N}^{pq}] = 0$ on $\mathcal{Y}_s \bigcap \mathcal{Y}_f$ also.

We recall that the components of $\mathbf{N}^{pq}$ are vectors; hence we designate the vector components by

$$\mathbf{N}^{pq} = [N_1^{pq}, N_2^{pq}]$$

The system of partial differential equations then takes the form

$$\begin{aligned}&(\lambda + 2\mu)\frac{\partial^2 N_k^{pq}}{\partial y_k \partial y_l} + 2\mu \triangle \mathbf{N}_l^{pq} = 0 \quad \text{in} \quad \mathcal{Y}_s \\ &(\eta + 2\xi)\frac{\partial^2 N_k^{pq}}{\partial y_k \partial y_l} + 2\eta \triangle \mathbf{N}_l^{pq} = 0 \quad \text{in} \quad \mathcal{Y}_f\end{aligned} \tag{3.8}$$

We may express these as matrix equations:

$$(\lambda + 2\mu)\mathbf{H}\mathbf{N}^{pq} + 2\mu \triangle \mathbf{N}^{pq} = 0 \quad \text{in} \quad \mathcal{Y}_s \tag{3.9}$$

$$(\eta + 2\xi)\mathbf{H}\mathbf{N}^{pq} + 2\eta \triangle \mathbf{N}^{pq} = 0 \quad \text{in} \quad \mathcal{Y}_f \tag{3.10}$$

where $\mathbf{H}$ is the *Hessian* operator, i.e.,

$$\mathbf{H} := \begin{pmatrix} \dfrac{\partial^2}{\partial y_1^2} & \dfrac{\partial^2}{\partial y_1 \partial y_2} \\ \dfrac{\partial^2}{\partial y_1 \partial y_2} & \dfrac{\partial^2}{\partial y_2^2} \end{pmatrix}$$

We now turn to determining the matrix solutions $\mathbf{M}^{pq}$. To this end, we introduce $Q^{pq} = \mathbf{M}^{pq} + \mathbf{N}^{pq}$, solve the problems for Q^{pq}, and then obtain the solutions for $\mathbf{M}^{pq}$. First we compute the terms $\left(A^f + i\omega B^f\right)\mathcal{E}^{pq}$ and $\left(A^f + i\omega B^f\right)e(\mathbf{M}^{pq})$:

$$\begin{aligned}&\left(A^f + i\omega B^f\right)_{ijkl}\mathcal{E}_{kl}^{pq} = \delta_{pq}\left(c^2\rho_f + i\omega\xi\right)\delta_{ij} + 2i\omega\eta\mathcal{E}_{ij}^{pq} \\ &\left(A^f + i\omega B^f\right)_{ijkl} e(\mathbf{M}^{pq})_{kl} = \left(c^2\rho_f + i\omega\xi\right)e_{kk}(\mathbf{M}^{pq})\delta_{ij} \\ &\qquad + 2i\omega\eta e_{ij}(\mathbf{M}^{pq})\end{aligned} \tag{3.11}$$

This leads to the following equation holding in $\mathcal{Y}_f$

$$i\omega\eta\triangle Q^{pq} + \left(c^2\rho_f + i\omega\xi + i\omega\eta\right)\mathbf{H}Q^{pq} = 0 \quad \text{in} \quad \mathcal{Y}_f \tag{3.12}$$

On the other hand, in the solid part $\mathcal{Y}_s$, we have

$$\begin{aligned}(A^s e(\mathbf{M}^{pq}))_{ij} &= \lambda\delta_{ij}e_{kk}(\mathbf{M}^{pq}) + 2\mu e_{ij}(\mathbf{M}^{pq}) \\ (A^s e(\mathcal{E}^{pq}))_{ij} &= \lambda\delta_{ij}\delta_{pq} + 2\mu\mathcal{E}_{ij}^{pq} \\ (B^s e(\mathbf{M}^{pq}))_{ij} &= \overline{\lambda}\delta_{ij}e_{kk}(\mathbf{M}^{pq}) + 2\overline{\mu}e_{ij}(\mathbf{M}^{pq}) \\ (B^s e(\mathcal{E}^{pq}))_{ij} &= \overline{\lambda}\delta_{ij}\delta_{pq} + 2\overline{\mu}\mathcal{E}_{ij}^{pq}\end{aligned}$$

which yields

$$(\mu + i\omega\overline{\mu})\triangle Q^{pq} + \left(\lambda + \mu + i\omega(\overline{\lambda} + \overline{\mu})\right)\mathbf{H}Q^{pq} = 0 \quad \text{in} \quad \mathcal{Y}_s \tag{3.13}$$

