Chapter 4

The Water Wave Problem and Hamiltonian Transformation Theory

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4.1 Introduction

About 70% of the Earth's surface is covered by water and about 40% of the world's population lives within 100 km of the coast and estuaries. It is thus not surprising that ocean waves, directly or indirectly, affect human activities and have been a constant subject of interest. They continue to fascinate us by their beauty, the broad range of phenomena that can be observed at various length scales, as well as the numerous applications most notably in oceanography and coastal engineering. For example, information on ocean waves is fundamental to the safe and economic design of ships, offshore structures, and coastal edifices, to better understanding air-sea interactions, ocean circulation and thus to improving weather forecasting. It is key to better understanding extreme wave phenomena such as tsunamis and rogue waves. It is needed for more efficient extraction of wave power which has drawn increasing attention in recent years as a potential source of renewable energy. The persistence of many open problems such as the phenomenon of wave breaking, which are still poorly understood, as well as the need for more accurate models to tackle the increasingly complex challenges raised by our modern society, also drive research in this area.

Although the mathematical study of ocean waves (or more generally water waves) goes back to the eighteenth century, the nonlinear problem remains challenging both analytically and computationally. It is only in the past two decades that significant progress has been made in the rigorous analysis of the full equations. A major difficulty has to do with the fact that, in the classical formulation, surface water waves are described by a free-boundary value problem. Not only does the usual nonlinearity of fluid dynamics enter the boundary conditions, the boundary itself (i.e. the free surface) is also to be found as part of the solution, and the velocity field depends nonlocally on the moving boundary of the fluid domain. In some cases, water waves may be viewed as a perturbation relative to the trivial geometry of a flat surface. This enables the derivation of reduced models for weakly nonlinear waves, which are more amenable to analysis and simulation. The water wave problem is notorious for its wealth of reduced models that arise in various asymptotic limits. Many of these models happen to have a universal character and appear in other areas of nonlinear science such as optics, plasma physics, and quantum physics. Consideration of these limiting regimes, including the development of perturbation methods to derive and analyze the resulting equations, has produced an enormous literature.

A well-known approach in this regard is to adopt a Hamiltonian point of view, based on the remarkable result of Zakharov [101] that the water wave problem has a canonical Hamiltonian formulation in terms of two conjugate variables, namely the surface elevation and the velocity potential evaluated at the free surface. This approach provides a unified framework where perturbation calculations are performed following common rules from Hamiltonian transformation theory. These include canonical transformations and reduction to normal forms. While Zakharov's idea has been around for some time, most studies in this framework either focus on the deep-water regime [52, 89] or show calculations preferably in the Fourier space [77, 104]. Furthermore, while transformation theory is a basic well-established tool for finite-dimensional Hamiltonian systems, it is less understood in the context of partial differential equations (PDEs) and infinite-dimensional Hamiltonian systems. This extension is an active area of research in mathematical analysis and significant progress has been made in recent years [55, 66]. There are important implications considering that Hamiltonian PDEs are widespread in science and engineering.

The main objective of this chapter is to give an overview of recent work by the authors and collaborators in developing a systematic perturbation method to derive asymptotic models for nonlinear water waves, which are all Hamiltonian PDEs. Most of these results have been obtained in the past decade and half, and concern both the shallow-water and deep-water scaling limits. Zakharov's Hamiltonian formulation of the governing equations, together with the Dirichlet–Neumann operator as introduced by Craig and Sulem [40], provides the basis for this asymptotic analysis. From a modeling point of view, reduced models should retain important structural properties of the original system, including energy conservation. It is thus desirable that they retain a Hamiltonian structure. Except for a number of generic examples such as the Korteweg–de Vries and nonlinear Schrödinger equations, many existing models (especially at high truncation order or in complex physical settings) are not known to be Hamiltonian PDEs. For this purpose, we introduce a set of canonical transformations that are relevant to limiting scaling regimes in the water wave problem, and develop associated rules to determine their effects on the Hamiltonian and symplectic structure of the system. These transformation rules are directly applicable in the physical space and allow a systematic point of view to be retained throughout the asymptotic procedure. As a consequence, a Hamiltonian model with a well-defined symplectic structure is obtained at each level of approximation. We illustrate the versatility of this approach by applying it to modeling surface gravity waves in the asymptotic shallow-water and deep-water regimes. Because three-wave resonances do not occur for deep-water gravity waves, all cubic terms may be eliminated from the Hamiltonian, as they are not relevant to the wave dynamics in this case. We discuss how this elimination can be implemented in the present framework via normal form transformations. We also take this opportunity to review rigorous mathematical results on the initial value problem for water waves, considering that the same basic formulation involving the Dirichlet–Neumann operator has played a key role in recent breakthroughs [79].

This chapter is organized as follows. Section 4.2 presents the basic mathematical formulation for surface gravity waves, including the Hamiltonian form of the governing equations and an analysis of the Dirichlet–Neumann operator. Section 4.3 describes the normal form transformations to eliminate non-resonant cubic and quartic terms, and discusses their mapping properties. Section 4.4 lays out the general procedure to derive reduced models in this Hamiltonian setting, including a description of canonical transformations that are relevant to the shallow-water and deep-water scaling limits. This approach is applied to the derivation of the Boussinesq and Korteweg–de Vries equations for long waves on shallow water, and the nonlinear Schrödinger equation for near-monochromatic waves on deep water. Because the latter equation only describes the wave envelope, a procedure for reconstruction of the actual free surface is also presented. Section 4.5 gives a review on the local and global existence theory for the water wave problem, providing a summary of how analytic tools have evolved, starting from the pioneering work of Nalimov [82] in the early seventies and leading to groundbreaking results in the last fifteen years. Finally, Sect. 4.6 outlines two different numerical methods to solve the full nonlinear equations for surface gravity waves. The first one focuses on steadily progressing wave solutions in a moving reference frame, while the second one considers the general time evolution problem by solving the Hamiltonian form of the governing equations. An application to the head-on collision of two solitary waves on shallow water is discussed, including a comparison with laboratory measurements.

This chapter is based on lectures given by Walter Craig at the summer school on *Waves in Flows* that took place in Prague in August 2018. Walter passed away on January 18, 2019 and it is with great sadness that we (P. G. and C. S.) lost our long-time collaborator and dear friend. We will forever be grateful to Walter for his inspiration and friendship.

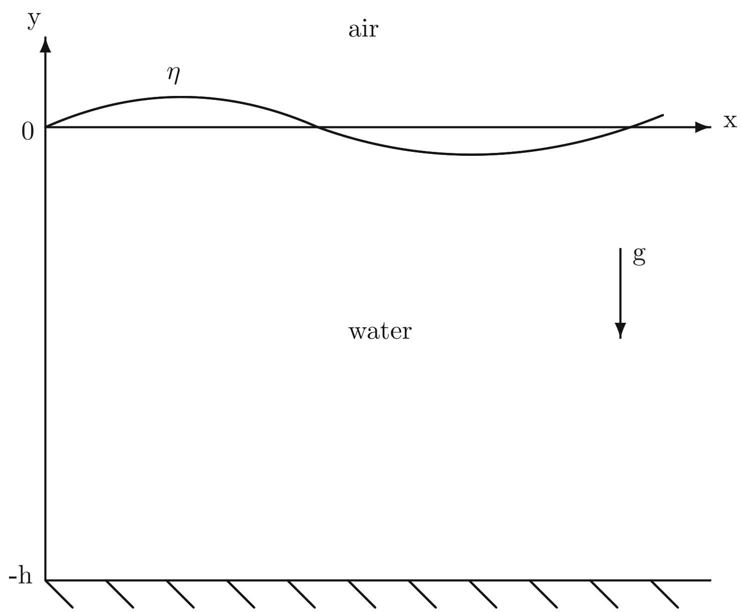

Figure 4.1: Two-dimensional sketch of the fluid domain

4.2 Water Waves and Hamiltonian PDEs

4.2.1 Physical Derivation of the Governing Equations

We consider the motion of a free surface on top of a d-dimensional body of water under the influence of gravity ($d = 2$ or 3). Gravity acts as a restoring force to disturbances at the free surface (see Fig. 4.1). Surface tension could also be incorporated into this formulation but it is neglected here. The fluid domain is defined by

$$\Omega = \{(x, y) : x \in \mathbb{R}^{d-1}, -h < y < \eta(x,t)\} ,$$

where we assume that the bottom is uniform (located at constant depth $y = -h$), namely

$$\Gamma_b = \{(x, y) : x \in \mathbb{R}^{d-1}, y = -h\} ,$$

and the free surface is given as the graph of a function η, namely

$$\Gamma = \{(x, y) : x \in \mathbb{R}^{d-1}, y = \eta(x,t)\} .$$

For an incompressible and inviscid flow, the fluid velocity field $\mathbf{u}(x, y, t)$ in Ω obeys Euler's equations

$$\nabla \cdot \mathbf{u} = 0 , \tag{4.1}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla \mathcal{P} - \mathbf{g} = 0 , \tag{4.2}$$

which are associated with mass and momentum conservation, respectively. Moreover, assuming that the flow is irrotational involves the additional constraint

$$\nabla \times \mathbf{u} = 0 . \tag{4.3}$$

In these equations, $\nabla = (\partial_x, \partial_y)^\top$ is the spatial gradient, ρ is the fluid density, $\mathcal{P}(x, y, t)$ is the fluid pressure, and $\mathbf{g} = (0, -g)^\top$ is the acceleration due to gravity. Noting that $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla|\mathbf{u}|^2/2$ in this case, Eq. (4.2) can be expressed as

$$\partial_t \mathbf{u} + \nabla \left(\frac{|\mathbf{u}|^2}{2} + \frac{\mathcal{P}}{\rho} + gy \right) = 0 .$$

In the absence of wind and surface tension, the boundary conditions on Γ are the dynamic condition

$$\mathcal{P} = \mathcal{P}_a , \tag{4.4}$$

where $\mathcal{P}_a$ denotes the constant atmospheric pressure, and the kinematic condition

$$\partial_t \eta = \sqrt{1 + |\partial_x \eta|^2}\, \mathbf{u} \cdot \mathbf{n} . \tag{4.5}$$

Equation (4.4) prescribes the continuity of normal stress across Γ, while Eq. (4.5) reflects the fact that the free surface moves according to the flow. On Γ_b (which is taken to be a rigid boundary and hence there is zero fluid flux across it), the impermeability condition amounts to

$$\mathbf{u} \cdot \mathbf{n} = 0 \,. \tag{4.6}$$

In (4.5) and (4.6), the vector $\mathbf{n}$ denotes the unit outward normal to the boundary of the fluid domain.

The irrotationality condition (4.3) implies that $\mathbf{u} = \nabla\varphi$ where the scalar function $\varphi(x, y, t)$ represents the velocity potential. In terms of φ, Eqs. (4.1) and (4.2) take the form

$$\Delta\varphi = 0 \,,$$

and

$$\nabla\left(\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} + \frac{\mathcal{P}}{\rho} + gy\right) = 0 \,,$$

the latter implying

$$\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} + \frac{\mathcal{P}}{\rho} + gy = C(t) \,, \tag{4.7}$$

where C is an arbitrary function of t, which can be discarded without loss of generality. This is equivalent to absorbing it into the definition of φ via the gauge transformation

$$\varphi \to \varphi + \int^t C(\tau)\, d\tau \,.$$

Evaluating (4.7) on Γ leads to

$$\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} + \frac{\mathcal{P}_a}{\rho} + g\eta = 0 \,,$$

by virtue of (4.4). Similarly to C, the constant pressure term $\mathcal{P}_a/\rho$ can also be discarded, yielding

$$\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0 \,.$$

This equation plays the same role as the dynamic condition (4.4) on Γ and is also called Bernoulli's condition. Substituting φ in (4.5) and (4.6), and collecting all the equations, we obtain the boundary value problem

$$\Delta\varphi = 0 \,, \quad \text{in} \quad \Omega \,, \tag{4.8}$$

$$\partial_t\eta = \partial_y\varphi - \partial_x\eta \cdot \partial_x\varphi \,, \quad \text{on} \quad \Gamma \,, \tag{4.9}$$

$$\partial_t\varphi = -g\eta - \frac{1}{2}|\nabla\varphi|^2 \,, \quad \text{on} \quad \Gamma \,, \tag{4.10}$$

$$\partial_y\varphi = 0 \,, \quad \text{on} \quad \Gamma_b \,, \tag{4.11}$$

which is referred to as the potential-flow formulation of Euler's equations for surface gravity water waves. To derive (4.9) and (4.11), we have used the fact that

$$\mathbf{n} = \frac{(-\partial_x\eta, 1)^\top}{\sqrt{1 + |\partial_x\eta|^2}} \,,$$

on Γ and $\mathbf{n} = -\mathbf{e}_y$ on Γ_b. Typical boundary conditions in the horizontal hyperplane are periodic boundary conditions (over a given periodic cell $\mathbb{T}^{d-1}$) or vanishing boundary conditions at infinity.

This system of equations admits a number of invariants of motion, due to the inviscid character of the flow [10]. These include the energy

$$\begin{aligned} H &= \int_\Omega \left(\frac{1}{2}|\mathbf{u}|^2 + gy\right) dV = K + P\,, \\ &= \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta} \frac{1}{2}|\mathbf{u}|^2 \, dydx + \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta} gy \, dydx\,, \\ &= \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta} \frac{1}{2}|\nabla\varphi|^2 \, dydx + \int_{\mathbb{R}^{d-1}} \frac{1}{2} g\eta^2 \, dx\,, \end{aligned} \tag{4.12}$$

where the first integral K represents kinetic energy, while the second integral P represents potential energy. Note that the constant contribution from the y-integration of the gravity term in P can be discarded because H is conserved in time and thus is determined up to a constant level. Other invariants of motion are the volume

$$V = \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta} dydx = \int_{\mathbb{R}^{d-1}} \eta \, dx\,, \tag{4.13}$$

where again the constant contribution from the y-integration can be omitted, and the impulse (i.e. the horizontal momentum)

$$I = \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta} \partial_x \varphi \, dydx\,.$$

Because

$$\int_{-h}^{\eta} \partial_x \varphi \, dy = \partial_x \left(\int_{-h}^{\eta} \varphi \, dy \right) - \varphi(x,\eta,t)\partial_x \eta\,, \tag{4.14}$$

by Leibniz's rule, the expression of I reduces to

$$I = -\int_{\mathbb{R}^{d-1}} \varphi(x,\eta,t)\partial_x \eta \, dx\,. \tag{4.15}$$

The total x-derivative in (4.14) does not contribute after integration by virtue of the (periodic or vanishing) boundary conditions.

This formulation of the water wave problem has the following remarkable property, which was first discovered by Zakharov [101, 103] and will be discussed in a subsequent section.

Theorem 4.2.1. *Equations* (4.8)–(4.11) *possess a canonical Hamiltonian structure in terms of Darboux coordinates, with Hamiltonian H given by the conserved energy* (4.12).

For this purpose, we first review basic notions on Hamiltonian systems, with an emphasis on Hamiltonian PDEs.

4.2.2 General Notions on Hamiltonian Systems

A Hamiltonian system is associated with a Hamiltonian function $H : \mathcal{M} \to \mathbb{R}$ where $\mathcal{M}$ is the phase space. We restrict ourselves to phase spaces that are Hilbert spaces, denoting the inner product between two vectors $v_1, v_2 \in \mathcal{V}(\mathcal{M})$ by $\langle v_1, v_2 \rangle$. The symplectic structure is given by a two-form ω on $\mathcal{M}$, which can be represented by the inner product

$$\omega(v_1, v_2) = \langle v_1, J^{-1} v_2 \rangle ,$$

where the invertible operator J satisfies $J^{-\top} = -J^{-1}$ due to the antisymmetry of two-forms, namely

$$\langle J^{-1} v_1, v_2 \rangle = -\langle v_1, J^{-1} v_2 \rangle .$$

The Hamiltonian vector field X_H that describes the initial value problem

$$\partial_t v = X_H , \quad v(0) = v_0 , \tag{4.16}$$

follows from the relation

$$dH(v_1) = \omega(v_1, X_H) , \quad \forall v_1 \in \mathcal{V}(\mathcal{M}) . \tag{4.17}$$

The inner product can be used to define the gradient of functions on $\mathcal{M}$; in particular, $\mathrm{grad}_v H$ is defined via the Gâteaux derivative

$$dH(v_1) = \frac{d}{ds} H(v + s v_1) \Big|_{s=0} = \langle \mathrm{grad}_v H, v_1 \rangle , \quad \forall v_1 \in \mathcal{V}(\mathcal{M}) . \tag{4.18}$$

Identifying (4.18) with (4.17) implies that $X_H = J \mathrm{grad}_v H$, therefore Hamilton's equations of motion take the form

$$\partial_t v = J \mathrm{grad}_v H . \tag{4.19}$$

With this at hand, it is straightforward to show that the Hamiltonian is conserved over time. Indeed, we have

$$\begin{aligned} \frac{dH}{dt} &= \langle \mathrm{grad}_v H, \partial_t v \rangle = \langle \mathrm{grad}_v H, J \mathrm{grad}_v H \rangle , \\ &= \frac{1}{2} \langle \mathrm{grad}_v H, J \mathrm{grad}_v H \rangle + \frac{1}{2} \langle \mathrm{grad}_v H, J \mathrm{grad}_v H \rangle , \\ &= \frac{1}{2} \langle \mathrm{grad}_v H, J \mathrm{grad}_v H \rangle - \frac{1}{2} \langle J \mathrm{grad}_v H, \mathrm{grad}_v H \rangle , \\ &= 0 , \end{aligned}$$

due to the skew symmetry of J. Similarly, the Poisson brackets between H and other functions F are defined by

$$\{H, F\} = \frac{dF}{dt} = \langle \mathrm{grad}_v F, \partial_t v \rangle = \langle \mathrm{grad}_v F, J \mathrm{grad}_v H \rangle .$$

This definition implies that, if $\{H, F\} = 0$ (i.e. F Poisson commutes with H), then $dF/dt = 0$ and consequently F is an invariant of motion for the Hamiltonian system (4.16). In particular, $\{H, H\} = dH/dt = 0$ (i.e. H Poisson commutes with itself) as shown above.

We denote the solution map, or the flow, for the initial value problem (4.16) by $v(t) = \chi_t(v_0)$. From the classical theory of ordinary differential equations (ODEs), whenever the Hamiltonian vector field $X_H \in C^1(\mathcal{M}, \mathcal{V}(\mathcal{M}))$, meaning that the Hamiltonian $H \in C^2(\mathcal{M}, \mathbb{R})$, then the flow is defined and unique, at least locally in time. We point out, however, that this regularity condition rarely holds when Eq. (4.16) is described by a PDE, and much effort has been devoted to the study of well-posedness and properties of the solution map for numerous examples of evolution equations. Furthermore, it is not clear that the property of being a Hamiltonian system is of crucial importance in this effort. Nonetheless, because of its interest in various special cases and because Hamiltonian PDEs appear naturally in many areas of physics, it is reasonable to take seriously the analogy between Hamiltonian dynamical systems and PDEs.

4.2.3 Examples of Hamiltonian PDEs

To illustrate the basic concepts introduced above, we review well-known examples of Hamiltonian PDEs. These will be relevant to a subsequent discussion on reduced models for water waves. It is assumed that suitable boundary and initial conditions are specified in each case.

Quasilinear Wave Equation

Consider a scalar field $p(x, t)$ satisfying the equation

$$\partial_t^2 p = \Delta p - \partial_p Q(p, x) \,, \quad x \in \Omega \subseteq \mathbb{R}^{d-1} \,. \tag{4.20}$$

This can be written in the form (4.19) with

$$v = \begin{pmatrix} p \\ q \end{pmatrix} , \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

and

$$H = \int_\Omega \left(\frac{1}{2} q^2 + \frac{1}{2} |\nabla p|^2 + Q(p, x) \right) dx \,.$$

More specifically, the second-order equation (4.20) is equivalent to a system of first-order equations

$$\begin{aligned} \partial_t p &= q \,, \\ \partial_t q &= \Delta p - \partial_p Q \,, \end{aligned}$$

such that $q = \mathrm{grad}_q H$ and $\Delta p - \partial_p Q = -\mathrm{grad}_p H$.

The gradient is taken with respect to the $L^2(\Omega)$ inner product, which also dictates what Hilbert space should be proposed for $\mathcal{M}$. In particular, by definition,

$$\begin{aligned}\langle \mathrm{grad}_p H, p_1\rangle &= \int_\Omega p_1(\mathrm{grad}_p H)\,dx = \frac{d}{ds}H(p+sp_1)\Big|_{s=0}, \qquad (4.21)\\ &= \frac{d}{ds}\int_\Omega \left(\frac{1}{2}q^2 + \frac{1}{2}|\nabla(p+sp_1)|^2 + Q(p+sp_1,x)\right)dx\Big|_{s=0},\\ &= \frac{d}{ds}\int_\Omega \left(\frac{1}{2}q^2 + \frac{1}{2}|\nabla(p+sp_1)|^2 + Q(p,x) + sp_1\partial_p Q + \dots\right)dx\Big|_{s=0},\\ &= \int_\Omega (\nabla p\cdot\nabla p_1 + p_1\partial_p Q)\,dx = \int_\Omega (-p_1\Delta p + p_1\partial_p Q)\,dx\,,\end{aligned}$$

via integration by parts, for any $p_1 \in \mathcal{V}(\mathcal{M})$ and with vanishing boundary conditions. Hence, by identification, $\mathrm{grad}_p H = -\Delta p + \partial_p Q$ as claimed above.