together with the following transmission conditions

$$\begin{aligned}&\nu_j^s\left[(\lambda + i\omega\overline{\lambda})\left(e_{kk}(Q^{pq}) + \delta_{pq}\right)\delta_{ij}\right. \\ &\left.+ 2(\mu + i\omega\overline{\mu})\left(e_{ij}(Q^{pq}) + \mathcal{E}_{ij}^{pq}\right)\right] \\ &= \nu_j^f\left[\left(c^2\rho_f + i\omega\xi\right)\left(e_{kk}(Q^{pq}) + \delta_{pq}\right)\delta_{ij}\right. \\ &\left.+ 2i\omega\eta\left(e_{ij}(Q^{pq}) + \mathcal{E}_{ij}^{pq}\right)\right]\end{aligned} \tag{3.14}$$

4. THE EFFECTIVE EQUATIONS

To obtain the effective equations, we substitute (2.3) into (2.1) and collect terms containing the same components of $e(u^0)$:

$$\int_\Omega \rho F \cdot \bar{\varphi} = -\omega^2 \int_\Omega \rho u^0 \cdot \bar{\varphi} + \int_\Omega \left(\int_\mathcal{Y} \theta (A^f + i\omega B^f)(\mathcal{E}^{pq} + e(\mathbf{N}^{pq} + \mathbf{M}^{pq})) \right) e(u^0)_{pq} : e(\bar{\varphi}) + \int_\Omega \left(\int_\mathcal{Y} (1-\theta)(A^s + i\omega B^s)(\mathcal{E}^{pq} + e(\mathbf{N}^{pq} + \mathbf{M}^{pq})) \right) \times e(u^0)_{pq} : e(\bar{\varphi}), \qquad \forall \varphi \in H_0^1(\Omega)^n \tag{4.1}$$

In (4.1), consider separately integrals over $\mathcal{Y}$ and recall that $\mathbf{N}^{pq}$ are independent of ω. This allows us to separate the following three groups of terms: terms independent of ω, terms that are linear in ω, and the rest of the terms. This yields

$$\int_\mathcal{Y} \theta (A^f + i\omega B^f)_{ijkl} (\mathcal{E}^{pq} + e(\mathbf{N}^{pq} + \mathbf{M}^{pq}))_{kl} + \int_\mathcal{Y} (1-\theta)(A^s + i\omega B^s)_{ijkl} \left(\mathcal{E}^{pq} + e(\mathbf{N}^{p\mathcal{A}q} + \mathbf{M}^{pq})\right)_{kl} = \mathcal{A}_{ijpq} + i\omega \mathcal{B}_{ijpq} + \mathcal{C}_{ijpq}(\omega) \tag{4.2}$$

where the effective materials tensors $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are defined by

$$\mathcal{A}_{ijpq} = \int_\mathcal{Y} (\theta A^f + (1-\theta) A^s)_{ijkl} (\mathcal{E}^{pq} + e(\mathbf{N}^{pq}))_{kl} \tag{4.3}$$

$$\mathcal{B}_{ijpq} = \int_\mathcal{Y} (\theta B^f + (1-\theta) B^s)_{ijkl} (\mathcal{E}^{pq} + e(\mathbf{N}^{pq}))_{kl} \tag{4.4}$$

$$\mathcal{C}_{ijpq} = \int_\mathcal{Y} \left[\theta (A^f + i\omega B^f) + (1-\theta)(A^s + i\omega B^s)\right]_{ijkl} \times (\mathcal{E}^{pq} + e(\mathbf{M}^{pq}))_{kl} \tag{4.5}$$

Combining (4.1) and (4.5) we obtain the weak formulation of the effective equation

$$\int_\Omega \rho F \cdot \bar{\varphi} = -\omega^2 \int_\Omega \rho u^0 \cdot \bar{\varphi} + \int_\Omega [\mathcal{A} + i\omega \mathcal{B} + \mathcal{C}(\omega)]\, e(u^0) : e(\bar{\varphi}) \tag{4.6}$$

The preceding can be summarized as the following.

Theorem 1. *Let u^ε be the unique solution in $H_0^1(\Omega)$ of*

$$\begin{aligned} \mathcal{L}_\varepsilon u^\varepsilon &= F\rho^\varepsilon \qquad \text{in} \quad \Omega \\ u^\varepsilon &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{4.7}$$

where $\mathcal{L}_\varepsilon$ denotes the second order partial differential operator

$$\mathcal{L}_\varepsilon u^\varepsilon = -\text{div}\left(((1-\theta^\varepsilon)\sigma^{s,\varepsilon} + \theta^\varepsilon \sigma^{f,\varepsilon}) e(u^\varepsilon)\right) - \omega^2 \rho^\varepsilon u^\varepsilon$$

$\theta^\varepsilon \equiv \theta\left(\frac{\varepsilon}{x}\right)$ is the characteristic function of the marrow part Ω_f^ε and

$$\rho^\varepsilon = \theta^\varepsilon \rho_f + \rho_s (1 - \theta^\varepsilon)$$

Then, there exists a subsequence $\{u^\varepsilon\}$, not relabeled, such that $\{u^\varepsilon\}$ converges weakly in $H_0^1(\Omega)$ to a limit $u^0 \in H_0^1(\Omega)$, and ρ^ε converges weakly in $L^\infty(\Omega)$ to ρ. The pair $\{u^0, \rho\}$ is a weak solution of the homogenized equation

$$\begin{aligned} \mathcal{L}u &= F\rho \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{4.8}$$

where $\mathcal{L}$ denotes the homogenized operator such that

$$\mathcal{L}u = -\text{div}_x \left\{ \mathcal{A} e(u^0) + i\omega \mathcal{B} e(u^0) + \mathcal{C}(\omega) e(u^0) \right\} - \omega^2 \rho u$$

The effective constant tensors $\mathcal{A}$ and $\mathcal{B}$ are defined, in (4.3), (4.4), respectively. The effective frequency dependent tensor $\mathcal{C}(\omega)$ is defined in (4.5). The vectors $\mathbf{N}^{pq}$, $\mathbf{M}^{pq}$ that appear in (4.3)–(4.5) are solutions of the auxiliary cell problems (2.5), (2.7), respectively.

The homogenized behavior of the slightly compressible viscous fluid in the elastic porous medium is described by the equations (4.8), which are the equations of linear viscoelasticity, i.e., our mixture of a slightly compressible viscous fluid and an elastic solid behaves on average as a single-phase viscoelastic material. The effective tensors $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ characterize, respectively, elastic moduli, viscous moduli, and long-time relaxation moduli of the effective material.

5. NUMERICAL EXPERIMENTS

Using the physical values given in Table 1 and the value for λ, which is computed by the formula below:

$$\lambda = K_b - \frac{2}{3}\mu + \frac{(K_r - K_b)^2 - 2\beta K_r(K_r - K_b) + \beta^2 K_r^2}{D - K_b} \quad (5.1)$$

where

$$D = K_r(1 + \beta(K_r/K_f - 1)) \quad (5.2)$$

we were able to compute the coefficients in the effective equations from the cell problem solutions $\mathbf{N}^{pq}$ and $\mathbf{M}^{pq}$ $(p, q = 1, 2)$. The equations for $\mathbf{N}^{pq}$ and $\mathbf{M}^{pq}$ were discretized using a second-order finite difference scheme, and the resulting linear systems were solved numerically by a direct method. The numerical model was tested in simple cases. We show two typical coefficients N^{11} and N^{12} in Fig. 2 and Fig. 3 respectively. These numerical results were then used to compute the effective coefficients $\mathcal{A}_{ijpq}$, $\mathcal{B}_{ijpq}$, and $\mathcal{C}_{ijpq}$. Numerical quadrature, such as the trapezoidal rule, was used for these computations. Typically, in our simulations, the unit cell was specified of length $L = 1$ in each direction, the trabeculae frame was $2/5$ of the total length of the unit cell in each direction, and the spatial resolution was $\Delta y_1 = \Delta y_2 = 0.0154$. It is important to realize that these coefficients are in themselves only used to compute the effective *constant* coefficients appearing in the effective Eq. (4.8).

If we introduce the notation

$$\mathcal{E}_{ijkl} = \mathcal{A}_{ijkl} + \omega\mathcal{B}_{ijkl} + \mathcal{C}_{ijkl}(\omega)$$

then the effective equations, in the original x- coordinates, take the form

$$\mathcal{E}_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} - \omega^2 u_i = 0$$

which in component form becomes

TABLE 1. Parameters used in our model

Symbol	Parameter
$\rho_f = 10^3$	density of the pore fluid
$K_b = 2.76 \times 10^9$	complex frame bulk modulus
$\mu = 1.15 \times 10^9$	frame shear modulus
$K_f = 2 \times 10^9$	fluid bulk modulus
$K_r = 2 \times 10^{10}$	frame material bulk modulus
$\beta = 0.76$	porosity
$\eta = 1.5$	first viscosity of pore fluid
$\xi = 0$	second viscosity of pore fluid
$\omega = \pi \times 10^6$	sound frequency
$c = 1483$	speed of sound