Considering that the Laplacian Δ is an unbounded operator, the initial value problem should be posed only on an appropriate subdomain of $\mathcal{M}$. Note the characteristic skew-symmetric form of the operator J in this case. We will say that a Hamiltonian system with J of this form is in Darboux coordinates.

Boussinesq System

Long waves on shallow water can be described by coupled nonlinear equations of the form

$$\begin{aligned}\partial_t p &= -\partial_x\big(q + \partial_q Q\big)\,,\\ \partial_t q &= -\partial_x\left(p \mp \frac{1}{3}\partial_x^2 p + \partial_p Q\right), \quad x\in\mathbb{T}\,,\end{aligned}$$

where $Q(p,q)$ is a nonlinear function of p and q. The variable $q(x,t)$ is related to the vertical displacement of the water surface, while the variable $p(x,t)$ is related to a horizontal velocity of the fluid. This is a Hamiltonian system of the form (4.19) with

$$v = \begin{pmatrix} p \\ q\end{pmatrix}, \quad J = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0\end{pmatrix},$$

and

$$H = \int_{\mathbb{T}}\left(\frac{1}{2}p^2 + \frac{1}{2}q^2 \pm \frac{1}{6}(\partial_x p)^2 + Q(p,q)\right)dx\,. \qquad (4.22)$$

It can be checked that

$$\mathrm{grad}_p H = p \mp \frac{1}{3}\partial_x^2 p + \partial_p Q\,, \quad \mathrm{grad}_q H = q + \partial_q Q\,.$$

The $-$ sign in (4.22) is ill-posed (the "bad" Boussinesq system), while the $+$ sign is well-posed (the "good" Boussinesq system) [80]. Completely integrable nonlinear cases include $Q(p,q) = p^3$ [102] and $Q(p,q) = p^2q/2$ [76, 86].

In the absence of dispersion (i.e. in the absence of derivatives of p in the equations and Hamiltonian), the Boussinesq system reduces to the shallow-water equations. The mathematically rigorous study of the shallow-water limit was initiated by Kano and Nishida [74]. The work of Lannes [79] extended these results from analytic spaces to Sobolev spaces, and to more subtle limiting situations such as the Green–Naghdi system.

Korteweg–de Vries Equation

The Korteweg–de Vries (KdV) equation was first derived as a model for water waves, following a reduction of the Boussinesq system. It is a classic example of a completely integrable infinite-dimensional nonlinear system, which admits explicit analytical solutions and arises in many scientific areas. For this reason, it has been extensively studied and has been extended to various settings, including other types of nonlinearity. A generalized KdV equation can be written as

$$\partial_t p = \frac{1}{3}\partial_x^3 p - \partial_x\big(\partial_p Q(p,x)\big)\,, \quad x \in \mathbb{T}\,,$$

which can cast into Hamiltonian form (4.19) by choosing $v = p$, $J = -\partial_x$ and

$$H = \int_{\mathbb{T}} \left(\frac{1}{6}(\partial_x p)^2 + Q(p,x)\right) dx\,.$$

Completely integrable cases include $Q(p,x) = p^3$ and $Q(p,x) = p^4$. The rigorous analysis of the water wave problem in the KdV limit can be found in work by Kano and Nishida [75], Craig [24], and Schneider and Wayne [87].

Nonlinear Schrödinger Equation

The nonlinear Schrödinger (NLS) equation was first derived to describe the modulation of periodic waves on deep water [101]. It is another example of a universal model for wave propagation in nonlinear media, and its study has also produced an abundant literature [90]. A generalized NLS equation can be written as

$$i\partial_t u = -\frac{1}{2}\Delta u + \partial_{\overline{u}} Q(u,\overline{u},x)\,, \quad x \in \mathbb{T}^{d-1}\,, \tag{4.23}$$

for the complex envelope $u(x,t)$ of periodic waves. The symbol $\bar{\cdot}$ denotes complex conjugation. This is a Hamiltonian PDE in the sense of (4.19) with

$$v = \begin{pmatrix} u \\ \overline{u} \end{pmatrix}\,, \quad J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,,$$

and

$$H = \int_{\mathbb{T}^d} \left(\frac{1}{2}|\nabla u|^2 + Q(u,\overline{u},x)\right) dx\,. \tag{4.24}$$

This system gives two equivalent equations, one for u and another for its complex conjugate $\overline{u}$. Similar to (4.21), it is clear that Eq. (4.23) is given by $\partial_t u = -i\,\mathrm{grad}_{\overline{u}} H$, considering that $|\nabla u|^2 = \nabla u \cdot \nabla \overline{u}$ in (4.24). The NLS equation often admits a gauge symmetry under phase translation, in which case $Q(u, \overline{u}, x) = Q(|u|^2, x)$. Completely integrable cases include $Q(|u|^2, x) = \pm |u|^4/2$ for $d = 2$. The rigorous justification of the NLS approximation for two-dimensional water waves was addressed by Totz and Wu [93] and Düll et al. [50] in the infinite- and finite-depth case, respectively.

In the following, we will introduce a unified perturbation approach based on the Hamiltonian formulation of the water wave problem, and we will present a detailed derivation of such models in appropriate scaling limits.

4.2.4 Zakharov's Hamiltonian for Water Waves

Zakharov [101] showed that the boundary value problem (4.8)–(4.11) can be recast as a closed Hamiltonian system in canonical form

$$\begin{pmatrix} \partial_t \eta \\ \partial_t \xi \end{pmatrix} = J \begin{pmatrix} \mathrm{grad}_\eta H \\ \mathrm{grad}_\xi H \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.25}$$

for the two Darboux coordinates $\eta(x, t)$ and

$$\xi(x, t) = \varphi(x, \eta(x, t), t)\,,$$

the latter being the trace of the velocity potential at the free surface. As shown below, it is a lower-dimensional system of equations compared to (4.8)–(4.11) in the sense that it involves "surface" variables alone, whose spatial dependence is only on the horizontal hyperplane. The Hamiltonian H in (4.25) is given by the energy (4.12). The equivalence between the two systems (4.8)–(4.11) and (4.25) is far from being obvious, in part because the fluid velocity depends nonlocally on the moving boundary of the domain and the variable ξ does not appear explicitly in the original formulation (4.8)–(4.11).

In subsequent work, Craig and Sulem [40] made the key observation that the dependence on η can be made more explicit by introducing the Dirichlet–Neumann operator (DNO)

$$G(\eta) : \xi \longmapsto \sqrt{1 + |\partial_x \eta|^2}\,(\mathbf{n} \cdot \nabla \varphi)\big|_{y=\eta}\,, \tag{4.26}$$

which is the singular integral operator that takes Dirichlet data ξ on Γ, solves Laplace's equation (4.8) subject to (4.11), and returns the corresponding Neumann data (i.e. the normal fluid velocity on Γ). This operation is linear in ξ but depends nonlinearly on η. It is also an inherent assumption of the Hamiltonian formulation (4.25) and of the definition (4.26) of the DNO that the function η representing the free surface be a graph of x, which implies that overturning waves (with a multi-valued profile) are not permitted here.

With this at hand, the right-hand sides of (4.25) can be expressed in terms of surface variables alone. By the chain rule, we have

$$\partial_t \xi = \partial_t \varphi + (\partial_t \eta)(\partial_y \varphi)\big|_{y=\eta} \,, \tag{4.27}$$

and

$$\partial_x \xi = \partial_x \varphi + (\partial_x \eta)(\partial_y \varphi)\big|_{y=\eta} \,. \tag{4.28}$$

Recognizing that

$$G(\eta)\xi = \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi\big|_{y=\eta} \,,$$

from the definition (4.26) of the DNO, and combining it with (4.28), leads to

$$\partial_y \varphi\big|_{y=\eta} = G(\eta)\xi + \partial_x \eta \cdot \partial_x \varphi\big|_{y=\eta} = G(\eta)\xi + \partial_x \eta \cdot \big(\partial_x \xi - (\partial_x \eta)(\partial_y \varphi)\big)\big|_{y=\eta} \,,$$

hence

$$\partial_y \varphi\big|_{y=\eta} = \frac{G(\eta)\xi + \partial_x \eta \cdot \partial_x \xi}{1 + |\partial_x \eta|^2} \,. \tag{4.29}$$

Substituting this expression back into (4.28) yields

$$\begin{aligned} \partial_x \varphi\big|_{y=\eta} &= \partial_x \xi - \frac{\partial_x \eta}{1 + |\partial_x \eta|^2}\big(G(\eta)\xi + \partial_x \eta \cdot \partial_x \xi\big) \,, \qquad (4.30) \\ &= \frac{\partial_x \xi - (\partial_x \eta) G(\eta)\xi + |\partial_x \eta|^2 \partial_x \xi - (\partial_x \eta \cdot \partial_x \xi)\partial_x \eta}{1 + |\partial_x \eta|^2} \,, \end{aligned}$$

from which we obtain

$$\begin{aligned} |\partial_x \varphi|^2 + (\partial_y \varphi)^2\big|_{y=\eta} &= \frac{|\partial_x \xi|^2 + (G(\eta)\xi)^2 - (\partial_x \eta \cdot \partial_x \xi)^2 + |\partial_x \eta|^2 |\partial_x \xi|^2}{1 + |\partial_x \eta|^2} \,, \\ &= |\partial_x \xi|^2 + \frac{(G(\eta)\xi)^2 - (\partial_x \eta \cdot \partial_x \xi)^2}{1 + |\partial_x \eta|^2} \,. \qquad (4.31) \end{aligned}$$

Substituting (4.29) into (4.27) gives

$$\partial_t \varphi\big|_{y=\eta} = \partial_t \xi - \frac{G(\eta)\xi}{1 + |\partial_x \eta|^2}\big(G(\eta)\xi + \partial_x \eta \cdot \partial_x \xi\big) \,, \tag{4.32}$$

where we have used the identity

$$\partial_t \eta = G(\eta)\xi \,, \tag{4.33}$$

by virtue of the kinematic condition (4.9). With (4.31) and (4.32) in mind, the dynamic condition (4.10) can be written as

$$\partial_t \xi = -g\eta - \frac{|\partial_x \xi|^2}{2} + \frac{\big(G(\eta)\xi + \partial_x \eta \cdot \partial_x \xi\big)^2}{2(1 + |\partial_x \eta|^2)} \,. \tag{4.34}$$

Equations (4.33) and (4.34) are the full expressions of the Hamiltonian formulation (4.25) for surface gravity water waves.

Similarly, the Hamiltonian (4.12) can be expressed as a lower-dimensional integral in terms of surface variables alone. Restricting our attention to the kinetic part, we note that

$$K = \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta} \frac{1}{2} |\nabla\varphi|^2 \, dydx = \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta} \frac{1}{2} \big(\nabla \cdot (\varphi \nabla \varphi) - \varphi \Delta \varphi \big) \, dydx \,,$$

where the last term vanishes due to (4.8). Then using the divergence theorem, we obtain

$$2K = \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta} \nabla \cdot (\varphi \nabla \varphi) \, dydx = \int_{\Gamma} (\varphi \nabla \varphi) \cdot \mathbf{n} \, dS + \int_{\Gamma_b} (\varphi \nabla \varphi) \cdot \mathbf{n} \, dS \,,$$

where the surface integral along Γ_b vanishes as a result of (4.11). It readily follows from (4.26) that

$$\begin{aligned} 2K &= \int_{\Gamma} (\varphi \nabla \varphi) \cdot \mathbf{n} \, dS = \int_{\Gamma} \varphi (\nabla \varphi \cdot \mathbf{n}) \, dS \,, \\ &= \int_{\mathbb{R}^{d-1}} \varphi (\mathbf{n} \cdot \nabla \varphi) \big|_{y=\eta} \sqrt{1 + |\partial_x \eta|^2} \, dx = \int_{\mathbb{R}^{d-1}} \xi G(\eta) \xi \, dx \,, \end{aligned}$$

hence

$$H = \frac{1}{2} \int_{\mathbb{R}^{d-1}} \big(\xi G(\eta) \xi + g \eta^2 \big) \, dx \,, \tag{4.35}$$

which is Zakharov's Hamiltonian given explicitly in terms of the two conjugate variables η and ξ. As for the impulse (4.15), it can be converted to

$$I = - \int_{\mathbb{R}^{d-1}} \xi \partial_x \eta \, dx = \int_{\mathbb{R}^{d-1}} \eta \partial_x \xi \, dx \,, \tag{4.36}$$

via integration by parts.

4.3 Dirichlet–Neumann Operator and Its Analysis

The DNO plays a key role in the present formulation and analysis. We recall here some of its properties as these will be relevant to our subsequent discussion. The reader is referred to [79] for a more detailed presentation of the DNO. Let $\eta \in C^1(\mathbb{R}^{d-1})$. Then $G(\eta)$ satisfies the following properties:

1. $G(\eta)$ is a continuous operator from $H^1(\mathbb{R}^{d-1})$ to $L^2(\mathbb{R}^{d-1})$, and more generally from $H^s(\mathbb{R}^{d-1})$ to $H^{s-1}(\mathbb{R}^{d-1})$.
2. $G(\eta)$ is self-adjoint and positive semi-definite, with $G(\eta)1 = 0$.

3. As an operator $G(\eta) : H^1(\mathbb{R}^{d-1}) \to L^2(\mathbb{R}^{d-1})$, it depends analytically upon $\eta \in B_R(0) \subseteq C^1(\mathbb{R}^{d-1})$ for some nonzero value of R.

Here, $H^s(\mathbb{R}^{d-1})$ denotes the Sobolev space of order s equipped with the norm

$$\|f\|^2_{H^s(\mathbb{R}^{d-1})} = \sum_{0\le|\alpha|\le s} \|D^\alpha f\|^2_{L^2(\mathbb{R}^{d-1})} .$$

The latter property entails questions related to the boundedness of singular integrals on hypersurfaces. It was proved in the case $d = 2$ by Coifman and Meyer [20] and in the case $d \geq 2$ by Craig et al. [39]. In particular, it implies the existence of a convergent Taylor expansion for the DNO. This will be discussed in more detail in a subsequent section.

4.3.1 Legendre Transform

In the water wave problem, the surface elevation η is a natural choice of dynamical variable and acts as the "angle" variable in the Hamiltonian formulation. On the other hand, the choice of the "action" variable ξ, which is canonically conjugate to η, is less obvious. Using the DNO, we show that this "action" variable can be easily deduced from first principles of classical mechanics. This is accomplished by expressing the Lagrangian as

$$L = K - P ,$$

through an analogy with classical mechanics. In terms of η and the corresponding tangent-space variable $\dot\eta = \partial_t\eta$, the Lagrangian takes the form

$$\begin{aligned} L &= \frac{1}{2}\int_{\mathbb{R}^{d-1}} \varphi(\mathbf{n}\cdot\nabla\varphi)\big|_{y=\eta}\sqrt{1+|\partial_x\eta|^2}\,dx - \frac{1}{2}\int_{\mathbb{R}^{d-1}} g\eta^2\,dx ,\\ &= \frac{1}{2}\int_{\mathbb{R}^{d-1}} \left(\dot\eta G(\eta)^{-1}\dot\eta - g\eta^2\right) dx , \end{aligned}$$

given the fact that

$$\dot\eta = \sqrt{1+|\partial_x\eta|^2}\,(\mathbf{n}\cdot\nabla\varphi)\big|_{y=\eta} = G(\eta)\varphi(x,\eta,t) ,$$

based on (4.9) and (4.26). The inverse of $G(\eta)$ can be defined because $\varphi(x,\eta,t) = G(\eta)^{-1}\dot\eta$ is determined up to an additive constant that is irrelevant to the dynamics. The two canonical conjugate variables then follow from the Legendre transform

$$\left(\eta, \mathrm{grad}_{\dot\eta} L\right) = \left(\eta, G(\eta)^{-1}\dot\eta\right) = (\eta,\xi) ,$$

which is precisely the choice of Darboux coordinates introduced by Zakharov [101].

4.3.2 Shape Derivative of H

It is easy to see that

$$\mathrm{grad}_\xi H = G(\eta)\xi\,,$$

since H is quadratic in ξ and $G(\eta)$ is self-adjoint. On the other hand, it is a more subtle calculation to show that $\mathrm{grad}_\eta H$ coincides with the right-hand side of (4.34). We provide here such a calculation along the lines of that given in [26]. Noting that

$$\mathrm{grad}_\eta H = \mathrm{grad}_\eta K + \mathrm{grad}_\eta P = \mathrm{grad}_\eta K + g\eta\,,$$

the gradient of the kinetic energy K with respect to η is the tricky part. Consider a fluid domain Ω with free surface η and a family of nearby domains Ω_1 with nearby free surfaces $\eta_1 = \eta + \delta\eta$, where $0 < \delta \ll 1$. Denote the corresponding outward unit normals by $\mathbf{n}$ and $\mathbf{n}_1$. We consider the Dirichlet integrals

$$K(\eta,\xi) = \frac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G(\eta)\xi\, dx\,, \quad K_1 = K(\eta_1,\xi) = \frac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G(\eta_1)\xi\, dx\,,$$

for which we impose that the traces of the velocity potentials φ_1 on η_1 and φ on η coincide, i.e.

$$\varphi(x,\eta,t) = \xi(x,t) = \varphi_1(x,\eta_1,t)\,,$$

while we vary the boundary curve from η to η_1. This is to say that we take the partial derivative of K with respect to variations of the domain, while fixing the boundary conditions of the velocity potential. For this purpose, the boundary values of φ on η_1 can be expressed as

$$\varphi(\eta_1) = \varphi(\eta) + \delta\eta\,\partial_y\varphi + O(\delta^2)\,,$$

and thus

$$\varphi_1(\eta_1) - \varphi(\eta_1) = -\delta\eta\,\partial_y\varphi + O(\delta^2)\,, \tag{4.37}$$

where we have dropped the dependence on x and t for convenience. Furthermore, considering a harmonic function φ defined in a neighborhood that includes $\Omega\cup\Omega_1$, the difference of boundary integral expressions for the Dirichlet integrals is given by

$$\begin{aligned}\int_{\Gamma_1}\varphi(\mathbf{n}_1\cdot\nabla\varphi)\,dS - \int_{\Gamma}\varphi(\mathbf{n}\cdot\nabla\varphi)\,dS &= \int_{\Omega_1}|\nabla\varphi|^2 dV - \int_{\Omega}|\nabla\varphi|^2 dV\,,\\ &= \int_{\Omega_1\setminus\Omega}|\nabla\varphi|^2 dV\,,\\ &\simeq \int_{\Gamma}|\nabla\varphi|^2\,\delta\eta dx\,,\end{aligned} \tag{4.38}$$

according to Green's first identity. The variation of the kinetic energy with fixed boundary data ξ is calculated as the limit $\delta \to 0$ of

$$
\begin{aligned}
K_1 - K &= \frac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G(\eta_1)\xi \, dx - \frac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G(\eta)\xi \, dx \,, \\
&= \frac{1}{2}\int_{\Gamma_1} \varphi_1(\mathbf{n}_1 \cdot \nabla\varphi_1) \, dS - \frac{1}{2}\int_{\Gamma} \varphi(\mathbf{n} \cdot \nabla\varphi) \, dS \,, \\
&= \int_{\Gamma_1} (\varphi_1 - \varphi)(\mathbf{n}_1 \cdot \nabla\varphi_1) \, dS - \frac{1}{2}\int_{\Gamma_1} \varphi_1(\mathbf{n}_1 \cdot \nabla\varphi_1) \, dS \\
&\quad + \int_{\Gamma_1} \varphi(\mathbf{n}_1 \cdot \nabla\varphi_1) \, dS - \frac{1}{2}\int_{\Gamma} \varphi(\mathbf{n} \cdot \nabla\varphi) \, dS \,,
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
K_1 - K &= \int_{\Gamma} -\delta\eta \, \partial_y\varphi(\mathbf{n} \cdot \nabla\varphi) \, dS + \frac{1}{2}\int_{\Gamma_1} \varphi(\mathbf{n}_1 \cdot \nabla\varphi) \, dS \\
&\quad - \frac{1}{2}\int_{\Gamma} \varphi \, \mathbf{n} \cdot \nabla(\delta\eta \, \partial_y\varphi) \, dS + \frac{1}{2}\int_{\Gamma} \delta\eta \, \partial_y\varphi(\mathbf{n} \cdot \nabla\varphi) \, dS \\
&\quad - \frac{1}{2}\int_{\Gamma} \varphi(\mathbf{n} \cdot \nabla\varphi) \, dS + O(\delta^2) \,,
\end{aligned}
$$

by using (4.37). Because the operator $\mathbf{n}\cdot\nabla$ is self-adjoint (similar to the DNO), the third and fourth integrals on the right-hand side above cancel out after integration by parts. Then appealing to (4.38) yields

$$
K_1 - K = \frac{1}{2}\int_{\Gamma} |\nabla\varphi|^2 \, \delta\eta dx - \int_{\Gamma} \delta\eta \, \partial_y\varphi(\mathbf{n} \cdot \nabla\varphi) \, dS + O(\delta^2) \,.
$$

Finally, we arrive at

$$
\begin{aligned}
K_1 - K &= \int_{\mathbb{R}^{d-1}} \delta\eta \left(\frac{|\partial_x\xi|^2}{2} + \frac{(G(\eta)\xi)^2 - (\partial_x\eta \cdot \partial_x\xi)^2}{2(1 + |\partial_x\eta|^2)} \right. \\
&\qquad \left. - \frac{G(\eta)\xi + \partial_x\eta \cdot \partial_x\xi}{1 + |\partial_x\eta|^2} G(\eta)\xi \right) dx + O(\delta^2) \,, \\
&= \int_{\mathbb{R}^{d-1}} \delta\eta \left(\frac{|\partial_x\xi|^2}{2} - \frac{\big(G(\eta)\xi + \partial_x\eta \cdot \partial_x\xi\big)^2}{2(1 + |\partial_x\eta|^2)} \right) dx + O(\delta^2) \,,
\end{aligned}
$$

via (4.26), (4.29), and (4.31). Identification with

$$
K_1 - K = \int_{\mathbb{R}^{d-1}} \delta\eta \, (\mathrm{grad}_\eta K) \, dx + O(\delta^2) = \langle \mathrm{grad}_\eta K, \delta\eta \rangle + O(\delta^2)
$$

confirms that

$$
\mathrm{grad}_\eta K = \frac{|\partial_x\xi|^2}{2} - \frac{\big(G(\eta)\xi + \partial_x\eta \cdot \partial_x\xi\big)^2}{2(1 + |\partial_x\eta|^2)} \,, \tag{4.39}
$$

hence $\partial_t\xi = -\mathrm{grad}_\eta H = -g\eta - \mathrm{grad}_\eta K$ as stated in (4.34).

Equation (4.39) allows us to write the shape derivative $\mathrm{grad}_\eta G(\eta)\xi$ as an operator acting on $\delta\eta$ defined by

$$\delta\eta \longmapsto \mathrm{grad}_\eta G(\eta)\xi \cdot \delta\eta = -G(\eta)\Big(\delta\eta\, \partial_y\varphi\big|_{y=\eta}\Big) - \partial_x \cdot \Big(\delta\eta\, \partial_x\varphi\big|_{y=\eta}\Big),$$

where $\partial_y\varphi\big|_{y=\eta}$ and $\partial_x\varphi\big|_{y=\eta}$ (the components of the fluid velocity evaluated at the free surface) are given by (4.29) and (4.30), respectively. Indeed, it follows from (4.39) that

$$\begin{aligned}\langle \xi, \mathrm{grad}_\eta G(\eta)\xi \cdot \delta\eta\rangle &= \int_{\mathbb{R}^{d-1}} \delta\eta \left(|\partial_x\xi|^2 - \frac{\big(G(\eta)\xi + \partial_x\eta \cdot \partial_x\xi\big)^2}{1 + |\partial_x\eta|^2} \right) dx\,, \\ &= \int_{\mathbb{R}^{d-1}} \delta\eta \Big(|\partial_x\xi|^2 - \big(G(\eta)\xi + \partial_x\eta \cdot \partial_x\xi\big) \partial_y\varphi\big|_{y=\eta} \Big) dx\,.\end{aligned}$$

Integrating by parts and using the fact that $G(\eta)$ is self-adjoint, we find

$$\begin{aligned}\langle \xi, \mathrm{grad}_\eta G(\eta)\xi \cdot \delta\eta\rangle &= -\int_{\mathbb{R}^{d-1}} \xi \partial_x \cdot (\delta\eta\, \partial_x\xi)\, dx \\ &\quad - \int_{\mathbb{R}^{d-1}} \xi \Big[G(\eta)\Big(\delta\eta\, \partial_y\varphi\big|_{y=\eta}\Big) - \partial_x \cdot \Big(\delta\eta\, (\partial_x\eta)\, \partial_y\varphi\big|_{y=\eta}\Big)\Big] dx\,, \\ &= -\int_{\mathbb{R}^{d-1}} \xi \Big[G(\eta)\Big(\delta\eta\, \partial_y\varphi\big|_{y=\eta}\Big) + \partial_x \cdot \Big(\delta\eta\, \partial_x\varphi\big|_{y=\eta}\Big)\Big] dx\,,\end{aligned}$$

which, by identification, yields the above formula for $\mathrm{grad}_\eta G(\eta)\xi \cdot \delta\eta$.

4.3.3 Invariants of Motion

It can be checked that the energy (4.35) is conserved over time for the Hamiltonian formulation (4.25). Indeed, given $v = (\eta, \xi)^\top$, we have

$$\begin{aligned}\frac{dH}{dt} &= \{H, H\} = \langle \mathrm{grad}_v H, J\mathrm{grad}_v H\rangle\,, \\ &= \int_{\mathbb{R}^{d-1}} \big((\mathrm{grad}_\eta H)(\mathrm{grad}_\xi H) - (\mathrm{grad}_\xi H)(\mathrm{grad}_\eta H)\big)\, dx\,, \\ &= 0\,.\end{aligned}$$

Similarly, the volume (4.13) is conserved because

$$\begin{aligned}\frac{dV}{dt} &= \{H, V\}\,, \\ &= \int_{\mathbb{R}^{d-1}} \big((\mathrm{grad}_\eta V)(\mathrm{grad}_\xi H) - (\mathrm{grad}_\xi V)(\mathrm{grad}_\eta H)\big)\, dx\,, \\ &= \int_{\mathbb{R}^{d-1}} \partial_t\eta\, dx = \int_{\mathbb{R}^{d-1}} G(\eta)\xi\, dx\,, \\ &= \int_{\mathbb{R}^{d-1}} \xi G(\eta) 1\, dx = 0\,,\end{aligned}$$

due to Property 2 of the DNO.

Verifying the conservation of the impulse (4.15) is a nontrivial task and, for this purpose, it is preferable to work directly with the original definition, yielding

$$\frac{dI}{dt} = \frac{d}{dt}\left(\int_{\mathbb{R}^{d-1}}\int_{-h}^{\eta}\partial_x\varphi\,dydx\right) = \int_{\mathbb{R}^{d-1}}\left((\partial_t\eta)(\partial_x\varphi)\big|_{y=\eta} + \int_{-h}^{\eta}\partial_{tx}\varphi\,dy\right)dx\,,$$
$$= \int_{\mathbb{R}^{d-1}}\left((\partial_t\eta)(\partial_x\varphi)\big|_{y=\eta} + \partial_x\int_{-h}^{\eta}\partial_t\varphi\,dy - (\partial_x\eta)(\partial_t\varphi)\big|_{y=\eta}\right)dx\,.$$

Again, the total x-derivative does not contribute because of the boundary conditions. Noting that $I \in \mathbb{R}^{d-1}$, it is more convenient to proceed with a componentwise calculation. Substituting (4.9) and (4.10) for $\partial_t\eta$ and $\partial_t\varphi$, respectively, leads to

$$\begin{aligned}\frac{dI_j}{dt} &= \int_{\mathbb{R}^{d-1}}\Big[(\partial_y\varphi)(\partial_j\varphi) - (\partial_j\eta)(\partial_j\varphi)^2 - (\partial_\ell\eta)(\partial_\ell\varphi)(\partial_j\varphi)\\ &\qquad + (\partial_j\eta)\Big(\frac{1}{2}|\nabla\varphi|^2 + g\eta\Big)\Big]_{y=\eta}dx\,,\\ &= \int_{\mathbb{R}^{d-1}}\Big[(\partial_y\varphi)(\partial_j\varphi) - \frac{1}{2}(\partial_j\eta)(\partial_j\varphi)^2 + \frac{1}{2}(\partial_j\eta)(\partial_\ell\varphi)^2\\ &\qquad + \frac{1}{2}(\partial_j\eta)(\partial_y\varphi)^2 - (\partial_\ell\eta)(\partial_\ell\varphi)(\partial_j\varphi) + \frac{1}{2}g\partial_j\eta^2\Big]_{y=\eta}dx\,,\end{aligned} \tag{4.40}$$

where ∂_j and ∂_ℓ denote the partial derivatives with respect to the horizontal coordinates x_j and x_ℓ, respectively, in the case $d > 2$. The gravity term integrates to zero for the same reason as above. Realizing that the above integral is a surface integral of the form $\int_\Gamma \mathbf{F}\cdot\mathbf{n}\,dS$ with

$$\mathbf{F} = \left(\frac{1}{2}(\partial_j\varphi)^2 - \frac{1}{2}(\partial_\ell\varphi)^2 - \frac{1}{2}(\partial_y\varphi)^2, (\partial_\ell\varphi)(\partial_j\varphi), (\partial_y\varphi)(\partial_j\varphi)\right)^\top\,,$$

and because

$$\nabla\cdot\mathbf{F} = (\partial_j, \partial_\ell, \partial_y)^\top\cdot\mathbf{F} = (\partial_j\varphi)(\partial_j^2 + \partial_\ell^2 + \partial_y^2)\varphi = (\partial_j\varphi)\Delta\varphi = 0\,,$$

by virtue of (4.8), this implies that $dI_j/dt = 0$ according to the divergence theorem. Alternatively, the conservation of I can be expressed by the Poisson brackets

$$\frac{dI}{dt} = 0 = \{H, I\}\,.$$

Although the calculation is less straightforward in this case, it can be recognized that

$$\begin{aligned}\{H, I\} &= \int_{\mathbb{R}^{d-1}}\big((\mathrm{grad}_\eta I)(\mathrm{grad}_\xi H) - (\mathrm{grad}_\xi I)(\mathrm{grad}_\eta H)\big)\,dx\,,\\ &= \int_{\mathbb{R}^{d-1}}\big((\partial_x\xi)(\partial_t\eta) - (\partial_x\eta)(\partial_t\xi)\big)\,dx\,,\\ &= \int_{\mathbb{R}^{d-1}}\left[(\partial_x\xi)G(\eta)\xi + (\partial_x\eta)\left(\frac{|\partial_x\xi|^2}{2} - \frac{\big(G(\eta)\xi + \partial_x\eta\cdot\partial_x\xi\big)^2}{2(1+|\partial_x\eta|^2)} + g\eta\right)\right]dx\end{aligned}$$

is a version of (4.40) that is written explicitly in terms of the surface variables η and ξ, for which we have used (4.33) and (4.34).

4.3.4 Taylor Expansion of G

Exploiting Property 3 of the DNO, Craig and Sulem [40] devised a perturbative approach for computing it based on a Taylor series expansion

$$G(\eta) = \sum_{j=0}^{\infty} G_j(\eta)\,, \tag{4.41}$$

about the quiescent state $\eta = 0$. Each Taylor term G_j is homogeneous of degree j in η and is determined recursively. This derivation was first given in the case $d = 2$ [40] and then extended to the case $d \geq 2$ by Craig et al. [39] and Nicholls [83]. We outline here the main steps.

Explicit formulas for the G_j's in (4.41) can be obtained by analyzing their action on suitable harmonic functions. For the boundary conditions that we have in mind (periodic or vanishing at infinity) and for such a linear equation as (4.8), we consider elementary solutions of the form

$$\varphi(x, y) = \cosh((h + y)|k|)e^{ik\cdot x}\,,$$

which satisfy (4.11) on Γ_b. Keep in mind that $k \in \mathbb{R}^{d-1}$ and thus $|k|$ denotes its Euclidian norm. For simplicity, because time is frozen in this calculation, we omit the dependence on t. We insert this form in the definition (4.26) where

$$\begin{aligned} \xi(x) &= \cosh((h + \eta)|k|)e^{ik\cdot x}\,, \\ \partial_y \varphi\big|_{y=\eta} &= \sinh((h + \eta)|k|)|k|e^{ik\cdot x}\,, \\ \partial_x \varphi\big|_{y=\eta} &= i\cosh((h + \eta)|k|)k e^{ik\cdot x}\,. \end{aligned}$$

We then adopt the representation (4.41) for the DNO and, accordingly, Taylor expand the hyperbolic functions about $\eta = 0$. This yields

$$\begin{aligned} &\left(\sum_{j=0}^{\infty} G_j(\eta)\right) \sum_{j=0}^{\infty} \frac{(\eta|k|)^j}{j!} \cosh^{(j)}(h|k|)e^{ik\cdot x} \\ &= \sum_{j=0}^{\infty} \frac{(\eta|k|)^j}{j!}|k| \sinh^{(j)}(h|k|)e^{ik\cdot x} - i(\partial_x \eta)\cdot \sum_{j=0}^{\infty} \frac{(\eta|k|)^j}{j!} k \cosh^{(j)}(h|k|)e^{ik\cdot x}\,. \end{aligned}$$

Equating terms of the same order in η leads to a recursion formula for each G_j. At zeroth order (corresponding to $\eta = 0$), it is easy to see that

$$\begin{aligned} G_0 \cosh(hk)e^{ik\cdot x} &= |k| \sinh(h|k|)e^{ik\cdot x}\,, \\ G_0 e^{ik\cdot x} &= |k| \tanh(h|k|)e^{ik\cdot x}\,, \end{aligned}$$

which defines the Fourier symbol of G_0 as applied to an elementary Fourier mode $e^{ik\cdot x}$. We can therefore introduce the symbolic notation

$$G_0 f(x) = |D| \tanh(h|D|) f(x) \,, \tag{4.42}$$

for G_0 acting on any sufficiently well-behaved function $f(x)$ that may be written in terms of a Fourier series or a Fourier transform, with $D = -i\partial_x$ (so that its Fourier symbol is k). For this reason, D and G_0 are also called Fourier multipliers.

At each order $j > 0$ in η, we find the operator relation

$$\begin{aligned} &G_j(\eta)\cosh(h|k|) + \sum_{\ell=0}^{j} G_{j-\ell}(\eta)\frac{(\eta|k|)^\ell}{\ell!}\cosh^{(\ell)}(h|k|) \\ &= \frac{(\eta|k|)^j}{j!}|k|\sinh^{(j)}(h|k|) - i(\partial_x\eta)\cdot\frac{(\eta|k|)^{j-1}}{(j-1)!}k\cosh^{(j-1)}(h|k|)\,, \end{aligned}$$

acting again on $e^{ik\cdot x}$ but, for convenience, this factor has now been dropped out. Noting that the derivatives of the hyperbolic functions satisfy

$$\cosh^{(j)}(h|k|) = \begin{cases} \cosh(h|k|)\,, & j \text{ even}\,, \\ \sinh(h|k|)\,, & j \text{ odd}\,, \end{cases} \qquad \sinh^{(j)}(h|k|) = \begin{cases} \sinh(h|k|)\,, & j \text{ even}\,, \\ \cosh(h|k|)\,, & j \text{ odd}\,, \end{cases}$$

we can split the contributions into two parts. For $j > 0$ even,

$$\begin{aligned} G_j(\eta)\cosh(h|k|) = &-\sum_{\ell=2,\text{even}}^{j} G_{j-\ell}(\eta)\frac{(\eta|k|)^\ell}{\ell!}\cosh(h|k|) \\ &-\sum_{\ell=1,\text{odd}}^{j-1} G_{j-\ell}(\eta)\frac{(\eta|k|)^\ell}{\ell!}\sinh(h|k|) \\ &+\frac{(\eta|k|)^j}{j!}|k|\sinh(h|k|) - i(\partial_x\eta)\cdot\frac{(\eta|k|)^{j-1}}{(j-1)!}k\sinh(h|k|)\,, \end{aligned}$$

hence

$$\begin{aligned} G_j(\eta) = &-\sum_{\ell=2,\text{even}}^{j} G_{j-\ell}(\eta)\frac{(\eta|k|)^\ell}{\ell!} - \sum_{\ell=1,\text{odd}}^{j-1} G_{j-\ell}(\eta)\frac{(\eta|k|)^\ell}{\ell!}\tanh(h|k|) \\ &+\frac{(\eta|k|)^j}{j!}|k|\tanh(h|k|) - i(\partial_x\eta)\cdot\frac{(\eta|k|)^{j-1}}{(j-1)!}k\tanh(h|k|)\,, \end{aligned}$$

which is equivalent to

$$\begin{aligned} G_j(\eta) = &-\sum_{\ell=2,\text{even}}^{j} G_{j-\ell}(\eta)\frac{\eta^\ell|D|^\ell}{\ell!} - \sum_{\ell=1,\text{odd}}^{j-1} G_{j-\ell}(\eta)\frac{\eta^\ell|D|^\ell}{\ell!}\tanh(h|D|) \\ &+\frac{\eta^j|D|^j}{j!}|D|\tanh(h|D|) + (D\eta)\cdot\frac{\eta^{j-1}|D|^{j-1}}{(j-1)!}D\tanh(h|D|)\,. \end{aligned}$$

In this equivalence, it is important to keep in mind that D is an operator acting on functions of x and therefore its position within each term above matters. By convention, unless parentheses are specified, operators such as D, $|D|$, and G_j act on all that is immediately located on their right side. Because

$$\begin{aligned} D \cdot \frac{\eta^j |D|^{j-1}}{j!} D = D \cdot \left(\frac{\eta^j |D|^{j-1}}{j!} D \right) &= j(D\eta) \cdot \frac{\eta^{j-1} |D|^{j-1}}{j!} D + \frac{\eta^j |D|^{j-1}}{j!} |D|^2 \,, \\ &= (D\eta) \cdot \frac{\eta^{j-1} |D|^{j-1}}{(j-1)!} D + \frac{\eta^j |D|^j}{j!} |D| \,, \end{aligned}$$

by the product rule of differentiation, we arrive at

$$\begin{aligned} G_j(\eta) = &- \sum_{\ell=2,\text{even}}^{j} G_{j-\ell}(\eta) \frac{\eta^\ell |D|^\ell}{\ell!} - \sum_{\ell=1,\text{odd}}^{j-1} G_{j-\ell}(\eta) \frac{\eta^\ell |D|^\ell}{\ell!} \tanh(h|D|) \\ &+ D \cdot \frac{\eta^j |D|^{j-1}}{j!} D \tanh(h|D|) \,, \end{aligned}$$

and by using (4.42), a full recursive form emerges as

$$\begin{aligned} G_j(\eta) = &- \sum_{\ell=2,\text{even}}^{j} G_{j-\ell}(\eta) \frac{\eta^\ell |D|^\ell}{\ell!} - \sum_{\ell=1,\text{odd}}^{j-1} G_{j-\ell}(\eta) \frac{\eta^\ell |D|^{\ell-1}}{\ell!} G_0 \\ &+ D \cdot \frac{\eta^j |D|^{j-2}}{j!} D G_0 \,. \end{aligned} \tag{4.43}$$

Similarly, for j odd,

$$\begin{aligned} G_j(\eta) \cosh(h|k|) = &- \sum_{\ell=2,\text{even}}^{j-1} G_{j-\ell}(\eta) \frac{(\eta|k|)^\ell}{\ell!} \cosh(h|k|) \\ &- \sum_{\ell=1,\text{odd}}^{j} G_{j-\ell}(\eta) \frac{(\eta|k|)^\ell}{\ell!} \sinh(h|k|) \\ &+ \frac{(\eta|k|)^j}{j!} |k| \cosh(h|k|) - i(\partial_x \eta) \cdot \frac{(\eta|k|)^{j-1}}{(j-1)!} k \cosh(h|k|) \,, \end{aligned}$$

which reduces to

$$\begin{aligned} G_j(\eta) = &- \sum_{\ell=2,\text{even}}^{j-1} G_{j-\ell}(\eta) \frac{(\eta|k|)^\ell}{\ell!} - \sum_{\ell=1,\text{odd}}^{j} G_{j-\ell}(\eta) \frac{(\eta|k|)^\ell}{\ell!} \tanh(h|k|) \\ &+ \frac{(\eta|k|)^j}{j!} |k| - i(\partial_x \eta) \cdot \frac{(\eta|k|)^{j-1}}{(j-1)!} k \,, \end{aligned}$$

and can be cast in the operator form

$$G_j(\eta) = -\sum_{\ell=2,\text{even}}^{j-1} G_{j-\ell}(\eta)\frac{\eta^\ell |D|^\ell}{\ell!} - \sum_{\ell=1,\text{odd}}^{j} G_{j-\ell}(\eta)\frac{\eta^\ell |D|^{\ell-1}}{\ell!} G_0 + D\cdot\frac{\eta^j |D|^{j-1}}{j!} D \,. \tag{4.44}$$

Finally, in light of the self-adjointness of the DNO (Property 2), slightly different but equivalent formulas can be written by flipping the sequence of application of the various operators in (4.43) and (4.44), yielding

$$G_j(\eta) = G_0 D\cdot\frac{|D|^{j-2}\eta^j}{j!} D - \sum_{\ell=2,\text{even}}^{j} \frac{|D|^\ell \eta^\ell}{\ell!} G_{j-\ell}(\eta) - \sum_{\ell=1,\text{odd}}^{j-1} G_0\frac{|D|^{\ell-1}\eta^\ell}{\ell!} G_{j-\ell}(\eta) \,, \tag{4.45}$$

for $j > 0$ even, and

$$G_j(\eta) = D\cdot\frac{|D|^{j-1}\eta^j}{j!} D - \sum_{\ell=2,\text{even}}^{j-1} \frac{|D|^\ell \eta^\ell}{\ell!} G_{j-\ell}(\eta) - \sum_{\ell=1,\text{odd}}^{j} G_0\frac{|D|^{\ell-1}\eta^\ell}{\ell!} G_{j-\ell}(\eta) \,, \tag{4.46}$$

for j odd, where $G_0(D) = |D|\tanh(h|D|)$. This observation was first made by Nicholls [83] and has important implications for numerical simulation. We will come back to this point in a subsequent section. Interestingly, in the infinite depth limit ($h \to +\infty$), G_0 reduces to $|D|$ but otherwise Eqs. (4.45) and (4.46) remain unchanged. Moreover, the overall derivation and expression of these recursion formulas are insensitive to the spatial dimension d.