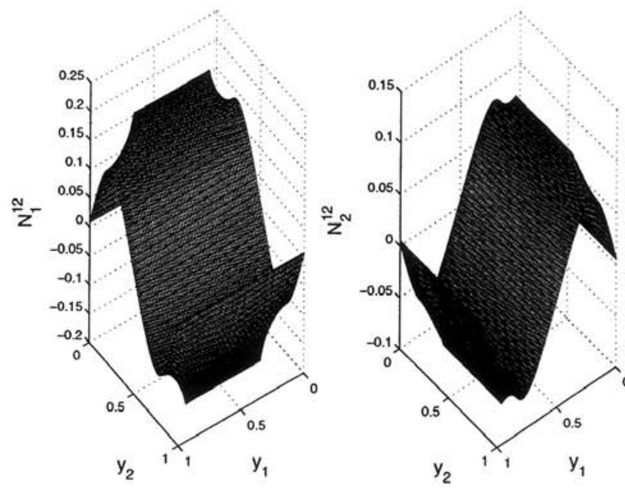

FIGURE 3. Plot of N_1^{12} and N_2^{12}. The unit cell is of length $L = 1$ in each direction, the trabeculae frame is $2/5$ of the total length of the unit cell in each direction, and $\beta = 0.76$

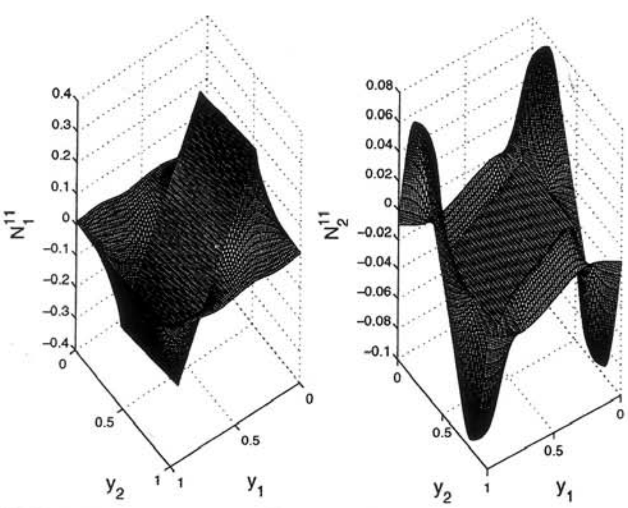

FIGURE 2. Plot of N_1^{11} and N_2^{11}. The unit cell is of length $L = 1$ in each direction, the trabeculae frame is $2/5$ of the total length of the unit cell in each direction, and $\beta = 0.76$

$$\mathcal{E}_{i111}\frac{\partial^2 u_1}{\partial x_1^2}+\mathcal{E}_{i211}\frac{\partial^2 u_1}{\partial x_1 \partial x_2}+\mathcal{E}_{i212}\frac{\partial^2 u_1}{\partial x_2^2}+\mathcal{E}_{i121}\frac{\partial^2 u_1}{\partial x_1^2}$$
$$+\mathcal{E}_{i122}\frac{\partial^2 u_1}{\partial x_1 \partial x_2}+\mathcal{E}_{i222}\frac{\partial^2 u_1}{\partial x_2^2}=2\pi\omega^2 u_i, \quad i=1,2$$