As an example, the two next-order contributions after G_0 are given by

$$G_1(\eta) = D\cdot\eta D - G_0\eta G_0 \,,$$

and

$$G_2(\eta) = \frac{1}{2}\Big(G_0 D\cdot\eta^2 D - |D|^2\eta^2 G_0\Big) - G_0\eta G_1(\eta) \,.$$

Applied to a function $f(x)$ as defined in (4.42), this second-order operator can be simplified to

$$\begin{aligned} G_2(\eta) &= \frac{1}{2}\Big(G_0 D\cdot\eta^2 D - |D|^2\eta^2 G_0\Big) - G_0\eta D\cdot\eta D + G_0\eta G_0\eta G_0 \,, \\ &= G_0\eta(D\eta)\cdot D + \frac{1}{2}G_0\eta^2|D|^2 - \frac{1}{2}|D|^2\eta^2 G_0 \\ &\quad - G_0\eta(D\eta)\cdot D - G_0\eta^2|D|^2 + G_0\eta G_0\eta G_0 \,, \\ &= -\frac{1}{2}\Big(G_0\eta^2|D|^2 + |D|^2\eta^2 G_0 - 2G_0\eta G_0\eta G_0\Big) \,. \end{aligned}$$

For $\eta \in C^1$, each Taylor term G_j is bounded from $H^1 \to L^2$ because it is related to a multiple commutator of the form

$$\underbrace{\big[\eta, \dots \big[\eta, |D|^j\big]\big]}_{j \text{ times}} \sim |D\eta|^j \,.$$

With this series for the DNO, the Hamiltonian (4.35) is itself analytic (in an appropriately chosen domain) and possesses a Taylor series expansion about the equilibrium state $(\eta, \xi) = 0$, namely

$$\begin{aligned} H(\eta,\xi) &= \frac{1}{2}\int_{\mathbb{R}^{d-1}} \left(\xi G_0 \xi + g\eta^2\right) dx + \sum_{\ell=1}^{\infty} \frac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G_\ell(\eta)\xi \, dx \,, \\ &= \sum_{j=2}^{\infty} H_j(\eta,\xi)\,, \end{aligned} \tag{4.47}$$

where H_j is homogeneous of degree j with respect to the variables (η, ξ).

4.4 Birkhoff Normal Forms

This section is devoted to normal form transformations for water waves. We present an overview based on the pioneering papers by Dyachenko and Zakharov [51] and Craig and Wolfork [44]. We concentrate mainly on their construction at a formal level and refer to the more recent works of Craig and Sulem [41, 42] for their analytic properties. We consider the case of gravity waves in a two-dimensional channel in either finite or infinite depth. Our starting point is the Hamiltonian formulation of the water wave problem. The quadratic part of H is

$$H_2(\eta,\xi) = \frac{1}{2}\int_{\mathbb{R}^{d-1}} \left(\xi G_0 \xi + g\eta^2\right) dx \,,$$

while the j-th term of its Taylor series about equilibrium

$$H_j(\eta,\xi) = \frac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G_{j-2}(\eta)\xi \, dx$$

is associated with j-wave interactions. The stationary solution $(\eta, \xi) = 0$, corresponding to a fluid at rest, is an elliptic stationary point in dynamical systems terms.

4.4.1 Significance of the Normal Form

The goal of Birkhoff normal form transformations is to eliminate non-resonant terms from the Hamiltonian, so that the original equations will only retain essential nonlinearities. The theory of normal forms produces a series of near-identity transformations which remove the non-resonant terms in the Hamiltonian degree by degree. It is a central step in many approaches to analytic studies of water wave equations, including questions of long-time existence of solutions for small initial data [12, 13] and the construction of periodic, quasi-periodic solutions [8, 14, 15]. In the following, we restrict ourselves to the two-dimensional problem ($d = 2$). To retain the structure of a Hamiltonian system in Darboux coordinates, we consider canonical transformations

$$\tau : v = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \longmapsto w\,, \tag{4.48}$$

in a neighborhood of the origin, so that the new equations become

$$\partial_t w = J\,\mathrm{grad}_w \widetilde{H}(w)\ , \quad \widetilde{H}(w) = H(\tau^{-1}(w))\,.$$

We can also iterate the process of transformation. The transformed Hamiltonian is said to be in *Birkhoff normal form* up to order $n \geq 3$ if the Taylor expansion of the Hamiltonian H up to order n contains only resonant terms

$$\widetilde{H}(w) = H_2(w) + \big(Z_3 + \cdots + Z_n\big) + R_{n+1}\,, \tag{4.49}$$

where each Z_j retains only resonant terms such that

$$\{H_2, Z_j\} = 0\,.$$

The new Hamiltonian $\widetilde{H}(w)$ is conserved by the flow $\widetilde{\chi}_t(w)$. Resonant terms $Z_3 + \cdots + Z_n$ describe an averaged system, which often has particular solutions of interest. For example, the remaining resonant terms after the third-order Birkhoff normal form transformation take the form of coupled resonant triads, related to the classical Wilton ripples. In [41, 42], a special effort was made to give a rigorous setting to normal form transformations of third and fourth order. On a formal level, this reduction to Birkhoff normal forms of order $n = 4$ for the water wave Hamiltonian in the case of infinite depth was carried out in [25, 44, 51], with the conclusions that $Z_3 = 0$ and that Z_4 has an expression in terms of action variables alone, despite the family of Benjamin–Feir resonances. This transformation procedure and reduction to Birkhoff normal form is part of the theory of averaging for dynamical systems. Consider $x \in \mathbb{T}$ on a torus with periodic boundary conditions and introduce the Fourier transform variables and complex symplectic coordinates

$$(\eta_k, \xi_k) := \frac{1}{\sqrt{|\mathbb{T}|}} \int_{\mathbb{T}} e^{-ikx} (\eta(x), \xi(x))\, dx\,,$$

$$z_k := \frac{1}{\sqrt{2}}\big(a_k \eta_k + i\, a_k^{-1} \xi_k\big)\,, \quad a_k = \sqrt[4]{\frac{g}{|k| \tanh(h|k|)}}\,. \tag{4.50}$$

Define the action-angle variables in the form

$$z_k = \sqrt{p_k} e^{i\theta_k}\,, \quad p_k = |z_k|^2\,.$$

A Hamiltonian $H(p)$ expressed in action variables alone is said to be integrable if

$$\partial_t \theta = \mathrm{grad}_p H(p)\,, \quad \theta(t) = \theta(0) + t\,\mathrm{grad}_p H(p)\,,$$
$$\partial_t p = -\mathrm{grad}_\theta H(p) = 0\,, \quad p(t) = p(0)\,.$$

Such flows conserve each p_k, hence every Sobolev norm

$$\|z(t)\|_r^2 = \sum_k \langle k\rangle^{2r} |z_k(t)|^2 = \sum_k \langle k\rangle^{2r} p_k = \|z(0)\|_r^2\,.$$

4.4.2 Complex Symplectic Coordinates and Poisson Brackets

We consider a periodic setting, i.e.

$$\eta(x + 2\pi k, t) = \eta(x,t)\,, \quad \xi(x + 2\pi k, t) = \xi(x,t)\,,$$

writing η and ξ as Fourier series

$$\eta(x) = \frac{1}{\sqrt{2\pi}} \sum_k \eta_k\, e^{ikx}\,, \quad \xi(x) = \frac{1}{\sqrt{2\pi}} \sum_k \xi_k\, e^{ikx}\,.$$

Since volume is conserved, we can assume, without loss of generality, that the zeroth Fourier coefficient η_0 vanishes. We now introduce the complex symplectic coordinates, in the general setting. In a finite-depth channel, the dispersion relation is

$$\omega_k^2 = gk \tanh(hk)\,.$$

Because η and ξ are real functions, the reality conditions are expressed as

$$\overline{z}_{-k} = \frac{1}{\sqrt{2}} (a_k \eta_k - i\, a_k^{-1} \xi_k)\,,$$

or equivalently,

$$\eta_k = \frac{1}{\sqrt{2}} a_k^{-1} (z_k + \overline{z}_{-k})\,, \quad \xi_k = \frac{1}{\sqrt{2}\, i} a_k (z_k - \overline{z}_{-k})\,.$$

The Hamiltonian has an expansion in the form

$$H(\eta,\xi) = H_2 + H_3 + \cdots + H_n + R_{n+1}\,, \tag{4.51}$$

where

$$H_2 = \frac{1}{2} \sum_k \left(k \tanh(hk) |\xi_k|^2 + g|\eta_k|^2 \right)\,,$$

$$H_3 = \frac{1}{2\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} (-k_1 k_3 - G_{k_1}^{(0)} G_{k_3}^{(0)})\, \xi_{k_1} \eta_{k_2} \xi_{k_3}\,,$$

and $G_k^{(0)} = G_0(k) = k\tanh(hk)$. Note that the zeroth Fourier coefficient ξ_0 of ξ does not appear in the Hamiltonian.

When $h \to +\infty$ (infinite depth), the dispersion relation reduces to

$$\omega_k^2 = g|k|\,,$$

and the coefficients a_k are defined as

$$a_k^2 = \left(\frac{g}{|k|}\right)^{1/2}.$$

In both cases, the quadratic part H_2 of the Hamiltonian written in the variables z_k reduces to

$$H_2 = \sum_k \omega_k |z_k|^2\,,$$

while the cubic (third-order) part H_3 is

$$H_3 = \frac{1}{8\sqrt{\pi}} \sum_{k_1+k_2+k_3=0} (k_1k_3 + G_1G_3)\frac{a_1a_3}{a_2}(z_1 - \overline{z}_{-1})(z_2 + \overline{z}_{-2})(z_3 - \overline{z}_{-3})\,, \quad (4.52)$$

where for simplicity, in this and subsequent formulas, we use the notation that $z_j = z_{k_j}$, $a_j = a_{k_j}$, $G_j = G_{k_j}^{(0)}$, and $\omega_j = \omega_{k_j}$.

We define the Poisson brackets of $K(\eta,\xi)$ and $H(\eta,\xi)$ in the usual way as

$$\{K,H\} = \int_0^{2\pi} \Big((\mathrm{grad}_\eta H)(\mathrm{grad}_\xi K) - (\mathrm{grad}_\xi H)(\mathrm{grad}_\eta K)\Big)dx\,. \qquad (4.53)$$

In terms of the Fourier coefficients of the (real) functions η and ξ, and assuming H, K are real,

$$\begin{aligned}
\{K,H\} &= \sum_k \Big((\partial_{\eta_k} H)(\overline{\partial_{\xi_k} K}) - (\partial_{\xi_k} H)(\overline{\partial_{\eta_k} K})\Big)\,,\\
&= \sum_k \Big((\partial_{\eta_k} H)(\overline{\partial_{\overline{\xi}_{-k}} K}) - (\partial_{\xi_k} H)(\overline{\partial_{\overline{\eta}_{-k}} K})\Big)\,,\\
&= \sum_{k_1+k_2=0} \Big((\partial_{\eta_2} H)(\overline{\partial_{\overline{\xi}_1} K}) - (\partial_{\xi_2} H)(\overline{\partial_{\overline{\eta}_1} K})\Big)\,,\\
&= \sum_{k_1+k_2=0} \Big((\partial_{\eta_2} H)(\partial_{\xi_1} K) - (\partial_{\xi_2} H)(\partial_{\eta_1} K)\Big)\,.
\end{aligned}$$

In terms of the z_k variables,

$$\{K,H\} = \frac{1}{i} \sum_{k_1+k_2=0} \Big((\partial_{\overline{z}_{-1}} K)(\partial_{z_2} H) - (\partial_{\overline{z}_{-2}} H)(\partial_{z_1} K)\Big)\,, \qquad (4.54)$$

or equivalently,

$$\{K,H\} = \frac{1}{i}\sum_k \Big((\partial_{z_k} H)(\partial_{\overline{z}_k} K) - (\partial_{\overline{z}_k} H)(\partial_{z_k} K) \Big) .$$

Indeed, the partial differential operators complexify in the standard way,

$$\partial_{z_k} = \frac{1}{\sqrt{2}}(a_k \partial_{\eta_k} - i\, a_k^{-1}\partial_{\xi_k})\,, \quad \partial_{\overline{z}_{-k}} = \frac{1}{\sqrt{2}}(a_k \partial_{\eta_k} + i\, a_k^{-1}\partial_{\xi_k})\,,$$

and the left-hand side of (4.54) is then rewritten as

$$\begin{aligned} &\frac{1}{2i}\sum_{k_1+k_2=0} \Big((a_1\partial_{\eta_1} + i\, a_1^{-1}\partial_{\xi_1})K\,(a_2\partial_{\eta_2} - i\, a_2^{-1}\partial_{\xi_2})H \\ &\qquad\qquad -(a_1\partial_{\eta_1} - i\, a_1^{-1}\partial_{\xi_1})K\,(a_2\partial_{\eta_2} + i\, a_2^{-1}\partial_{\xi_2})H \Big) \\ &= \sum_{k_1+k_2=0} \Big((\partial_{\xi_1}K)(\partial_{\eta_2}H) - (\partial_{\eta_1}K)(\partial_{\xi_2}H) \Big) . \end{aligned}$$

4.4.3 Resonances

A term in the Hamiltonian $H(z_k, \overline{z}_{-k})$ is resonant at order n when

$$\sum_{j=1}^{\ell} \omega_j - \sum_{j=\ell+1}^{n} \omega_j = 0\,,$$

and $k_1 + \cdots + k_\ell + k_{\ell+1} + \cdots + k_n = 0$. We do not include $k = 0$ in the sums because we have assumed that the zeroth modes of η and ξ vanish.

Proposition 4.4.1. *In the pure gravity case, there are no resonant triads.*

Equivalently, there are no triplets (k_1, k_2, k_3), with k_j being nonzero, positive or negative integers, such that $k_1 + k_2 + k_3 = 0$ and $\omega_1 \pm \omega_2 \pm \omega_3 = 0$ for any choice of sign. This is due to the increasing and concave character of the dispersion relation ω_k for finite and infinite depth. This means that formally $Z_3 = 0$.

Remark 4.4.2. On the other hand, in the presence of surface tension σ, there are possible nontrivial resonant triads. These resonant interactions are related to Wilton ripples, in reference to observations by Wilton [95]. In the case of a periodic domain, generically these resonant triads do not appear, but for certain choices of parameters (g, h, σ) there can be a finite number of such triads. The maximum wave number k_j involved in a resonant triad is bounded by a constant $C(g, h, \sigma)$ that depends locally and uniformly upon these parameters.

At the next order in the expansion of the water wave Hamiltonian ($n = 4$), and in the case of pure gravity waves on infinite depth, Dyachenko and Zakharov [51] and Craig and Worfolk [44] made the remarkable observation that, in addition to $Z_3 = 0$, Z_4 has an expression in terms of action variables alone, despite the family of Benjamin–Feir resonances. Precisely, in deep water $h \to +\infty$ with $d = 2$ and frequencies $\omega_k = \sqrt{g|k|}$,

$$\omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4 = 0\,, \quad k_1 + k_2 + k_3 + k_4 = 0\,,$$

have integrable solutions

$$\omega_1 + \omega_2 - \omega_3 - \omega_4 = 0\,, \quad \{k_1, k_2\} = \{k_3, k_4\}\,,$$

and non-integrable *Benjamin–Feir* resonant interactions

$$\begin{aligned} k_1 : k_2 : k_3 : k_4 &= n^2 : (n+1)^2 : n^2(n+1)^2 : -(n^2+n+1)^2\,, \\ \omega_1 : \omega_2 : \omega_3 : \omega_4 &= n : -(n+1) : -n(n+1) : (n^2+n+1)\,. \end{aligned}$$

We will discuss this property in Sect. 4.4.6.

4.4.4 Formal Transformation Theory and Birkhoff Normal Form

The transformation τ is defined by (4.48) which gives rise to the reduced Hamiltonian (4.49) and will be constructed as the flow at time $s = -1$ associated with an auxiliary Hamiltonian K such that

$$\partial_s \chi_s = X_K(\chi_s)\,,$$

with

$$\chi_s(w)\big|_{s=0} = w\,, \quad \widetilde{H}(w) = H(\chi_s(w))\big|_{s=-1}\,.$$

This is a canonical transformation preserving the Hamiltonian character of the system. The Taylor series expansion near $s = 0$ of the new Hamiltonian $\widetilde{H}$ is

$$\widetilde{H}(v) = H(\chi_s(v))\big|_{s=0} - \frac{dH}{ds}(\chi_s(v))\big|_{s=0} + \frac{1}{2}\frac{d^2H}{ds^2}(\chi_s(v))\big|_{s=0} - \dots\,,$$

where terms in this expansion are given by

$$\begin{aligned} H(\chi_s(v))\big|_{s=0} &= H(v), \\ \frac{dH}{ds}(\chi_s(v))\big|_{s=0} &= \int_0^{2\pi} \Big((\mathrm{grad}_\eta H)(\partial_s \eta) + (\mathrm{grad}_\xi H)(\partial_s \xi)\Big)dx\,, \\ &= \int_0^{2\pi} \Big((\mathrm{grad}_\eta H)(\mathrm{grad}_\xi K) - (\mathrm{grad}_\xi H)(\mathrm{grad}_\eta K)\Big)dx\,, \\ &= \{K, H\}\,. \end{aligned}$$

Similar formulas for the higher s-derivatives can be obtained, yielding

$$\widetilde{H}(v) = H(v) - \{K, H\}(v) + \frac{1}{2}\{K, \{K, H\}\}(v) + \dots$$

Returning to the expansion (4.51) in the original variables, the transformed Hamiltonian becomes

$$\begin{aligned}\widetilde{H}(v) = & H_2(v) + H_3(v) + \dots \\ & -\{K, H_2\}(v) - \{K, H_3\}(v) - \{K, H_4\}(v) - \dots \\ & + \frac{1}{2}\{K, \{K, H_2\}\}(v) + \frac{1}{2}\{K, \{K, H_3\}\}(v) + \dots \end{aligned} \tag{4.55}$$

If K is homogeneous of degree n, and H_j homogeneous of degree j, then $\{K, H_j\}$ will be of degree $n + j - 2$. Thus, if we construct an auxiliary Hamiltonian K_3 homogeneous of degree 3 satisfying the relation

$$H_3 - \{K_3, H_2\} = 0\,, \tag{4.56}$$

we will have eliminated all cubic terms in the transformed Hamiltonian $\widetilde{H}$. Equation (4.56) is referred to as the third-order cohomological equation.

4.4.5 Solving the Third-Order Cohomological Equation

A central property of the complex symplectic coordinates $(z_k, \overline{z}_{-k})$ is that they diagonalize the coadjoint operator $\mathrm{coad}_{H_2} := \{H_2, \cdot\}$, that is, the linear operation of taking Poisson brackets with H_2. Indeed, the Poisson brackets of H_2 acting on monomials of the form $z_{k_1} z_{k_2} \overline{z}_{-k_3}$ are simply a multiplicative factor.

Lemma 4.4.3.

$$\{H_2, z_1 z_2 \overline{z}_{-3}\} = \frac{1}{i}(\omega_1 + \omega_2 - \omega_3)\, z_1 z_2 \overline{z}_{-3}\,.$$

Proof. Let $H_2 = \sum_k \omega_k |z_k|^2$ and $K = z_1 z_2 \overline{z}_{-3}$, then

$$\partial_{z_k} H_2 = \omega_k \overline{z}_k\,, \quad \partial_{\overline{z}_k} H_2 = \omega_k z_k\,,$$

and thus

$$\partial_{z_k} K = (z_2 \delta_{1k} + z_1 \delta_{2k}) \overline{z}_{-3}\,, \quad \partial_{\overline{z}_k} K = z_1 z_2 \delta_{k(-3)}\,.$$

Consequently,

$$\begin{aligned}\{K, H_2\} &= \frac{1}{i}\Big(\omega_k \overline{z}_k (z_1 z_2 \delta_{k(-3)}) - \omega_k z_k (z_2 \delta_{1k} + z_1 \delta_{2k}) \overline{z}_{-3}\Big)\,, \\ &= \frac{1}{i}\Big(\omega_3 z_1 z_2 \overline{z}_{-3} - \omega_1 z_1 z_2 \overline{z}_{-3} - \omega_2 z_1 z_2 \overline{z}_{-3}\Big)\,, \\ &= i\,(\omega_1 + \omega_2 - \omega_3)\, z_1 z_2 \overline{z}_{-3}\,.\end{aligned}$$

That is, $z_1 z_2 \overline{z}_{-3}$ is an eigenvector of coad_{H_2} with eigenvalue $-i(\omega_1 + \omega_2 - \omega_3)$. Monomials associated with zero eigenvalues correspond to resonant terms. □

The next proposition states that it is indeed possible to solve the cohomological equation (4.56) explicitly, removing all cubic terms except, if they are present, the resonant terms of H_3.