Using MAPLE output, we list below these equations for the porosities of normal, osteoporotic, and severely osteoporotic bone, namely, $\beta = 0.76, 0.83, 0.90$, respectively. For $\beta = 0.76$, we have

```
2.189433292*(diff(u(x, y),x,x))+(0.8033696226e-1*I)
*(diff(v(x,y),x,x))+1.284543528*10^(-10)*E[1121]
*(diff(v(x,y),x,x))+1.284543528*10^(-10)*E[1121]
*(diff(u(x,y),x,x))+0.9614372843e-1*(diff(v(x,y),x,x))
+(.2368914469*I)*(diff(u(x,y),y,y))+(1.470789632*I)
*(diff(v(x,y),y,x))+1.370591759*(diff(v(x,y),y,x))
-(0.2891357240e-3*I)*(diff(v(x,y),y,y))
-0.3221249804e-4*(diff(u(x,y),y,x))+(1.899325828*I)
*(diff(u(x,y),x,x))+ 1.284543528*10^(-10)*E[1221]
*(diff(v(x,y),y,x))+1.284543528*10^(-10)* E[1221]
*(diff(u(x,y),y,y))+.2364477255*(diff(u(x,y),y,y))
-(0.1738102475e-4*I)*(diff(u(x,y),y,x))
-0.4034931056e-3*(diff(v(x,y),y,y))
- .9999999998*u(x,y)=0
```

```
-0.3221249804e-4*(diff(u(x,y),x,x))+(.2368914469*I)
*(diff(v(x,y),x,x))+1.284543528*10^(-10)*E[1221]
*(diff(v(x,y),x,x))+1.284543528*10^(-10)* E[1221]
*(diff(u(x,y),y,x))+.2364477255*(diff(v(x,y),x,x))
+(0.7992352755e-1*I)*(diff(u(x,y),y, y))+1.379624669
*(diff(u(x,y),y,x))+(0.7963439184e-1*I)*(diff(v(x,y),
y,x))+0.9608269463e-1*(diff(v(x,y),y,x))
+(2.184662236*I)*(diff(v(x,y),y,y))-(0.1738102475e-4*I)
*(diff(u(x,y),x,x))+ 1.284543528*10^(-10)*E[2221]
*(diff(v(x,y),y,x))+1.284543528*10^(-10)* E[2221]
*(diff(u(x,y),y,y))+(1.104265223*I)
*(diff(u(x,y),y,x))+0.9648618774e-1*(diff(u(x,y),y,y))
+2.086150070*(diff(v(x,y),y,y))-.9999999998*u(x,y)=0.
```

For $\beta = 0.83$ we have

```
(2.08+1.7823*I)*(diff(u(x,y),x,x))+(0.94e-1
-0.73e-2*I)*(diff(u(x,y),y,x))+(0.941e-1
+0.77e-2*I)*(diff(v(x,y),y,y))+(1.6066
+1.3043*I)*(diff(v(x,y),y,x))+(.4732+.4736*I)
*(diff(u(x,y),y,y))+(0.7e-4+0.1e-4*I)
*(diff(v(x,y),x,x))-1.*u(x,y)=0,
```

```
(0.941e-1-0.73e-2*I)*(diff(u(x,y),x,
x))+(1.5157+1.3209*I)*(diff(u(x,y),y,x))
+(1.9929+1.7959*I)*(diff(v(x,y),y,y))
+(0.9411e-1-0.76e-2*I)*(diff(v(x,y),y,x))
+(.4736+.4732*I)*(diff(v(x,y),x,x))
+(0.5e-4+0.7e-4*I)*(diff(u(x,y),y,y))-1.*u(x,y).
```

For $\beta = 0.90$ we have

```
(2.08+1.7823*I)*(diff(u(x,y),x,x))
+(0.94e-1-0.73e-2*I)*(diff(u(x,y),y,x))
+(0.941e-1+0.77e-2*I)*(diff(v(x,y),y,y))
+(1.6066+1.3043*I)*(diff(v(x,y),y,x))
+(.4732+.4736*I)*(diff(u(x,y),y,y))
+(0.7e-4+0.1e-4*I)*(diff(v(x,y),x,x))-1.*u(x,y)=0,
```

```
(0.941e-1-0.73e-2*I)*(diff(u(x,y),x,x))+(1.5157
+1.3209*I)*(diff(u(x,y),y,x))+(1.9929+1.7959*I)
*(diff(v(x,y),y,y))+(0.9411e-1-0.76e-2*I)
*(diff(v(x,y),y,x))+(.4736+.4732*I)*(diff(v(x,y),x,
x))+(0.5e-4+0.7e-4*I)*(diff(u(x,y),y,y))-1.*u(x,y)=0.
```

To interpret the meaning of our bone coefficients, it is useful to compare these with those introduced by [6,7]. The Biot equations for the propagation of acoustic disturbances in a porous media were obtained using mixture theory. Identification of some our parameters with Biot's physical parameters would give a hint as to which will play a significant role for the inverse problem. However, we must add the disclaimer that the Biot model was not meant for uniphasic vibrations, and our comparison becomes specious as it reduces to an elliptic system having only a principal part. Nevertheless, a comparison shows that we obtain coefficients of the same order of magnitude. For simplicity, we consider the case where all parameters are constant in the Biot model. This leads to the system (5.3) below:

$$\mu\nabla^2\mathbf{u}+\nabla[(\lambda+\mu)e+Q\epsilon]=\frac{\partial^2}{\partial t^2}(\rho_{11}\mathbf{u}+\rho_{12}\mathbf{U}) + b\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}) \qquad (5.3)$$

$$\nabla[Qe+R\epsilon]=\frac{\partial^2}{\partial t^2}(\rho_{12}\mathbf{u}+\rho_{22}\mathbf{U})-b\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U})$$

where the coefficients λ and μ are Lamé coefficients and where $e := \nabla\cdot\mathbf{u}$, $\epsilon := \nabla\cdot\mathbf{U}$.

The form of the dissipation parameter b is complicated but not necessary for the uniphasic case. When the medium is undergoing time-harmonic oscillations of angular frequency ω, this becomes

$$\mu\nabla^2\mathbf{u}+\nabla[(\lambda+\mu)e+Q\epsilon]=-\omega^2(\rho_{11}\mathbf{u}+\rho_{12}\mathbf{U}) + i\omega b(\mathbf{u}-\mathbf{U}) \qquad (5.4)$$

$$\nabla[Qe+R\epsilon]=-\omega^2(\rho_{12}\nabla+\rho_{22}\mathbf{U})-bi\omega(\mathbf{u}-\mathbf{U})$$

For the case of uniphasic oscillations, this system reduces to a single equation

$$\mu\nabla^2\mathbf{u}+\mathrm{grad}\left(\lambda+\mu+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)\mathrm{div}\mathbf{u}\right)=0 \qquad (5.5)$$

In component form, this leads to

$$\mu\,\triangle u_1+\left(\lambda+\mu+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)\right)\times\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial y_1 \partial x_2}\right)=0$$

TABLE 2. Coefficients $\mathcal{A}_{ijkl}, \mathcal{B}_{ijkl}, \mathcal{C}_{ijkl}$, for $\beta = 0.76$

Effective Elasticity Coefficients	Effective Viscosity Coefficients	Effective Relaxation Coefficients
$\mathcal{A}_{1111} = 4.58467e+09$	$\mathcal{B}_{1111} = 1341.05$	$\mathcal{C}_{1111} = 3.56332e+09+2.86726e+09i$
$\mathcal{A}_{1211} = -1.0815e-06$	$\mathcal{B}_{1211} = -0.0270966$	$\mathcal{C}_{1211} = -125385+17472i$
$\mathcal{A}_{2111} = -1.0815e-06$	$\mathcal{B}_{2111} = -0.0270966$	$\mathcal{C}_{2111} = -125385+17472i$
$\mathcal{A}_{2211} = 2.33171e+09$	$\mathcal{B}_{2211} = 623.405$	$\mathcal{C}_{2211} = 2.11803e+09+1.41771e+09i$
$\mathcal{A}_{1112} = 1.10921e+07$	$\mathcal{B}_{1112} = 0.0151305$	$\mathcal{C}_{1112} = 7.37374e+08+6.25365e+08i$
$\mathcal{A}_{1212} = 1.11852e+09$	$\mathcal{B}_{1212} = 356.193$	$\mathcal{C}_{1212} = 7.22194e+08+7.25155e+08i$
$\mathcal{A}_{2112} = 1.11852e+09$	$\mathcal{B}_{2112} = 356.193$	$\mathcal{C}_{2112} = 7.22194e+08+7.25155e+08i$
$\mathcal{A}_{2212} = 1.10921e+07$	$\mathcal{B}_{2212} = -5.04045e-14$	$\mathcal{C}_{2212} = 7.4004e+08+6.22194e+08i$
$\mathcal{A}_{1122} = 1.90032e+09$	$\mathcal{B}_{1122} = 623.326$	$\mathcal{C}_{1122} = 2.51426e+09+2.84463e+09i$
$\mathcal{A}_{1222} = 1.00583e-07$	$\mathcal{B}_{1222} = 0.109289$	$\mathcal{C}_{1222} = -1.57057e+06-782100i$
$\mathcal{A}_{2122} = 1.00583e-07$	$\mathcal{B}_{2122} = 0.109289$	$\mathcal{C}_{2122} = -1.57057e+06-782100i$
$\mathcal{A}_{2222} = 4.15328e+09$	$\mathcal{B}_{2222} = 1340.55$	$\mathcal{C}_{2222} = 3.96692e+09+4.29219e+09i$

TABLE 3. Coefficients $\mathcal{A}_{ijkl}, \mathcal{B}_{ijkl}, \mathcal{C}_{ijkl}$, for $\beta = 0.83$