Proposition 4.4.4. *The cohomological equation (4.56) has a unique solution, given in complex symplectic coordinates by the expression*

$$\begin{aligned} K_3 = \frac{1}{8i\sqrt{\pi}} & \sum_{k_1+k_2+k_3=0} (k_1k_3 + G_1G_3)\frac{a_1a_3}{a_2}\frac{z_1z_2z_3 - \overline{z}_{-1}\overline{z}_{-2}\overline{z}_{-3}}{\omega_1+\omega_2+\omega_3} \\ & - \sum_{k_1+k_2+k_3=0} 2(k_1k_3 + G_1G_3)\frac{a_1a_3}{a_2}\frac{z_1z_2\overline{z}_{-3} - \overline{z}_{-1}\overline{z}_{-2}z_3}{\omega_1+\omega_2-\omega_3} \\ & + \sum_{k_1+k_2+k_3=0} (k_1k_3 + G_1G_3)\frac{a_1a_3}{a_2}\frac{z_1\overline{z}_{-2}z_3 - \overline{z}_{-1}z_2\overline{z}_{-3}}{\omega_1-\omega_2+\omega_3} + R\,, \end{aligned} \tag{4.57}$$

where the three sums are performed for triads (k_1, k_2, k_3), with $k_1 + k_2 + k_3 = 0$ excluding the resonant terms for which the corresponding denominator vanishes. The term R consists of a finite sum of exceptional terms, that is, of the nonresonant terms of K_3 for which (k_1, k_2, k_3) possesses a resonant triad. Generically, $R = 0$. This is the case in particular, in the absence of surface tension and for infinite depth.

Proof. Write H_3 as given by (4.52) in the form of a linear combination of third-order monomials in z_k and $\overline{z}_{-k}$ and look for K_3 in the form of a linear combination of the same monomials. Then identify the coefficients, which is possible as long as the corresponding multiplicative factor $(\omega_1 \pm \omega_2 \pm \omega_3)$ does not vanish. We also use symmetry considerations to regroup terms. The term R on the right-hand side of (4.57) contains terms of the first and last sums in (4.57) corresponding to triads for which the denominator $\omega_1 + \omega_2 - \omega_k$ of the second sum vanishes, and terms of the first and second sums in (4.57) corresponding to triads for which the denominator $\omega_1 - \omega_2 - \omega_k$ of the third sum vanishes. □

It is useful for the analysis to rewrite K_3 in the variables (η, ξ).

Proposition 4.4.5. *In these variables, K_3 takes the form*

$$\begin{aligned} K_3 = \frac{1}{\sqrt{2\pi}} & \sum_{k_1+k_2+k_3=0} \frac{k_1k_3 + G_1G_3}{d(\omega_1,\omega_2,\omega_3)}\Big[a_1^2\omega_1(\omega_1^2 - \omega_2^2 - \omega_3^2)\,\eta_1\eta_2\xi_3 \\ & + \frac{a_1^2a_3^2}{a_2^2}\omega_1\omega_2\omega_3\,\eta_1\xi_2\eta_3 + \frac{1}{2a_2^2}\omega_2(\omega_1^2 - \omega_2^2 + \omega_3^2)\,\xi_1\xi_2\xi_3\Big] + R\,, \end{aligned} \tag{4.58}$$

where the denominator $d(\omega_1, \omega_2, \omega_3)$ is given by

$$d(\omega_1,\omega_2,\omega_3) = (\omega_1+\omega_2+\omega_3)(\omega_1+\omega_2-\omega_3)(\omega_1-\omega_2+\omega_3)(\omega_1-\omega_2-\omega_3)\,.$$

4.4.6 Normal Forms for Gravity Waves on Infinite Depth

In this section, we consider the pure gravity wave problem in a channel of infinite depth ($h \to +\infty$). In this case, expressions for K_3 and other quantities simplify significantly and the third-order normal form transformation defining our new system of coordinates is expressed in terms of the time-one flow of Burgers' equation. We give below details of the calculations.

First, we notice that in (4.58), $G_1 = |k_1|$ and $G_3 = |k_3|$, thus

$$k_1 k_3 + G_1 G_3 = k_1 k_3 (1 + \text{sgn}(k_1)\text{sgn}(k_3)) \,,$$

and it vanishes when k_1, k_3 are of opposite signs. Recalling that $\omega_k = \sqrt{g|k|}$, we see that in sectors of the space of wavenumbers $(k_1, k_2, k_3) \in \mathbb{Z}^3$ where $k_1 + k_2 + k_3 = 0$ and k_1 and k_3 have the same sign, the expression $(\omega_1^2 - \omega_2^2 + \omega_3^2)$ that appears in the last term on the right-hand side of (4.58) vanishes. Indeed,

$$\omega_1^2 - \omega_2^2 + \omega_3^2 = g(|k_1| - |k_2| + |k_3|) \,.$$

Then, in the region of $(k_1, k_2, k_3) \in \mathbb{Z}^3$ where $k_1 + k_2 + k_3 = 0$ and k_1 and k_3 have the same sign,

$$|k_1| + |k_3| = |k_1 + k_3| = |k_2| \,.$$

Lemma 4.4.6. *In the region of the wavenumber lattice $(k_1, k_2, k_3) \in \mathbb{Z}^3$ where $k_1 + k_2 + k_3 = 0$ and k_1 and k_3 have the same sign, the expression for the denominator reduces to*

$$d(\omega_1, \omega_2, \omega_3) = -4g^2 k_1 k_3 \,.$$

Proof.

$$\begin{aligned} d_{123} := d(\omega_1, \omega_2, \omega_3) &= \Big((\omega_1 + \omega_2)^2 - \omega_3^2\Big)\Big((\omega_1 - \omega_2)^2 - \omega_3^2\Big) \,, \\ &= (\omega_1^2 + \omega_2^2 - \omega_3^2)^2 - 4\omega_1^2\omega_2^2 \,, \\ &= g^2\Big((|k_1| + |k_2| - |k_3|)^2 - 4|k_1||k_2|\Big) \,. \end{aligned}$$

If $k_1 > 0$, $k_3 > 0$, then $k_2 < 0$ and

$$\frac{1}{g^2} d_{123} = (k_1 - k_2 - k_3)^2 + 4k_1 k_2 = (2k_1)^2 + 4k_1 k_2 = -4k_1 k_3 \,.$$

Similarly, if $k_1 < 0$, $k_3 < 0$, then $k_2 > 0$ and

$$\frac{1}{g^2} d_{123} = (-k_1 + k_2 + k_3)^2 + 4k_1 k_2 = (-2k_1)^2 + 4k_1 k_2 = -4k_1 k_3 \,.$$

□

Proposition 4.4.7. *The expression for K_3 simplifies to*

$$K_3 = -\frac{1}{4\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} (1 + \text{sgn}(k_1)\text{sgn}(k_3))(-2|k_3|\eta_1\eta_2\xi_3 + |k_2|\eta_1\xi_2\eta_3) \,. \quad (4.59)$$

Proof. We start from (4.58) that expresses K_3 in terms of variables (η, ξ) and we implement the above observations, that is: (i) the sum over $k_1 + k_2 + k_3 = 0$ reduces the sectors in wavenumbers such that k_1 and k_3 have the same sign; (ii) in this region the denominator $d(\omega_1, \omega_2, \omega_3) = -4g^2 k_1 k_3$; (iii) the last term in (4.58) vanishes. We also use the identities $\omega_k^2 = g|k|$, $a_k^2 \omega_k = g$, $\omega_k / a_k^2 = |k|$. This leads to

$$\begin{aligned}
K_3 &= -\frac{1}{4\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} (\mathrm{sgn}(k_1)\mathrm{sgn}(k_2) + \mathrm{sgn}(k_2)\mathrm{sgn}(k_3)) \\
&\qquad \times \Big(2|k_3|\eta_1\eta_2\xi_3 - |k_2|\eta_1\xi_2\eta_3 \Big) , \\
&= -\frac{1}{4\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} \Big(2\mathrm{sgn}(k_1)\eta_1 \, \mathrm{sgn}(k_2)\eta_2 |k_3|\xi_3 + 2\eta_1 \, \mathrm{sgn}(k_2)\eta_2 k_3 \xi_3 \\
&\qquad - \mathrm{sgn}(k_1)\eta_1 k_2 \xi_2 \eta_3 - \eta_1 k_2 \xi_2 \, \mathrm{sgn}(k_3)\eta_3 \Big) .
\end{aligned}$$

In the region of $(k_1, k_2, k_3) \in \mathbb{Z}^3$ where $k_1 + k_2 + k_3 = 0$ such that $k_1 > 0$, $k_3 > 0$ and $k_2 < 0$,

$$|k_1| - |k_2| - |k_3| = k_1 + k_2 - k_3 = -2k_3 = -2|k_3| ,$$

while in the region of the (k_1, k_2, k_3) plane where $k_1 + k_2 + k_3 = 0$ and $k_1 < 0$, $k_3 < 0$, and $k_2 > 0$,

$$|k_1| - |k_2| - |k_3| = -k_1 - k_2 + k_3 = 2k_3 = -2|k_3| ,$$

which leads to (4.59). □

A further simplification of the Hamiltonian K_3 arises from a simple identity.

Lemma 4.4.8. *For any $(k_1, k_2, k_3) \in \mathbb{Z}^3$ such that $k_1 + k_2 + k_3 = 0$, we have the equality*

$$\mathrm{sgn}(k_1)\mathrm{sgn}(k_2) + \mathrm{sgn}(k_2)\mathrm{sgn}(k_3) + \mathrm{sgn}(k_3)\mathrm{sgn}(k_1) = -1 .$$

Proof. In each of the sectors, either two wavenumbers are positive and the other one is negative, or the opposite. The equality is true in all cases. In the case where one wavenumber vanishes, the other two have opposite signs. □

Theorem 4.4.9. *The expression for K_3 is given by*

$$K_3 = -\frac{1}{2\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} \mathrm{sgn}(k_1)\mathrm{sgn}(k_2) \, \eta_1\eta_2|k_3|\xi_3 . \tag{4.60}$$

In coordinates from physical space rather than Fourier space, this expression for K_3 becomes

$$K_3 = \frac{1}{2} \int_0^{2\pi} (-i \, \mathrm{sgn}(D)\eta)^2 |D| \xi \, dx . \tag{4.61}$$

Proof. Using Lemma 4.4.8,

$$
\begin{aligned}
K_3 &= -\frac{1}{4\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} (\mathrm{sgn}(k_1)\mathrm{sgn}(k_2) + \mathrm{sgn}(k_2)\mathrm{sgn}(k_3)) \\
&\qquad \times\Big(2|k_3|\eta_1\eta_2\xi_3 - |k_2|\eta_1\xi_2\eta_3\Big)\,, \\
&= -\frac{1}{4\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} \Big(2\mathrm{sgn}(k_1)\,\eta_1\,\mathrm{sgn}(k_2)\,\eta_2|k_3|\xi_3 + 2\eta_1\,\mathrm{sgn}(k_2)\,\eta_2 k_3\xi_3 \\
&\qquad -\mathrm{sgn}(k_1)\,\eta_1 k_2\xi_2\eta_3 - \eta_1 k_2\xi_2\,\mathrm{sgn}(k_3)\,\eta_3\Big)\,.
\end{aligned}
$$

By symmetry considerations, the two last terms on the right-hand side of the above equation are the same, thus

$$
\begin{aligned}
K_3 &= -\frac{1}{4\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} \Big(2\mathrm{sgn}(k_1)\,\eta_1\,\mathrm{sgn}(k_2)\,\eta_2|k_3|\xi_3 \\
&\qquad +2\eta_1\,\mathrm{sgn}(k_2)\,\eta_2 k_3\xi_3 - 2\mathrm{sgn}(k_1)\,\eta_1 k_2\xi_2\eta_3\Big)\,, \\
&= -\frac{1}{4\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} \Big(2\mathrm{sgn}(k_1)\,\eta_1\,\mathrm{sgn}(k_2)\,\eta_2|k_3|\xi_3 \\
&\qquad +2\eta_1\,\mathrm{sgn}(k_2)\,\eta_2 k_3\xi_3 - 2\mathrm{sgn}(k_1)\,\eta_1\eta_2 k_3\xi_3\Big)\,.
\end{aligned}
$$

By symmetry, the two last terms on the right-hand side of this equation are the same but of opposite sign, and therefore cancel out, leading to

$$
K_3 = -\frac{1}{2\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} \mathrm{sgn}(k_1)\,\eta_1\,\mathrm{sgn}(k_2)\,\eta_2|k_3|\xi_3\,,
$$

which is precisely (4.60). We rewrite K_3 as

$$
K_3 = \frac{1}{2\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} (-i\,\mathrm{sgn}(k_1))\eta_1\,(-i\,\mathrm{sgn}(k_2))\eta_2\,|k_3|\xi_3\,.
$$

Returning to physical-space coordinates, it gives the expression (4.61). □

Third-Order Normal Form and Burgers' Equation

We are now able to write the third-order normal form transformation that defines our new coordinates, obtained as the solution map at $s = -1$ of the Hamiltonian flow

$$
\begin{pmatrix} \partial_s\eta \\ \partial_s\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{grad}_\eta K_3 \\ \mathrm{grad}_\xi K_3 \end{pmatrix} := X_{K_3}\,,
$$

with the (initial) condition at $s = 0$ being the original variables (η, ξ). Equivalently, in Fourier space,

$$\partial_s \eta_{-k} = \mathrm{grad}_{\xi_k} K_3 \,,$$
$$\partial_s \xi_{-k} = -\mathrm{grad}_{\eta_k} K_3 \,.$$

From (4.60), we compute

$$\partial_{\xi_k} K_3 = \frac{1}{2\sqrt{2\pi}} \sum_{k_1+k_2+k=0} |k|(-i\,\mathrm{sgn}(k_1))\eta_1\,(-i\,\mathrm{sgn}(k_2))\eta_2 \,,$$

leading to

$$\partial_s \eta_k = \frac{1}{2\sqrt{2\pi}} \sum_{k_1+k_2=k} |k|(-i\,\mathrm{sgn}(k_1))\eta_1\,(-i\,\mathrm{sgn}(k_2))\eta_2 \,.$$

Equivalently in the physical-space coordinates,

$$\partial_s \eta = \frac{1}{2}|D|(-i\,\mathrm{sgn}(D)\eta)^2 \,.$$

It is convenient to introduce the new variables

$$\widetilde{\eta} = -i\,\mathrm{sgn}(D)\eta \,, \quad \widetilde{\xi} = -i\,\mathrm{sgn}(D)\xi \,,$$

where the operator $-i\,\mathrm{sgn}(D)$ is the Hilbert transform associated with the infinite lower half-plane. Using $-i\,\mathrm{sgn}(D)|D| = -\partial_x$, the auxiliary Hamiltonian K_3 can be rewritten as

$$K_3 = \frac{1}{2}\int_0^{2\pi} (\widetilde{\eta})^2 \partial_x \widetilde{\xi}\, dx \,, \tag{4.62}$$

and the quantity $\widetilde{\eta}$ satisfies Burgers' equation

$$\partial_s \widetilde{\eta} = \mathrm{grad}_{\widetilde{\xi}} K_3 = -\frac{1}{2}\partial_x (\widetilde{\eta})^2 \,,$$

which is to say

$$\partial_s \widetilde{\eta} + \widetilde{\eta}\partial_x \widetilde{\eta} = 0 \,.$$

We now turn to

$$\partial_{\eta_k} K_3 = \frac{1}{\sqrt{2\pi}} \sum_{k+k_1+k_2=0} (-i\,\mathrm{sgn}(k))(-i\,\mathrm{sgn}(k_1))\eta_1 |k_2| \xi_2 \,,$$

so that

$$\partial_s \xi_k = \frac{1}{\sqrt{2\pi}} \sum_{k_1+k_2=k} (-i\,\mathrm{sgn}(k))(-i\,\mathrm{sgn}(k_1))\eta_1 |k_2| \xi_2 \,,$$

or equivalently in the physical space,

$$\partial_s \xi = (-i\,\mathrm{sgn}(D)\big((-i\,\mathrm{sgn}(D)\eta)|D|\xi\big)\,,$$

which after Hilbert transformation reduces to

$$\partial_s \widetilde{\xi} = -\mathrm{grad}_{\widetilde{\eta}} K_3 = -\widetilde{\eta}\partial_x \widetilde{\xi}\,.$$

Theorem 4.4.10. *The Hamiltonian system that defines the third-order Birkhoff normal form transformation τ_3 takes the form of two coupled PDEs*

$$\partial_s \widetilde{\eta} + \widetilde{\eta}\partial_x \widetilde{\eta} = 0\,, \tag{4.63a}$$

$$\partial_s \widetilde{\xi} + \widetilde{\eta}\partial_x \widetilde{\xi} = 0\,, \tag{4.63b}$$

where $(\widetilde{\eta}, \widetilde{\xi})$ are the Hilbert transforms of (η, ξ). The first equation for $\widetilde{\eta}$ is Burgers' equation. The second equation for $\widetilde{\xi}$ is its linearization along Burgers' flow.

An important related PDE question concerns the mapping properties of the transformation τ_3, in particular whether this transformation is well defined, and on which Banach spaces.

Theorem 4.4.11. *There exists $R_0 > 0$ such that for any $R < R_0$, on every neighborhood $B_R(0) \subseteq E_r = H^r_\eta \oplus H^r_\xi$ $(r > 3/2)$, the canonical Birkhoff normal form transformation τ_3 is well defined and continuous,*

$$\tau_3 : B_R(0) \longrightarrow B_{2R}(0)\,,$$
$$\tau_3^{-1} : B_{R/2}(0) \longrightarrow B_R(0)\,.$$

The Jacobian of this transformation is given by the variational equations for $(\delta\eta, \delta\xi)$,

$$\partial_t(\delta\eta) = -\widetilde{\eta}\partial_x(\delta\eta) - (\delta\eta)\partial_x\widetilde{\eta}\,,$$
$$\partial_t(\delta\xi) = -\widetilde{\eta}\partial_x(\delta\xi) - (\delta\eta)\partial_x\widetilde{\xi}\,.$$

Theorem 4.4.12. *The mapping τ_3 is smooth on the scale of Hilbert spaces, in particular the Jacobian*

$$\partial_v \tau_3 \;:\; H^{r-1}_\eta \oplus H^{r-1}_\xi \longrightarrow H^{r-1}_\eta \oplus H^{r-1}_\xi$$

is continuous.

Remark 4.4.13. There is a contrast between the energy spaces $H^r_\eta \oplus H^{r+1/2}_\xi$ for the linearized equations of water waves and the function spaces in which system (4.63) is well-posed. The evolution given by (4.63) is well defined for $\eta \in H^r$ $(r > 3/2)$ and the function ξ is simply transported along the characteristics defined by the evolution of η, thus the regularity of η and ξ coincides.

Fourth-Order Normal Form

After reduction to the third-order normal form, the transformed Hamiltonian (4.55) becomes

$$\widetilde{H}(w) = H_2(w) + H_4(w) - \frac{1}{2}\{K_3, H_3\}(w) + R_5\,.$$

Since K_3 is homogeneous of degree 3, its Poisson brackets with H_3 (homogeneous of degree 3) are of degree 4 and combine with H_4 to give the quartic contributions in the transformed Hamiltonian $\widetilde{H}$. We denote the new quartic (fourth-order) term by

$$H_4^+ = H_4 - \frac{1}{2}\{K_3, H_3\}\,.$$

Recalling that

$$H_4 = \frac{1}{2}\int \xi G_2(\eta)\xi\, dx\ ,$$

where

$$G_2(\eta) = -\frac{1}{2}\Big(D^2\eta^2|D| + |D|\eta^2 D^2 - 2|D|\eta|D|\eta|D|\Big)\,,$$

the next proposition shows that H_4^+ takes a very special form.

Proposition 4.4.14. *The fourth-order term H_4^+ is given by*

$$H_4^+ = H^{(4)}(\eta,\xi) - H^{(4)}(\widetilde{\eta},\xi)\,.$$

This expression can be further simplified.

Proposition 4.4.15.

$$H_4^+ = -\sum_{k_1+k_2+k_3+k_4=0} d_{1234}|k_1||k_4|\xi_1\eta_2\eta_3\xi_4\,, \tag{4.64}$$

where

$$\begin{aligned} d_{1234} = \frac{1}{16\pi}&\Big(1+\mathrm{sgn}(k_2)\mathrm{sgn}(k_3)\Big)\Big(1+\mathrm{sgn}(k_1)\mathrm{sgn}(k_4)\Big)\\ &\times\Big(|k_1+k_4|+|k_2+k_3|-|k_1+k_2|-|k_1+k_3|-|k_2+k_4|-|k_3+k_4|\Big)\,. \end{aligned}$$

The proofs of Propositions 4.4.14 and 4.4.15 with detailed computations can be found in [42] (Sect. 4.1). The transformed Hamiltonian now reads

$$\widetilde{H}(w) = H_2(w) + H_4^+(w) + R_5\,, \tag{4.65}$$

where $H_4^+(w)$ is given by (4.64). The fourth-order Birkhoff normal form transformation is more complicated. To construct the next change of variables, we need to identify the resonant terms Z_4 in $H_4^+(w)$. The term $[H_4^+] := Z_4$ is the average of H_4^+ over the flow of H_2. These arise in quartets of wavenumbers (k_1, k_2, k_3, k_4) such that $k_1 + k_2 + k_3 + k_4 = 0$ and $\omega_1 \pm \omega_3 \pm \omega_3 \pm \omega_4 = 0$. Two such families of wavenumbers exist:

(i) Special resonances, referred to as *Benjamin–Feir resonances* of the form

$$\begin{aligned}
&\left[k_j : (j = 1, 2, 3, 4)\right] = q\left[n^2, (n+1)^2, n^2(n+1)^2, -(n^2+n+1)^2\right], \\
&\left[\omega_j = \sqrt{g|k_j|} : (j = 1, 2, 3, 4)\right] = \sqrt{q}\left[n, n+1, n(n+1), n^2+n+1\right], \\
&\omega_1 - \omega_2 - \omega_3 + \omega_4 = 0,
\end{aligned}$$

where $q \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{N}$.

(ii) Generic quartets of wavenumbers $(k_1, -k_1, -k_4, k_4)$ and $(k_1, -k_4, -k_1, k_4)$ (k_1, k_4 are integers of the same sign). These are referred to as *benign* in [51] because the resulting Hamiltonian system, truncated to fourth order, is completely integrable. The structure of solutions to the latter was analyzed in detail in [44]. In the lattice $\mathbb{Z}^4$, the set of points harboring a potential resonant term is denoted by

$$\begin{aligned}
R = \Big\{&(k_1, k_2, k_3, k_4) : k_1 + k_2 + k_3 + k_4 = 0\,,\ k_1 k_4 > 0\,, \\
&k_2 = -k_1 \text{ and } k_3 = -k_4 \text{ or } k_3 = -k_1 \text{ and } k_2 = -k_4\Big\}.
\end{aligned}$$

Theorem 4.4.16. *Benjamin–Feir resonances are not present in H_4^+. The normal form transformation that removes the cubic terms also removes all Benjamin–Feir quartic terms.*

Proof. In the expression (4.64) for H_4^+, the sum is performed over (k_1, k_2, k_3, k_4) where two of the wavenumbers are positive and the two others are negative. This excludes the presence of Benjamin–Feir resonances, as these resonance relations only involve quartets of three positive and one negative wavenumbers, or vice versa. Thus the wavenumber quartets in R, consisting of those such that $(k_1, -k_1, -k_4, k_4)$ and $(k_1, -k_4, -k_1, k_4)$, with $\mathrm{sgn}(k_1 k_4) > 0$, are the only ones giving rise to fourth-order resonances. □

Now that we have identified the resonant terms in H_4^+, we can construct a second canonical change of variables that will eliminate the non-resonant fourth-order terms in the Hamiltonian. Let

$$\tau_4 \ : \ w' = \begin{pmatrix} \eta' \\ \xi' \end{pmatrix} \longmapsto \begin{pmatrix} \eta'' \\ \xi'' \end{pmatrix} = w''$$

be the time-one flow associated with a Hamiltonian K,

$$\partial_s \chi_s = X_K(\chi_s)\,,$$

with

$$\chi_s(w'')\big|_{s=0} = w''\,, \quad H''(w'') = H'(\chi_s(w''))\big|_{s=-1}\,.$$

The new Hamiltonian resulting from this change of variables is

$$H''(w'') = H'(w') - \{K, H'\}(w') + \frac{1}{2}\{K, \{K, H'\}\}(w') + \dots$$

Returning to the expansion (4.65) and dropping the primes

$$H(w) = H_2(w) + H_4^+(w) - \{K, H_2\}(w) - \{K, H_4^+\}(w) + \dots$$

We seek an auxiliary Hamiltonian K_4 that will define the second change of variables. It is homogeneous of degree 4 and satisfies the cohomological equation

$$\{K_4, H_2\} = H_4^+ - [H_4^+]\,,$$

where $[H_4^+] := Z_4$ in order to eliminate non-resonant quartic terms from the transformed Hamiltonian H, leaving only the resonant terms Z_4.

We first express H_4^+ as given by (4.64) in terms of the complex symplectic coordinates $(z_k, \overline{z}_{-k})$ in order to use the property of diagonalization of the coadjoint operator in these coordinates, e.g.

$$\{H_2, z_1 z_2 z_3 \overline{z}_{-4}\} = \frac{1}{i}(\omega_1 + \omega_2 + \omega_3 - \omega_4) z_1 z_2 z_3 \overline{z}_{-4}\,.$$

The calculation is lengthy and is presented in detail in [42]. The Hamiltonian K_4 is obtained as the sum of two contributions,

$$K_4 = K_4^N + K_4^R\,,$$

where

$$K_4^N = \sum_{\substack{(k_1,k_2,k_3,k_4)\in N \\ k_1+k_2+k_3+k_4=0}} d_{1234}\left(|k_4| \frac{k_1k_4 + k_2k_3}{k_1k_4 - k_2k_3} \eta_1\eta_2\eta_3\xi_4 + \frac{2}{g}\frac{|k_1||k_2||k_3||k_4|}{k_1k_4 - k_2k_3}\xi_1\eta_2\xi_3\xi_4 \right)\,,$$

where the sum is performed over the set $N = \mathbb{Z}^4 \backslash R$, which we define to be those $(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4$ such that $k_1 + k_2 + k_3 + k_4 = 0$ but excluding the quartets $(k_1, -k_1, -k_4, k_4)$ and $(k_1, -k_4, -k_1, k_4)$ that harbor potential resonant terms. Note that the denominator $k_1k_4 - k_2k_3$ in the above expression vanishes on the set R. Indeed, $k_1k_4 - k_2k_3 = -(k_1 + k_2)(k_1 + k_3)$.

Secondly, we extract the non-resonant terms involving wavenumbers in the set R, and separate the diagonal and off-diagonal terms denoted by $H_4^{\rm diag}$ and $H_4^{\rm off}$, respectively. The result is

$$K_4^R = K_4^{\rm diag} + K_4^{\rm off}\,,$$

with

$$K_4^{\rm diag} = \frac{1}{16\pi i}\sum_{k\neq 0}\frac{|k|^2}{\omega_k}\left(z_k^2 z_{-k}^2 - \overline{z}_{-k}^2\overline{z}_k^2\right),$$

and

$$K_4^{\rm off} = \frac{1}{2\pi i}\sum_{k_1\neq k_4}|k_1||k_4|\min(|k_1|,|k_4|)\frac{z_1z_{-1}z_4z_{-4}-\overline{z}_1\overline{z}_{-1}\overline{z}_4\overline{z}_{-4}}{\omega_1+\omega_4},$$

which can be expressed in terms of Fourier modes of η and ξ.

Integrable Birkhoff Normal Form

We now compute the resonant terms Z_4 of order four that remain in the modified Hamiltonian H_4^+ after formal reduction to the fourth-order Birkhoff normal form. As shown in [44], we will express them in terms of two action variables and check that the corresponding Hamiltonian system is completely integrable.

We start from H_4^+ as given by (4.64), written in complex symplectic coordinates $(z_k,\overline{z}_{-k})$, and extract the terms that involve monomials of the form $z_1z_2\overline{z}_{-3}\overline{z}_4$, with conditions on the quartets $\{k_j\}$ that they give rise to resonant terms, namely either $(k_1,-k_1,-k_4,k_4)$ or $(k_1,-k_4,-k_1,\ k_4)$ for nonzero integers k_1, k_4. This sum Z_4 of resonant terms is

$$Z_4 = \sum_{k_1,k_4\in\mathbb{Z}\setminus\{0\}}\frac{1}{4}d_{1234}|k_1||k_4|\frac{a_1a_4}{a_2a_3}(-4z_1z_2\overline{z}_{-3}\overline{z}_{-4}+\overline{z}_{-1}z_2z_3\overline{z}_{-4}+z_1\overline{z}_{-2}\overline{z}_{-3}z_4),$$

with

$$\begin{aligned} d_{1234} = {} & \frac{1}{16\pi}(1+\operatorname{sgn}(k_2k_3))(1+\operatorname{sgn}(k_1k_4)) \\ & \times(|k_1+k_4|+|k_2+k_3|-|k_1+k_2|-|k_1+k_3|-|k_2+k_4|-|k_3+k_4|).\end{aligned}$$

After some additional algebraic manipulations as shown in [42], we obtain the Hamiltonian truncated at order four.

Theorem 4.4.17. *After two normal form transformations, the Hamiltonian up to fourth order reduces to*

$$\begin{aligned}\widetilde{H} = {} & \sum_k \omega_k|z_k|^2 + \frac{1}{4\pi}\sum_k |k|^3\Big(|z_k|^4+|z_{-k}|^4-4|z_k|^2|z_{-k}|^2\Big) \\ & +\frac{2}{\pi}\sum_{|k_4|<|k_1|,k_1k_4>0}|k_1||k_4|^2\Big(-z_{-1}\overline{z}_{-1}z_4\overline{z}_4+\frac{1}{2}(|z_1|^2|z_4|^2+|z_{-1}|^2|z_{-4}|^2)\Big).\end{aligned} \tag{4.66}$$

To recover the Hamiltonian $\widetilde{H}$ in action-angle variables as given in [44], we introduce the quantities

$$p_1(k) = \frac{1}{2}(z_k\overline{z}_k+z_{-k}\overline{z}_{-k}),\quad p_2(k) = \frac{1}{2}(z_k\overline{z}_k-z_{-k}\overline{z}_{-k}).$$

The action variables for this system are given by the set $\{p_1(k), p_2(k)\}_{k\in\mathbb{Z}\setminus\{0\}}$ and the corresponding angles are defined as

$$\theta_1(k) = \frac{1}{2}\tan^{-1}\left(\frac{\mathrm{Im}(z_k z_{-k})}{\mathrm{Re}(z_k z_{-k})}\right), \quad \theta_2(k) = \frac{1}{2}\tan^{-1}\left(\frac{\mathrm{Im}(z_k \overline{z}_{-k})}{\mathrm{Re}(z_k \overline{z}_{-k})}\right),$$

where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real and imaginary parts, respectively. The time evolution of the angles is governed by

$$\partial_t\theta_1 = \partial_{p_1}\widetilde{H}, \quad \partial_t\theta_2 = \partial_{p_2}\widetilde{H}.$$

The action variables satisfy $p_1(-k) = p_1(k)$ and $p_2(-k) = -p_2(k)$, which implies

$$|z_k|^4 + |z_{-k}|^4 - 4|z_k|^2|z_{-k}|^2 = -2p_1^2(k) + 6p_2^2(k),$$

and

$$\begin{aligned} -z_{-1}\overline{z}_{-1}z_4\overline{z}_4 + \frac{1}{2}(|z_1|^2|z_4|^2 + |z_{-1}|^2|z_{-4}|^2) \\ = 2p_1(k_1)p_2(k_4) + p_2(k_1)p_1(k_4) - p_1(k_1)p_2(k_4). \end{aligned}$$

After substitution in (4.66), we find

Theorem 4.4.18. *The Hamiltonian up to fourth order is completely integrable and takes the form*

$$\begin{aligned} \widetilde{H} &= \sum_k \omega_k p_1(k) - \frac{1}{2\pi}\sum_k |k|^3\big(p_1(k)^2 - 3p_2(k)^2\big) + \frac{4}{\pi}\sum_{|k_4|<|k_1|} p_2(k_1)p_2(k_4), \\ &= H_2(p) + \widetilde{H}_4(p). \end{aligned}$$

4.5 Model Equations for Water Waves

The water wave problem is notorious for its wealth of model equations that arise in various asymptotic regimes. As part of this effort, different perturbation methods have been used by various investigators. In this section, we present a systematic perturbation approach for obtaining weakly nonlinear Hamiltonian models, and we apply it to deriving the KdV equation and NLS equation for two-dimensional gravity waves in the shallow-water long-wave limit and deep-water modulational limit, respectively. We further illustrate the approach by proposing higher-order versions of these generic equations. An important consequence is that, at each level of approximation, the resulting model automatically inherits a Hamiltonian structure from the original system (4.25).

Generally speaking, it is desirable that a reduced model retains important structural properties of the original system, including energy conservation. From the computational side, such a property comes in handy when testing the numerical solution of the reduced model, and opens up an avenue for applications in the exciting field of symplectic integrators for Hamiltonian PDEs [16]. This point of view also provides a unified theoretical framework for further analysis of such models (e.g. the stability analysis of solitary wave solutions [59]) as well as a suitable basis for possible inclusion of additional physical mechanisms. Another appeal of the proposed strategy is that the calculations are made more convenient due to the choice of a lower-dimensional formulation of the governing equations, together with a more explicit dependence on η (the free surface) via the series form of the DNO.

4.5.1 Linearized Problem

Before we consider models in the weakly nonlinear regime, we first examine the linearized problem for water waves of infinitesimal amplitude around the quiescent state $\eta = 0$. This is a relatively straightforward analysis by using the closed system (4.33)–(4.34) for η and ξ, while neglecting the nonlinear terms. The resulting equations are

$$\begin{pmatrix} \partial_t \eta \\ \partial_t \xi \end{pmatrix} = \begin{pmatrix} 0 & G_0(D) \\ -g & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \tag{4.67}$$

which can be combined into

$$\partial_t^2 \eta + g G_0(D) \eta = 0\,. \tag{4.68}$$

The corresponding Hamiltonian is simply the quadratic part H_2 in (4.47). Recall that $G_0(D) = |D| \tanh(h|D|)$. Seeking traveling wave solutions of the form $e^{i(k \cdot x - \omega t)}$ and inserting it in (4.68) leads to

$$\omega^2 = g G_0(k) = g|k| \tanh(h|k|)\,, \tag{4.69}$$

which is the well-known linear dispersion relation for gravity waves on water of finite depth.

In the two-dimensional case ($d = 2$), this dispersion relation reduces to

$$\omega = \sqrt{gk \tanh(hk)}\,,$$

and for the purposes of this discussion, it is sufficient to consider $k > 0$ and $\omega > 0$. Accordingly, a phase speed and a group speed can be defined as

$$c_p = \frac{\omega}{k} = \sqrt{\frac{g \tanh(hk)}{k}}\,, \quad c_g = \partial_k \omega = \frac{1}{2} c_p \left(1 + \frac{2hk}{\sinh(2hk)} \right),$$

respectively. In the shallow-water limit $hk \to 0$, these two wave speeds coincide

$$c_p = c_g = \sqrt{gh}\,, \tag{4.70}$$

and are independent of k (i.e. shallow-water waves are non-dispersive). In the deep-water limit $hk \to +\infty$, their limiting values are

$$c_g = \frac{1}{2} c_p = \frac{1}{2} \sqrt{\frac{g}{k}}\,, \tag{4.71}$$

which differ by a factor of 2 ($c_p > c_g$) and retain a dependence on k (i.e. deep-water waves are dispersive). In this regime, the smaller the k (i.e. the longer the wavelength), the larger the wave speed.

The general linear solution can be expressed as

$$\begin{pmatrix} \eta(x,t) \\ \xi(x,t) \end{pmatrix} = \begin{pmatrix} \cos\big(t\,\omega(D)\big) & \frac{\omega(D)}{g} \sin\big(t\,\omega(D)\big) \\ -\frac{g}{\omega(D)} \sin\big(t\,\omega(D)\big) & \cos\big(t\,\omega(D)\big) \end{pmatrix} \begin{pmatrix} \eta(x,0) \\ \xi(x,0) \end{pmatrix},$$

in terms of the initial conditions at $t = 0$. The coefficient matrix in this expression is the fundamental matrix for the linear system (4.67). Its entries involve

$$\omega(D) = \big(gG_0(D)\big)^{1/2} = \big(g|D| \tanh(h|D|)\big)^{1/2},$$

which denotes the Fourier multiplier operator associated with the linear dispersion relation (4.69). Further discussion on the linearized problem will be given in Sect. 4.6.1 when we review results on local well-posedness.

4.5.2 Non-dimensionalization

To introduce the various asymptotic scaling regimes that arise in the water wave problem, it is helpful to perform a non-dimensionalization of the full governing equations (4.33) and (4.34). Rather than introducing a new (possibly cumbersome) notation for all the variables, we choose to write, e.g. $x \to \lambda_0 x$, which means that x is replaced by $\lambda_0 x$ so that x now denotes a dimensionless variable. A natural choice of non-dimensionalization is defined by

$$x \to \lambda_0 x\,, \quad y \to hy\,, \quad t \to \frac{\lambda_0}{c_0} t\,, \quad \eta \to a_0 \eta\,, \quad \varphi \to \frac{a_0 c_0 \lambda_0}{h} \varphi\,, \tag{4.72}$$

where (a_0, λ_0, c_0) are characteristic values of the wave amplitude, wavelength, and wave speed, respectively. Making these substitutions in (4.33) and (4.34) leads to the dimensionless equations

$$\partial_t \eta = \frac{1}{\mu} G^{\mu}(\varepsilon\eta)\xi\,, \tag{4.73}$$

$$\partial_t \xi = -\eta - \frac{\varepsilon}{2} |\partial_x \xi|^2 + \mu\varepsilon \frac{\Big(\frac{1}{\mu} G^{\mu}(\varepsilon\eta)\xi + \varepsilon \partial_x \eta \cdot \partial_x \xi\Big)^2}{2(1 + \mu\varepsilon^2 |\partial_x \eta|^2)}\,, \tag{4.74}$$

where the dimensionless DNO is defined by

$$G^{\mu}(\varepsilon\eta)\xi = \partial_y \varphi - \mu\varepsilon \partial_x \eta \cdot \partial_x \varphi\big|_{y=\varepsilon\eta}\,,$$

in terms of the dimensionless parameters

$$\varepsilon = \frac{a_0}{h}\,, \quad \mu = \frac{h^2}{\lambda_0^2}\,.$$

In doing so, we have also made the choice $c_0 = \sqrt{gh}$, which coincides with (4.70), so that the coefficient of the linear term $-\eta$ on the right-hand side of (4.74) is normalized to unity.

Such a procedure helps identify two relevant independent dimensionless parameters in this problem. The parameter ε represents wave amplitude (relative to water depth) and is a measure of nonlinearity. The parameter μ represents water shallowness (relative to wavelength) and is a measure of wave dispersion. Clearly, the weakly nonlinear regime corresponds to $\varepsilon \ll 1$, with the limiting regime $\varepsilon = 0$ being the linear problem as discussed in the previous section. As for μ, the limit $\mu \to 0$ corresponds to the shallow-water or long-wave regime (where waves are non-dispersive), while the limit $\mu \to +\infty$ corresponds to the deep-water or short-wave regime (where waves are dispersive). Depending on the choice of range of values for μ and ε, a variety of asymptotic models can be derived via perturbation calculations. A few cases will be considered in the next section.

For waves on deep water, a more appropriate choice of non-dimensionalization consists in replacing both λ_0 and h by $1/k_0$ in (4.72), where k_0 is a characteristic wavenumber, and setting $c_0 = \sqrt{g/k_0}$. Not surprisingly, this choice of c_0 coincides with the linear phase speed in (4.71). The resulting dimensionless equations are similar to (4.73)–(4.74), with the exception that $\mu = 1$ and the remaining dimensionless parameter is given by $\varepsilon = k_0 a_0$, which represents wave steepness and is again a measure of nonlinearity. Note that a general choice of scaling factors can be introduced to accommodate the full range of possible values for ε and μ [79]. We also remark that, while it is common to non-dimensionalize the equations of motion, which is helpful at identifying suitable dimensionless parameters and associated scaling regimes, this procedure is not quite relevant to our own perturbation method as the latter is based on performing expansions directly in the Hamiltonian. Details are provided below. For convenience, we will switch back to dimensional variables in order to show more clearly how the physical parameters enter the coefficients of the resulting models.

4.5.3 Canonical Transformation Theory

Our approach to the systematic derivation of limiting models is from the viewpoint of Hamiltonian perturbation theory, in which the Hamiltonian $H(v;\varepsilon)$ is a function of a small dimensionless parameter $\varepsilon \in \mathcal{E}$ and the reduced equations are also Hamiltonian systems of the form (4.19). The small parameter ε is introduced through choices of scaling for the independent variables x and dependent variables v, as well as through further transformations of these dependent variables, corresponding to asymptotic regimes of interest. We consider a variety of scaling regimes for wave propagation on shallow or deep water, where both dispersive and nonlinear effects are brought into play. The parameter ε enters these various regimes in different ways; however, the systematic point of view is retained throughout the asymptotic procedure.

There are two essential steps in this approach: expansion of the Hamiltonian and modification of the symplectic structure. It is indeed natural to expand H in powers of ε via (4.47) and approximate orbits $v(t;\varepsilon)$ of (4.16) by those of the truncated problem

$$\partial_t v = J\mathrm{grad}_v\Big(H^{(0)} + \varepsilon H^{(1)} + \cdots + \varepsilon^n H^{(n)}\Big)\,, \quad v(0;\varepsilon,n) = v_0\,. \tag{4.75}$$

For convenience, the spatial dependence is omitted here. The solution $v(t;\varepsilon,n)$ clearly depends upon ε and the truncation order n. For at least finite intervals of time, there is the natural expectation that $v(t;\varepsilon,n)$ of (4.75) approximates solutions of the full problem (4.16), with a better approximation for larger n. This is indeed true for C^2 Hamiltonians.