Effective Elasticity Coefficients	Effective Viscosity Coefficients	Effective Relaxation Coefficients
$\mathcal{A}_{1111} = 4.57327e+09$	$\mathcal{B}_{1111} = 1333.46$	$\mathcal{C}_{1111} = 3.61561e+09+2.56873e+09i$
$\mathcal{A}_{1211} = -4.475e-07$	$\mathcal{B}_{1211} = -0.0338359$	$\mathcal{C}_{1211} = 66134.7-296249i$
$\mathcal{A}_{2111} = -4.475e-07$	$\mathcal{B}_{2111} = -0.0338359$	$\mathcal{C}_{2111} = 66134.7-296249i$
$\mathcal{A}_{2211} = 2.32038e+09$	$\mathcal{B}_{2211} = 615.814$	$\mathcal{C}_{2211} = 2.16911e+09+1.11889e+09i$
$\mathcal{A}_{1112} = 1.7835e+07$	$\mathcal{B}_{1112} = 0.0243283$	$\mathcal{C}_{1112} = -1.18135e+08+8.60127e+07i$
$\mathcal{A}_{1212} = 1.11853e+09$	$\mathcal{B}_{1212} = 356.247$	$\mathcal{C}_{1212} = 7.22853e+08+7.2481e+08i$
$\mathcal{A}_{2112} = 1.11853e+09$	$\mathcal{B}_{2112} = 356.247$	$\mathcal{C}_{2112} = 7.22853e+08+7.2481e+08$
$\mathcal{A}_{2212} = 1.7835e+07$	$\mathcal{B}_{2212} = -8.12677e-13$	$\mathcal{C}_{2212} = -1.17765e+08+8.65174e+07i$
$\mathcal{A}_{1122} = 1.88741e+09$	$\mathcal{B}_{1122} = 615.75$	$\mathcal{C}_{1122} = 2.00175e+09+1.18415e+09i$
$\mathcal{A}_{1222} = 3.41516e-06$	$\mathcal{B}_{1222} = 0.0522359$	$\mathcal{C}_{1222} = -261286+80028.9i$
$\mathcal{A}_{2122} = 3.41516e-06$	$\mathcal{B}_{2122} = 0.0522359$	$\mathcal{C}_{2122} = -261286+80028.9i$
$\mathcal{A}_{2222} = 4.14031e+09$	$\mathcal{B}_{2222} = 1332.93$	$\mathcal{C}_{2222} = 3.44738e+09+2.63387e+09i$

TABLE 4. Coefficients $\mathcal{A}_{ijkl}, \mathcal{B}_{ijkl}, \mathcal{C}_{ijkl}$, for $\beta = 0.90$

Effective Elasticity Coefficients	Effective Viscosity Coefficients	Effective Relaxation Coefficients
$\mathcal{A}_{1111} = 4.5152e+09$	$\mathcal{B}_{1111} = 1333.62$	$\mathcal{C}_{1111} = 3.0628e+09+2.78353e+09i$
$\mathcal{A}_{1211} = -1.88127e-07$	$\mathcal{B}_{1211} = 0.0353856$	$\mathcal{C}_{1211} = 332428+1.1014e+06i$
$\mathcal{A}_{2111} = -1.88127e-07$	$\mathcal{B}_{2111} = 0.0353856$	$\mathcal{C}_{2111} = 332428+1.1014e+06i$
$\mathcal{A}_{2211} = 2.2623e+09$	$\mathcal{B}_{2211} = 616.051$	$\mathcal{C}_{2211} = 1.61722e+09+1.33605e+09i$
$\mathcal{A}_{1112} = -2.227e+07$	$\mathcal{B}_{1112} = -0.030378$	$\mathcal{C}_{1112} = -1.97821e+09-1.02599e+09i$
$\mathcal{A}_{1212} = 1.11853e+09$	$\mathcal{B}_{1212} = 356.175$	$\mathcal{C}_{1212} = 7.21096e+08+7.29656e+08i$
$\mathcal{A}_{2112} = 1.11853e+09$	$\mathcal{B}_{2112} = 356.175$	$\mathcal{C}_{2112} = 7.21096e+08+7.29656e+08i$
$\mathcal{A}_{2212} = -2.227e+07$	$\mathcal{B}_{2212} = 4.89682e-13$	$\mathcal{C}_{2212} = -1.98257e+09-1.01749e+09i$
$\mathcal{A}_{1122} = 1.98421e+09$	$\mathcal{B}_{1122} = 616.117$	$\mathcal{C}_{1122} = 2.08593e+09+1.11011e+09i$
$\mathcal{A}_{1222} = -7.16595e-07$	$\mathcal{B}_{1222} = 0.00621678$	$\mathcal{C}_{1222} = -191283+99330i$
$\mathcal{A}_{2122} = -7.16595e-07$	$\mathcal{B}_{2122} = 0.00621678$	$\mathcal{C}_{2122} = -191283+99330i$
$\mathcal{A}_{2222} = 4.2371e+09$	$\mathcal{B}_{2222} = 1333.17$	$\mathcal{C}_{2222} = 3.53124e+09+2.55947e+09i$