Proposition 4.5.1. *Suppose that the Hamiltonian* $H \in C^{2,n+1}(\mathcal{M}\times\mathcal{E})$*. Then, at least for finite time intervals* $|t| \le T_0$*, orbits* $v(t;\varepsilon,n)$ *of the truncated system* (4.75) *are* ε^n *close to orbits of the full Hamiltonian system* (4.16)*.*

As pointed out earlier, Hamiltonian PDEs are rarely given by smooth vector fields. Therefore, the above proposition is not generally applicable. Nonetheless, it serves as a basic guiding principle for the modeling problem being addressed here. Such a method has been developed by Craig and Groves [27] and Moldabayev et al. [81] for long waves on water of uniform depth, Craig et al. [30, 31, 45] for long waves on water of variable depth, and Craig et al. [29, 33, 35, 36] for long waves at the surface and within stratified fluids. More recently, it has been applied to the nonlinear modulation of near-monochromatic waves on infinite or finite depth [32, 34, 37].

In addition to expanding the Hamiltonian, the changes of variables associated with a particular asymptotic regime modify the symplectic structure of the original problem. Accordingly, the operator J is different in the various settings but, from our perspective, it is independent of v and always homogeneous in ε, unlike certain cases considered by Olver [84] where J is nontrivially dependent on v and ε. Details on this point are provided below, following the systematic procedure introduced in [27, 29].

4.5.4 Calculus of Transformations

Consider two phase spaces $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ with a symplectic structure on $\mathcal{M}$ given by the operator J. Let $H : \mathcal{M} \to \mathbb{R}$ be a Hamiltonian. A transformation

$$\begin{array}{cccc} \tau & : & \mathcal{M} & \longrightarrow & \widetilde{\mathcal{M}} \\ & & v & \longmapsto & w \end{array}$$

gives rise to a Hamiltonian defined on $\widetilde{\mathcal{M}}$, namely

$$\widetilde{H}(w) = \widetilde{H}(\tau(v)) = H(v)\,.$$

By the chain rule, the Hamiltonian vector field $X_H = J\mathrm{grad}_v H$ on $\mathcal{M}$ is transformed to a vector field on $\widetilde{\mathcal{M}}$ as follows

$$\partial_t w = (\partial_v \tau)\partial_t v = (\partial_v \tau) J \mathrm{grad}_v H\,,$$

while, on the other hand,

$$\mathrm{grad}_v H = (\partial_v \tau)^\top \mathrm{grad}_w \widetilde{H}\,.$$

Equating these two expressions leads to

$$\partial_t w = (\partial_v \tau) J (\partial_v \tau)^\top \mathrm{grad}_w \widetilde{H}\,,$$

which implies that the transformation τ induces a symplectic structure on $\widetilde{\mathcal{M}}$, given by the structure map $\widetilde{J}' = (\partial_v \tau) J (\partial_v \tau)^\top$, and the transformed vector field $\widetilde{J}' \,\mathrm{grad}_w \widetilde{H}$ is Hamiltonian in the phase space $\widetilde{\mathcal{M}}$. When $\widetilde{\mathcal{M}}$ already has a symplectic structure $\widetilde{J}$ and the transformation τ is such that

$$(\partial_v \tau) J (\partial_v \tau)^\top = \widetilde{J}\,,$$

then it is called a canonical transformation from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$. In particular, when $\mathcal{M} = \widetilde{\mathcal{M}}$ and $J = \widetilde{J}$, e.g. with J given by (4.25) for Darboux coordinates, these are the usual canonical transformations which play a special role in Hamiltonian mechanics.

While the subject of canonical transformations and their generating functions is basic knowledge in finite-dimensional Hamiltonian systems, it is less developed for PDEs and other infinite-dimensional cases. In the following sections, we review some of the elementary transformations that occur for Hamiltonian PDEs, putting them into context. In each case, we show how the transformation affects the structure map and Hamiltonian of the water wave problem (4.25), and whenever suitable we introduce a small parameter to define asymptotic regimes of interest. In this way, we illustrate the approximation procedure invoked earlier through a series expansion of the Hamiltonian, together with an adjustment in the symplectic structure of the vector field. Recall that $v = (\eta, \xi)^\top$ with $\mathcal{M} = L^2(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$.

Amplitude Scaling

Consider the transformation $\tau : v \longmapsto w$ such that

$$w = \begin{pmatrix} \widetilde{\eta} \\ \widetilde{\xi} \end{pmatrix} = \begin{pmatrix} \alpha\eta \\ \beta\xi \end{pmatrix} = \tau(v)\,,$$

for α, $\beta \in \mathbb{R}^+$. The Jacobian of this transformation is given by

$$\partial_v \tau = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

and, accordingly, the transformed symplectic form is

$$\widetilde{J} = (\partial_v \tau) J (\partial_v \tau)^\top = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \alpha\beta J .$$

The effects of such a transformation are easily restored to the usual Darboux coordinates through a time change $\widetilde{t} = t/(\alpha\beta)$.

The small-amplitude limit of the water wave problem is introduced by an amplitude scaling, which is a transformation of this kind, namely

$$\begin{pmatrix} \varepsilon^2 \widetilde{\eta} \\ \varepsilon \widetilde{\xi} \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \quad 0 < \varepsilon \ll 1 .$$

Hereafter, the range of ε is defined as above. This means that we are seeking solutions for which the amplitude η is small and represented in this asymptotic regime by an $O(1)$ quantity $\widetilde{\eta}$ times ε^2, and similarly for $\xi = \varepsilon\widetilde{\xi}$. The resulting change of symplectic form is

$$\widetilde{J} = \varepsilon^{-3} J ,$$

which is equivalent to rescaling by a slow time variable, and the Taylor expansion (4.47) of the Hamiltonian reads

$$\widetilde{H} = \frac{1}{2} \int_{\mathbb{R}^{d-1}} \left(\varepsilon^2 \widetilde{\xi} G_0 \widetilde{\xi} + \varepsilon^4 g \widetilde{\eta}^2 \right) dx + \sum_{j=1}^{\infty} \frac{1}{2} \int_{\mathbb{R}^{d-1}} \varepsilon^{2+2j} \widetilde{\xi} G_j(\widetilde{\eta}) \widetilde{\xi}\, dx .$$

Because ε is a small parameter, we may express the Hamiltonian as an infinite power series in ε and consider approximations by retaining a finite number of terms. For instance, up to order $O(\varepsilon^4)$, we have

$$\varepsilon^2 \widetilde{H}^{(2)} + \varepsilon^4 \widetilde{H}^{(4)} = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^{d-1}} \widetilde{\xi} G_0 \widetilde{\xi}\, dx + \frac{\varepsilon^4}{2} \int_{\mathbb{R}^{d-1}} \left(g \widetilde{\eta}^2 + \widetilde{\xi} G_1(\widetilde{\eta}) \widetilde{\xi} \right) dx . \quad (4.76)$$

Spatial Scaling

The long-wave limit of the water wave problem highlights solutions whose typical wavelength is asymptotically long. This is accomplished by introducing the spatial scaling

$$x \longmapsto X = \varepsilon x .$$

The corresponding transformation τ of phase space $\mathcal{M}$ is

$$v(x) \longmapsto w(X) = v\left(\frac{X}{\varepsilon}\right) = \tau(v)(X) ,$$

with the Jacobian best described by its action on a vector field $v_2 \in \mathcal{V}(\mathcal{M})$,

$$(\partial_v \tau) v_2(X) = \frac{d}{ds}\tau(v + s\, v_2)\Big|_{s=0} = \frac{d}{ds}\left(v\left(\frac{X}{\varepsilon}\right) + s\, v_2\left(\frac{X}{\varepsilon}\right)\right)\Big|_{s=0} = v_2\left(\frac{X}{\varepsilon}\right).$$

The transpose is slightly less obvious and its action is inferred via the following identity

$$\begin{aligned}\langle v_1, (\partial_v \tau) v_2\rangle &= \int_{\mathbb{R}^{d-1}} v_1(X)\, v_2\left(\frac{X}{\varepsilon}\right) dX = \int_{\mathbb{R}^{d-1}} v_1(\varepsilon x)\, v_2(x)\, \varepsilon^{d-1} dx\,,\\ &= \langle (\partial_v \tau)^\top v_1, v_2\rangle\,.\end{aligned}$$

Therefore, $(\partial_v \tau)^\top v_1(x) = \varepsilon^{d-1} v_1(\varepsilon x)$ and the induced symplectic form is

$$\widetilde{J} = (\partial_v \tau) J (\partial_v \tau)^\top = \varepsilon^{d-1} J\,.$$

The original symplectic structure is again recovered via a time change $\widetilde{t} = \varepsilon^{d-1} t$.

It is necessary to study the effect that this transformation has on the Hamiltonian.

Lemma 4.5.2. *Let $\tau(v)(X) = v(X/\varepsilon) = w(X)$ be the transformation associated with spatial scaling, and let $m(D)$ be a Fourier multiplier operator defined by*

$$(m(D)v)(x) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{ik\cdot(x-x')} m(k) v(x')\, dx' dk\,.$$

Under τ, the Fourier multiplier operator is transformed to

$$\tau(m(D)v)(X) = (m(\varepsilon D_X)\tau(v))(X)\,.$$

Proof. Using the definition of the Fourier multiplier together with the Fourier inversion theorem, we have

$$\begin{aligned}\tau(m(D)v)(X) &= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{ik\cdot(X/\varepsilon - x')} m(k) v(x')\, dx' dk\,,\\ &= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{ik\cdot(X/\varepsilon - X'/\varepsilon)} m(k) v\left(\frac{X'}{\varepsilon}\right) \frac{dX' dk}{\varepsilon^{d-1}}\,,\\ &= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{iK\cdot(X - X')} m(\varepsilon K) v\left(\frac{X'}{\varepsilon}\right) dX' dK\,,\\ &= (m(\varepsilon D_X)\tau(v))(X)\,.\end{aligned}$$

□

This implies, for example, that the first contribution in the Taylor expansion of the DNO

$$G_0(D) = |D| \tanh(h|D|)$$

acts like $G_0(\varepsilon D_X)$ which can be expanded in powers of ε,

$$G_0(\varepsilon D_X) = \varepsilon|D_X|\tanh(\varepsilon h|D_X|) = \varepsilon^2 h|D_X|^2 - \frac{\varepsilon^4 h^3}{3}|D_X|^4 + \dots$$

when applied to functions that vary over the long spatial scale X. Using this expression, the Hamiltonian (4.76) becomes

$$\widetilde{H} = \frac{\varepsilon^4}{2}\int_{\mathbb{R}^{d-1}} \left[h\widetilde{\xi}|D_X|^2\widetilde{\xi} + g\widetilde{\eta}^2 + \varepsilon^2\left(\widetilde{\xi}D_X\cdot\widetilde{\eta}D_X\widetilde{\xi} - \frac{h^3}{3}\widetilde{\xi}|D_X|^4\widetilde{\xi}\right)\right]\frac{dX}{\varepsilon^{d-1}} + O(\varepsilon^{9-d}). \tag{4.77}$$

Surface Elevation-Velocity Coordinates

It is often convenient to write model equations for long waves in terms of the variables

$$w = (\eta, u)^\top = \tau(v) = (\eta, \partial_x\xi)^\top,$$

rather than $v = (\eta, \xi)^\top$. The second variable $u(x,t)$ essentially represents the horizontal velocity of the fluid at the free surface Γ. We say "essentially" because the actual horizontal velocity $\partial_x\varphi\big|_{y=\eta}$ on Γ is not exactly $\partial_x\xi$ as shown by (4.30). Nevertheless, these two quantities are quite similar in the long-wave limit where the wave slope is assumed to be small, i.e. $|\partial_x\eta| = O(\varepsilon^3)$. For simplicity, we now restrict ourselves to the two-dimensional case ($d = 2$). The Jacobian of this transformation is

$$\partial_v\tau = \begin{pmatrix} 1 & 0 \\ 0 & \partial_x \end{pmatrix},$$

and the induced symplectic form is represented by the operator

$$\widetilde{J} = (\partial_v\tau)J(\partial_v\tau)^\top = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}.$$

Returning to the Hamiltonian (4.77) with the tildes dropped off η and ξ, and phrasing it in surface elevation-velocity coordinates yields

$$\widetilde{H} = \frac{\varepsilon^3}{2}\int_{\mathbb{R}}\left[hu^2 + g\eta^2 + \varepsilon^2\left(\eta u^2 - \frac{h^3}{3}(\partial_X u)^2\right)\right]dX + O(\varepsilon^7), \tag{4.78}$$

where we have used the fact that

$$\int_{\mathbb{R}}\xi D_X^2\xi\,dX = -\int_{\mathbb{R}}(D_X\xi)^2\,dX = \int_{\mathbb{R}}(\partial_X\xi)^2\,dX = \int_{\mathbb{R}}u^2\,dX,$$

$$\int_{\mathbb{R}}\xi D_X\eta D_X\xi\,dX = -\int_{\mathbb{R}}\eta(D_X\xi)^2\,dX = \int_{\mathbb{R}}\eta(\partial_X\xi)^2\,dX = \int_{\mathbb{R}}\eta u^2\,dX,$$

$$\int_{\mathbb{R}}\xi D_X^4\xi\,dX = \int_{\mathbb{R}}(D_X^2\xi)^2\,dX = \int_{\mathbb{R}}(\partial_X^2\xi)^2\,dX = \int_{\mathbb{R}}(\partial_X u)^2\,dX,$$

via integration by parts. The truncated expression (4.78) up to order $O(\varepsilon^5)$ is precisely the Hamiltonian (4.22) for the Boussinesq system as mentioned earlier, with $(p,q) \sim (u,\eta)$.

Moving Reference Frame

It is common when studying asymptotic regimes of the water wave problem to work in coordinate systems that move with a characteristic speed of solutions, namely

$$\widetilde{v}(x,t) = v(x-ct,t)\,,$$

for appropriate choices of c. However, the time variable t plays a special role in our point of view of PDEs as Hamiltonian systems, so at first consideration this transformation, which mixes space and time variables, is not accommodated in the present picture. An alternative is to recall that the impulse (4.36) is a conserved quantity and Poisson commutes with the Hamiltonian [10]. Accordingly, their respective flows also commute

$$\chi_t^H \circ \chi_s^I(v) = \chi_s^I \circ \chi_t^H(v)\,.$$

Because the vector field associated with the impulse is given by $\partial_s v = J\mathrm{grad}_v I$, or more explicitly,

$$\partial_s \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \xi \\ -\partial_x \eta \end{pmatrix} = \begin{pmatrix} -\partial_x \eta \\ -\partial_x \xi \end{pmatrix}\,,$$

the corresponding flow is simply constant unit-speed translation

$$\chi_s^I(v)(x) = v(x-s)\,,$$

which implies that the flow along the diagonal is

$$\chi_t^H \circ \chi_{-ct}^I(v) = \chi_t^{H-cI}(v)\,.$$

It can be inferred that the Hamiltonian flow of $H(v) - cI(v)$ is the Hamiltonian flow of $H(v)$ observed in a coordinate frame translating with speed c.

In the context of long waves on shallow water, a characteristic speed is $c_0 = \sqrt{gh}$ as suggested by (4.70). Working in a reference frame moving at speed c_0 is thus accommodated from our point of view by looking at the flow whose Hamiltonian is $\widetilde{H} - c_0\widetilde{I}$. In terms of surface elevation-velocity coordinates scaled appropriately, we find

$$\widetilde{I} = \varepsilon^3 \int_{\mathbb{R}} \eta u\, dX\,,$$

and therefore

$$\begin{aligned} \widetilde{H} - c_0\widetilde{I} &= \frac{\varepsilon^3}{2} \int_{\mathbb{R}} \left[hu^2 - 2\sqrt{gh}\eta u + g\eta^2 + \varepsilon^2\Big(\eta u^2 - \frac{h^3}{3}(\partial_X u)^2\Big)\right] dX\,, \\ &= \frac{\varepsilon^3}{2} \int_{\mathbb{R}} \left[(\sqrt{g}\eta - \sqrt{h}u)^2 + \varepsilon^2\Big(\eta u^2 - \frac{h^3}{3}(\partial_X u)^2\Big)\right] dX\,, \end{aligned} \tag{4.79}$$

for the Boussinesq system. Although we still use the same notation for convenience, it is understood that X now plays the role of $X - c_0\widetilde{t}$.

Characteristic Coordinates

As shown in (4.78), the long-wave scaling regime typically features a Hamiltonian with a quadratic part of the form

$$\widetilde{H}_2 = \frac{1}{2}\int_{\mathbb{R}} (Au^2 + B\eta^2)\, dX \,,$$

where A, $B > 0$. The corresponding equations of motion are given by

$$\begin{pmatrix} \partial_t \eta \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix} \begin{pmatrix} B\eta \\ Au \end{pmatrix} = \begin{pmatrix} 0 & -A \\ -B & 0 \end{pmatrix} \begin{pmatrix} \partial_X \eta \\ \partial_X u \end{pmatrix} , \qquad (4.80)$$

which reduce to a wave equation for either η or u. This reduction suggests performing an additional transformation τ that will accomplish three tasks:

- Diagonalize the symplectic form

$$\widehat{J} = (\partial_v \tau)\widetilde{J}(\partial_v \tau)^\top = (\partial_v \tau) \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix} (\partial_v \tau)^\top = \begin{pmatrix} -\partial_X & 0 \\ 0 & \partial_X \end{pmatrix} .$$

- Transform the Hamiltonian to normal form

$$\widehat{H}_2 = \frac{1}{2}\int_{\mathbb{R}} \sqrt{AB}\,(r^2 + s^2)\, dX \,.$$

- Transform (4.80) to characteristic form

$$\begin{pmatrix} \partial_t r \\ \partial_t s \end{pmatrix} = \begin{pmatrix} -C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} \partial_X r \\ \partial_X s \end{pmatrix} .$$

All three are accomplished by the transformation $w = \tau(v) = \Theta v$ where $\Theta = \partial_v \tau$ is the matrix

$$\Theta = \begin{pmatrix} \sqrt[4]{\frac{B}{4A}} & \sqrt[4]{\frac{A}{4B}} \\ \sqrt[4]{\frac{B}{4A}} & -\sqrt[4]{\frac{A}{4B}} \end{pmatrix} ,$$

with the result that $C = \sqrt{AB}$. From (4.78), we have $A = h$ and $B = g$ so that

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \sqrt[4]{\frac{g}{4h}} & \sqrt[4]{\frac{h}{4g}} \\ \sqrt[4]{\frac{g}{4h}} & -\sqrt[4]{\frac{h}{4g}} \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix} , \qquad (4.81)$$

and $C = c_0 = \sqrt{gh}$. The variable $r(X,t)$ represents the solution's component that travels primarily to the right (in the positive X-direction), while $s(X,t)$ represents the component that travels primarily to the left (in the negative X-direction).

Of course, the nonlinear contributions will change accordingly. From (4.79), the resulting Hamiltonian reads

$$\widehat{H} - c_0\widehat{I} = \varepsilon^3 \int_{\mathbb{R}} c_0 s^2\, dX \qquad (4.82)$$
$$+\varepsilon^5 \int_{\mathbb{R}} \left(\frac{1}{4}\sqrt[4]{\frac{g}{4h}}(r+s)(r-s)^2 - \frac{h^3}{6}\sqrt{\frac{g}{4h}}(\partial_X r - \partial_X s)^2 \right) dX\,.$$

The fact that the r^2 term is absent is precisely due to the subtraction of $c_0\widehat{I}$ from the quadratic part $\widehat{H}_2$ of $\widehat{H}$. Indeed, we expect the corresponding advection term $c_0\partial_X r$ to be absent from the equation of motion, since we are looking at the Hamiltonian flow in a reference frame moving at speed c_0 in the positive X-direction.

4.5.5 Boussinesq and KdV Scaling Limits

The previous calculations pave the way for the derivation of the Boussinesq system and the KdV equation which model the evolution of two-dimensional weakly non-linear gravity waves on shallow water. Referring to the non-dimensionalization performed earlier, the Boussinesq system typically corresponds to the scaling regime where the small dimensionless parameter ε is defined in such a way that

$$\varepsilon^2 = \frac{a_0}{h} = \frac{h^2}{\lambda_0^2}\,.$$

The corresponding Hamiltonian up to order $O(\varepsilon^5)$ is given by (4.78) in terms of the surface elevation η and the horizontal fluid velocity u evaluated on Γ. Their dynamics obey

$$\begin{pmatrix} \partial_t\eta \\ \partial_t u \end{pmatrix} = \varepsilon^{-2} \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix} \begin{pmatrix} \mathrm{grad}_\eta H \\ \mathrm{grad}_u H \end{pmatrix},$$

where the factor ε^{-2} takes into account adjustments of the symplectic form due to the amplitude and spatial scalings. For simplicity, the tilde or hat notation has been dropped. These equations read more explicitly

$$\partial_t\eta = -\varepsilon\partial_X \left((h+\varepsilon^2\eta)u + \frac{\varepsilon^2}{3}h^3\partial_X^2 u \right),$$
$$\partial_t u = -\varepsilon\partial_X \left(g\eta + \frac{\varepsilon^2}{2}u^2 \right).$$

The extra factor ε on the right-hand sides may be removed by rescaling time as $T = \varepsilon t$, which is the typical long time scale associated with the long-wave limit. The resulting Boussinesq system

$$\partial_T\eta = -\partial_X\big((h+\varepsilon^2\eta)u\big) - \frac{\varepsilon^2}{3}h^3\partial_X^3 u\,, \qquad (4.83)$$
$$\partial_T u = -g\partial_X\eta - \varepsilon^2 u\partial_X u \qquad (4.84)$$

coincides with the completely integrable model investigated by Kaup [76].