$$\mu \triangle u_2 + \left(\lambda+\mu+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)\right) \times \left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2}+\frac{\partial^2 u_2}{\partial x_2^2}\right)=0$$

or

$$\left(2\mu+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)\right)\frac{\partial^2 u_1}{\partial x_1^2}+\mu\frac{\partial^2 u_1}{\partial x_2^2} + \left(\mu+\lambda+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)\right)\frac{\partial^2 u_2}{\partial x_1 \partial x_2}=0$$

$$\left(2\mu+\lambda+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)\right)\frac{\partial^2 u_2}{\partial x_2^2}+\mu\frac{\partial^2 u_2}{\partial x_1^2} + \left(\mu+\lambda+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)\right)\frac{\partial^2 u_1}{\partial x_1 \partial x_2}=0$$

The parameter μ, the complex frame shear modulus, is measured. The other parameters λ, R, and Q occurring in the constitutive equations are calculated from the measured or estimated values of the parameters given in Table 1 using the formulas

$$R=\frac{\beta^2 K_r^2}{D-K_b}$$

$$Q=\frac{\beta K_r\left((1-\beta)K_r-K_b\right)}{D-K_b}$$

In the Biot theory, the bulk and shear moduli K_b and μ are often given imaginary parts to account for frame inelasticity. Here ρ_{11} and ρ_{22} are density parameters for the solid and fluid, and ρ_{12} is a density coupling parameter. These are calculated from the inputs of Table 5 using the formulas

$$\rho_{11}=(1-\beta)\rho_r-\beta(\rho_f-m\beta)$$

$$\rho_{12}=\beta(\rho_f-m\beta)$$

$$\rho_{22}=m\beta^2$$

where

$$m=\frac{\alpha\rho_f}{\beta}$$

This suggests that we might try to associate, at least up to orders of magnitude, our coefficients $\mathcal{E}_{ijkl}$ with the Biot coefficients. The general form of the coefficients for the Biot uniphasic model are

$$\mathcal{E}_{1111}=2\mu+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)$$

$$\mathcal{E}_{1222}=\mu,\quad \mathcal{E}_{1121}=0$$

$$\mathcal{E}_{1121}=0,\quad \mathcal{E}_{1211}=0$$

$$\mathcal{E}_{1122}=\mu+\lambda+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)$$

$$\mathcal{E}_{2222}=2\mu+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)$$

$$\mathcal{E}_{2121}=\mu,\quad \mathcal{E}_{2111}=0$$

$$\mathcal{E}_{2212}=0,\quad \mathcal{E}_{2122}=0$$

$$\mathcal{E}_{2211}=\mu+\lambda+Q-\frac{\rho_{11}+\rho_{12}}{\rho_{12}+\rho_{22}}(Q+R)$$

$$\mathcal{E}_{1111}=\mathcal{E}_{2222}=6.112071192e\times 10^{10}$$

$$\mathcal{E}_{1122}=\mathcal{E}_{2121}=2.998535596e\times 10^{10}$$

$$\mathcal{E}_{1222}=\mathcal{E}_{2121}=1.150000000e\times 10^{9}$$

The Biot coefficients above that are zero correspond to our terms that are 10^{-4} smaller than the remaining terms. These remaining terms have real parts of the same order of magnitude as the other Biot coefficients, which, for the uniphasic model, are real.

6. CONCLUSION

We have shown that the method of two scale convergence may be used to compute effective bone parameters in the monophasic case that are comparable to those found in the Biot model. We are planning to use this model for the inverse problem for determining the bone coefficients. Further work is planned to complete the biphasic case. The biphasic is more suitable for comparison with the Biot model. Homogenization leads to systems similar to the Biot model but having more coefficients. Changing some of the physical assumptions about the fluid solid interaction may lead to new models more suitable for the ultrasound interrogation of bone.

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TABLE 5. Additional parameters needed in the Biot model when $\beta = 0.76$

Symbol	Parameter
$\rho_{11} = 5.304000 \times 10^2$	density
$\rho_{12} = -60.00000$	density
$\rho_{22} = 8.20000000 \times 10^2$	density
$\lambda = 2.02034969 \times 10^9$	Lamé coefficient
$R = 1.499870163 \times 10^9$	solid-fluid coupling coefficient
$Q = 1.5680000 \times 10^{11}$	solid-fluid coupling coefficient

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