Then, by introducing characteristic coordinates and looking for solutions in a coordinate system moving at speed $c_0 = \sqrt{gh}$, the Hamiltonian takes the reduced form (4.82). Furthermore, by restricting our attention to right-moving solutions in the region of phase space $\{s = 0\} \subseteq \mathcal{M}$, the Hamiltonian becomes

$$H - c_0 I = \varepsilon^5 \int_{\mathbb{R}} \left(\frac{1}{4} \sqrt[4]{\frac{g}{4h}} r^3 - \frac{h^3}{6} \sqrt{\frac{g}{4h}} (\partial_X r)^2 \right) dX \,.$$

It follows from diagonalization of the symplectic map, as shown in the previous section, that the equation of motion for r is given by

$$\partial_t r = -\varepsilon^{-2} \partial_X \mathrm{grad}_r (H - c_0 I) = -\varepsilon^3 \partial_X \left(\frac{h^3}{3} \sqrt{\frac{g}{4h}} \partial_X^2 r + \frac{3}{4} \sqrt[4]{\frac{g}{4h}} r^2 \right) ,$$

or equivalently

$$\partial_{\mathcal{T}} r = -\frac{h^3}{3} \sqrt{\frac{g}{4h}} \partial_X^3 r - \frac{3}{2} \sqrt[4]{\frac{g}{4h}} r \partial_X r \,, \tag{4.85}$$

which is the usual form of the KdV equation in terms of the long time scale $\mathcal{T} = \varepsilon^3 t$. The free-surface profile η is recovered from r by simply inverting (4.81).

Although the Kaup–Boussinesq equations (4.83) and (4.84) are completely integrable and admit exact analytical solutions, they have been shown to be ill-posed [7]. It is thus of interest to consider higher-order corrections. Retaining terms of up to order $O(\varepsilon^7)$ in (4.78) yields

$$\begin{aligned} H = \frac{\varepsilon^3}{2} \int_{\mathbb{R}} \Big[& hu^2 + g\eta^2 + \varepsilon^2 \Big(\eta u^2 - \frac{h^3}{3} (\partial_X u)^2 \Big) \\ & + \varepsilon^4 \Big(\frac{2}{15} h^5 (\partial_X^2 u)^2 - h^2 \eta (\partial_X u)^2 \Big) \Big] \, dX \,, \end{aligned} \tag{4.86}$$

which gives rise to the higher-order Boussinesq system

$$\begin{aligned} \partial_T \eta &= -\partial_X \left((h + \varepsilon^2 \eta) u + \frac{\varepsilon^2}{3} h^3 \partial_X^2 u + \frac{2\varepsilon^4}{15} h^5 \partial_X^4 u + \varepsilon^4 h^2 \partial_X (\eta \partial_X u) \right) , \\ \partial_T u &= -\partial_X \left(g\eta + \frac{\varepsilon^2}{2} u^2 - \frac{\varepsilon^4}{2} h^2 (\partial_X u)^2 \right) . \end{aligned}$$

Repeating the additional steps that are required to accommodate a moving reference frame and the unidirectional propagation of wave solutions, the Hamiltonian (4.86) successively becomes

$$\begin{aligned} H - c_0 I = {} & \varepsilon^3 \int_{\mathbb{R}} c_0 s^2 \, dX \\ & + \varepsilon^5 \int_{\mathbb{R}} \left(\frac{1}{4} \sqrt[4]{\frac{g}{4h}} (r+s)(r-s)^2 - \frac{h^3}{6} \sqrt{\frac{g}{4h}} (\partial_X r - \partial_X s)^2 \right) dX \\ & + \varepsilon^7 \int_{\mathbb{R}} \left(\frac{h^5}{15} \sqrt{\frac{g}{4h}} (\partial_X^2 r - \partial_X^2 s)^2 - \frac{h^2}{4} \sqrt[4]{\frac{g}{4h}} (r+s)(\partial_X r - \partial_X s)^2 \right) dX \,, \end{aligned}$$

and

$$H - c_0 I = \varepsilon^5 \int_{\mathbb{R}} \Big[\frac{1}{4} \sqrt[4]{\frac{g}{4h}} r^3 - \frac{h^3}{6} \sqrt{\frac{g}{4h}} (\partial_X r)^2 + \varepsilon^2 \Big(\frac{h^5}{15} \sqrt{\frac{g}{4h}} (\partial_X^2 r)^2 - \frac{h^2}{4} \sqrt[4]{\frac{g}{4h}} r (\partial_X r)^2 \Big) \Big] \, dX \, .$$

The corresponding equation of motion for r

$$\partial_{\mathcal{T}} r = -\frac{h^3}{3} \sqrt{\frac{g}{4h}} \partial_X^3 r - \frac{3}{2} \sqrt[4]{\frac{g}{4h}} r \partial_X r - \varepsilon^2 \left(\frac{2h^5}{15} \sqrt{\frac{g}{4h}} \partial_X^5 r + \frac{h^2}{2} \sqrt[4]{\frac{g}{4h}} r \partial_X^3 r + h^2 \sqrt[4]{\frac{g}{4h}} (\partial_X r)(\partial_X^2 r) \right) ,$$

is the so-called fifth-order KdV equation, due to the presence of the fifth derivative in X. Unlike the KdV equation (4.85), it is not integrable. It admits generalized solitary wave solutions which, unlike KdV solitons, are not truly localized but exhibit a central pulse that connects to smaller periodic waves on both sides, as found in computations of Champneys et al. [18].

These results have been extended in a similar systematic fashion to modeling long surface waves over variable topography [31] or over a vertically sheared current [94], long internal waves in two-layer flows [35] and long hydroelastic waves in floating ice sheets [49, 62]. In all these cases, a Hamiltonian reformulation of the full governing equations provides a basis for the asymptotic analysis and the proposed reduced models are all Hamiltonian PDEs. While the present calculations focus on the two-dimensional setting ($d = 2$), extensions of this approach to three dimensions ($d = 3$) have also been explored in [27, 30, 36, 63, 65].

4.5.6 Modulational Scaling Limit and the NLS Equation

Another asymptotic regime of interest in the water wave problem is the modulational limit for weakly nonlinear periodic waves on deep water. Recall from the non-dimensionalization performed earlier that the small dimensionless parameter ε in this case is defined as a characteristic wave steepness $\varepsilon = k_0 a_0$, where $k_0 > 0$ denotes the carrier (i.e. dominant) wavenumber of near-monochromatic waves. We will again restrict our attention to the two-dimensional case. Compared to the long-wave limit, additional transformations are required here as part of the asymptotic procedure, and will be described below. In particular, a modulational Ansatz makes it possible to derive reduced models for the wave envelope. This step introduces multiscale functions whose role needs to be carefully examined.

Normal Form Transformation

Because three-wave resonances do not occur in the context of deep-water gravity waves, the Hamiltonian can be simplified by eliminating all cubic terms through an appropriate canonical transformation. As explained in Sect. 4.4.6, this reduction to third-order normal form can be framed as

$$\begin{pmatrix} \widetilde{\eta} \\ \widetilde{\xi} \end{pmatrix} = \Theta_0 \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \tau(v)\,, \tag{4.87}$$

with

$$\Theta_0 = \partial_v \tau = \begin{pmatrix} -i\,\mathrm{sgn}(D) & 0 \\ 0 & -i\,\mathrm{sgn}(D) \end{pmatrix},$$

so that

$$\partial_s \begin{pmatrix} \widetilde{\eta} \\ \widetilde{\xi} \end{pmatrix} = J_0 \begin{pmatrix} \mathrm{grad}_{\widetilde{\eta}} K_3 \\ \mathrm{grad}_{\widetilde{\xi}} K_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{grad}_{\widetilde{\eta}} K_3 \\ \mathrm{grad}_{\widetilde{\xi}} K_3 \end{pmatrix}, \tag{4.88}$$

upon the adjustment $J_0 = \overline{\Theta}_0^{\top} \Theta_0^{\top} = J$. The auxiliary Hamiltonian K_3 is given by (4.62) and the transformation (4.88) takes the form (4.63). In particular, the first equation for $\widetilde{\eta}$ is Burgers' equation. The new Hamiltonian (4.65), after this normal form transformation, reduces to

$$H = H_2(\eta,\xi) + H_4(\eta,\xi) - H_4(-i\,\mathrm{sgn}(D)\eta,\xi) + R_5\,, \tag{4.89}$$

where the leading nonlinear contributions are quartic terms.

Modulational Ansatz

We now describe the scaling regime that embodies the special form of wave solutions that we are interested in. First, we transform to (complex) normal modes $z(x,t)$ and $\overline{z}(x,t)$ as defined by

$$\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \Theta_1 \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a(D) & i\,a^{-1}(D) \\ a(D) & -i\,a^{-1}(D) \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \tag{4.90}$$

where

$$a(D) = \sqrt[4]{\frac{g}{G_0(D)}}\,,$$

and the symbol $\bar{\cdot}$ stands for complex conjugation. These are the physical-space counterparts to the complex symplectic coordinates (4.50) that were introduced in the Fourier space earlier. As a result, the original system (4.25) becomes

$$\partial_t \begin{pmatrix} z \\ \overline{z} \end{pmatrix} = J_1 \begin{pmatrix} \mathrm{grad}_z H \\ \mathrm{grad}_{\overline{z}} H \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \mathrm{grad}_z H \\ \mathrm{grad}_{\overline{z}} H \end{pmatrix},$$

with the transformed symplectic map $J_1 = \Theta_1 J \Theta_1^{\top}$.

The next step introduces our modulational Ansatz

$$\begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \Theta_2 \begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \varepsilon^{-1} \begin{pmatrix} e^{-ik_0 x} & 0 \\ 0 & e^{ik_0 x} \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \end{pmatrix}, \tag{4.91}$$

which is to say that we are looking for solutions in the form of near-monochromatic waves with carrier wavenumber $k_0 > 0$ and with slowly varying complex envelope $u(X, t)$ depending on $X = \varepsilon x$. The small dimensionless parameter $\varepsilon = k_0 a_0$ is a measure of wave steepness and, equivalently, it is also a measure of the wave spectrum's narrowness around $k = k_0$. The corresponding equations of motion are

$$\partial_t \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = J_2 \begin{pmatrix} \mathrm{grad}_u H \\ \mathrm{grad}_{\overline{u}} H \end{pmatrix} = \varepsilon^{-1} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \mathrm{grad}_u H \\ \mathrm{grad}_{\overline{u}} H \end{pmatrix}, \tag{4.92}$$

where $J_2 = \varepsilon \Theta_2 J_1 \Theta_2^\top$. The extra factor ε in the definition of J_2 reflects the change in symplectic structure associated with the spatial scaling $x \to X = \varepsilon x$.

Expansion and Homogenization of Multiscale Functions

The Hamiltonian (4.89) is also transformed through the changes of variables (4.90) and (4.91). The first transformation (4.90) diagonalizes the quadratic (i.e. linear) part

$$H_2 = \int_{\mathbb{R}} \overline{z}\, \omega(D) z \, dx \,,$$

in terms of normal modes $(z, \overline{z})$ associated with the exact linear dispersion relation

$$\omega(D) = \big(g G_0(D)\big)^{1/2} = (g|D|)^{1/2} \,,$$

for deep-water gravity waves ($h \to +\infty$) in its operator form. Note the identity

$$\omega(D) = g a^{-2}(D) = G_0(D) a^2(D) \,.$$

Modulo the fact that complex amplitudes are now involved, this basically produces the same effect as the change (4.81) to characteristic coordinates for long waves on shallow water.

The second transformation (4.91) paves the way for the expansion of H in powers of ε. Due to the multiscale nature of this problem (fast oscillations in x and slow modulation in X), it is important to understand the action of Fourier multiplier operators on multiscale functions [43].

Theorem 4.5.3. *Assume that the Fourier multiplier $m(D)$ has the property*

$$|\partial_k^j m(k)| \le C_j (1 + k^2)^{(\ell - j)/2} \,, \quad 0 \le j \le \ell \,.$$

Then its action on a multiscale function $f(x, X)$, where $X = \varepsilon x$, has the asymptotic expansion

$$(m(D) f)(x, X) = \sum_{j=0}^{n} \frac{\varepsilon^j}{j!} \partial_k^j m(D_x) D_X^j f(x, X) + R_{n+1} f \,.$$

Proof. Using the definition of the Fourier multiplier together with the Fourier inversion theorem, we have

$$
\begin{aligned}
(m(D)f)(x,X) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ik'x} m(k')\widehat{f}(k';\varepsilon)\,dk' , \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{ik'x} e^{-i(k'-(k+\varepsilon K))x'} m(k')\widehat{f}(k,K)\,dkdKdx'dk' , \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i(k+\varepsilon K)x} m(k+\varepsilon K)\widehat{f}(k,K)\,dkdK .
\end{aligned}
$$

Because $m(D)$ satisfies the above property, i.e. it is a classical pseudodifferential operator of order ℓ, we can Taylor expand $m(k')$ about $k'=k$ and obtain

$$
\begin{aligned}
(m(D)f)(x,X) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ikx}e^{iKX}\left(\sum_{j=0}^{n}\partial_k^j m(k)\frac{(\varepsilon K)^j}{j!} + \widehat{R}_{n+1}\right)\widehat{f}(k,K)\,dkdK , \\
&= \sum_{j=0}^{n}\frac{\varepsilon^j}{j!}\partial_k^j m(D_x)D_X^j f(x,X) + R_{n+1}f .
\end{aligned}
$$

A more detailed proof with a precise estimate for the remainder $R_{n+1}f$ can be found in [43]. □

Omitting the t-dependence, a typical multiscale function is given by $f(x,X) = e^{ik_0x}u(X)$ according to the modulational Ansatz (4.91). This implies that e.g. $G_0(D)f$ admits the following asymptotic expansion in ε,

$$
\begin{aligned}
G_0\left(e^{ik_0x}u(X)\right) &= |D_x+\varepsilon D_X|\left(e^{ik_0x}u(X)\right) = e^{ik_0x}|k_0+\varepsilon D_X|u(X) \\
&= e^{ik_0x}(k_0^2+2\varepsilon k_0D_X+\varepsilon^2D_X^2)^{1/2}u(X) \\
&= e^{ik_0x}(k_0+\varepsilon D_X+\dots)u(X) ,
\end{aligned}
$$

given that $|D|^2 = D^2$. Hence, we may symbolically write

$$
|D_x+\varepsilon D_X| = \left(D_x^2+2\varepsilon D_xD_X+\varepsilon^2D_X^2\right)^{1/2} = |D_x| + \varepsilon|D_x|^{-1}D_xD_X + \dots
$$

Moreover, because our focus is on describing nontrivial dynamics of the wave envelope, the presence of multiple scales needs to be appropriately dealt with. This is a homogenization problem which is addressed in the present Hamiltonian framework via the scale separation lemma of Craig et al. [30].

Lemma 4.5.4. *Suppose that $p(x)$ is a continuous and periodic function of period γ, and $q(X)$ is a Schwarz-class function. Then the short scales represented in $p(x)$ and the long scales represented by $X=\varepsilon x$ in $q(X)$ are asymptotically separated. That is, for all $n>0$, we have the estimate*

$$
\int_{\mathbb{R}} p(x)q(\varepsilon x)\,dx = \langle p\rangle \int_{\mathbb{R}} q(X)\,\frac{dX}{\varepsilon} + O(\varepsilon^n) ,
$$

where

$$\langle p \rangle = \frac{1}{\gamma} \int_0^\gamma p(x)\, dx$$

denotes the average value of $p(x)$ *over a period* γ.

The reader is referred to [30] for the proof of this lemma. In the present setting, $p(x)$ is a function of the form $e^{i\ell k_0 x}$ ($\ell \in \mathbb{Z}$) for which

$$\langle e^{i\ell k_0 x} \rangle = \begin{cases} 1, & \text{if} \quad \ell = 0, \\ 0, & \text{otherwise}. \end{cases}$$

This implies that terms with fast oscillations essentially homogenize to zero and thus do not contribute to the effective Hamiltonian. This homogenization naturally selects four-wave resonances among all the possible quartic interactions because the corresponding fast oscillations exactly cancel out. We are left with the following contributions in terms of the slowly varying envelope u,

$$H_2(\eta, \xi) = \varepsilon \int_{\mathbb{R}} \overline{u}\, \omega(k_0 + \varepsilon D_X) u\, dX,$$

$$H_4(\eta, \xi) = \frac{\varepsilon^3}{4} \int_{\mathbb{R}} k_0^3 |u|^4\, dX,$$

$$H_4(-i\,\mathrm{sgn}(D)\eta, \xi) = -\frac{\varepsilon^3}{4} \int_{\mathbb{R}} k_0^3 |u|^4\, dX,$$

up to order $O(\varepsilon^3)$. Note that more significant differences between $H_4(\eta, \xi)$ and $H_4(-i\,\mathrm{sgn}(D)\eta, \xi)$, such as contributions from the wave-induced mean flow, only arise at higher orders.

NLS Equation

If we Taylor expand the dispersion relation $\omega(k_0 + \varepsilon D_X)$ in ε, and retain terms of order up to $O(\varepsilon^3)$, then the reduced Hamiltonian (4.89) reads

$$\begin{aligned} H &= H_2(\eta, \xi) + H_4(\eta, \xi) - H_4(-i\,\mathrm{sgn}(D)\eta, \xi), \qquad (4.93) \\ &= \frac{\varepsilon}{2} \int_{\mathbb{R}} \left[\overline{u} \Big(\omega_0 + \varepsilon \omega_0' D_X + \frac{\varepsilon^2}{2} \omega_0'' D_X^2 \Big) u + \text{c.c.} + \varepsilon^2 k_0^3 |u|^4 \right] dX + O(\varepsilon^4), \\ &= \varepsilon \int_{\mathbb{R}} \left(\omega_0 |u|^2 + \varepsilon \omega_0'\, \mathrm{Im}(\overline{u} \partial_X u) + \frac{\varepsilon^2}{2} \omega_0'' |\partial_X u|^2 + \frac{\varepsilon^2}{2} k_0^3 |u|^4 \right) dX + O(\varepsilon^4), \end{aligned}$$

where $\omega_0 = \omega(k_0)$ and similarly for its derivatives. The initials "c.c." stand for the complex conjugate of all the preceding terms on the right-hand side of the equation. In (4.93), we have used the fact that

$$\int_{\mathbb{R}} \big(\overline{u} D_X u + u \overline{D_X u} \big)\, dX = -i \int_{\mathbb{R}} \big(\overline{u} \partial_X u - u \overline{\partial_X u} \big)\, dX = 2 \int_{\mathbb{R}} \mathrm{Im}(\overline{u} \partial_X u)\, dX,$$

via integration by parts. It follows from (4.92) that the evolution equation for u is given by

$$\begin{aligned} \partial_t u &= -i\varepsilon^{-1}\mathrm{grad}_{\overline{u}} H \ , \\ &= -i\omega_0 u - \varepsilon\omega_0' \partial_X u + i\frac{\varepsilon^2}{2}\omega_0'' \partial_X^2 u - i\varepsilon^2 k_0^3 |u|^2 u \, , \end{aligned} \tag{4.94}$$

which is the cubic NLS equation for deep-water gravity waves [1, 43, 101], associated with the Hamiltonian (4.93). Note that the prefactors ε^{-1} in (4.92) and ε in (4.93) suitably cancel out. This equation describes right-moving waves as indicated by the linear advection term. If we switched the sign in the normal mode decomposition (4.90) of ξ, we would obtain a model for left-moving waves, with an advection term having the opposite sign.

The Hamiltonian (4.93) can be further simplified by subtracting a multiple of the wave action

$$M = \varepsilon \int_{\mathbb{R}} |u|^2 \, dX \ ,$$

together with a multiple of the impulse

$$I = \int_{\mathbb{R}} \eta \partial_x \xi \, dx = \varepsilon \int_{\mathbb{R}} \Big(k_0 |u|^2 + \frac{\varepsilon}{2} \big(\overline{u} D_X u + u \overline{D_X u} \big) \Big) \, dX + \dots \ ,$$

yielding

$$H - \omega_0' I - (\omega_0 - k_0\omega_0') M = \frac{\varepsilon^3}{2} \int_{\mathbb{R}} \big(\omega_0'' |\partial_X u|^2 + k_0^3 |u|^4 \big) dX \, .$$

This reduction is made possible by the fact that both I and M are conserved quantities (and thus Poisson commute with H) at this level of approximation. The basic conservation of I was already established in a previous section. As for the conservation of M, it is inherent to the modulational Ansatz and follows from the identity

$$\begin{aligned} \frac{dM}{dt} &= \{H, M\} \ , \\ &= \int_{\mathbb{R}} \big((\mathrm{grad}_u M)(-i \, \mathrm{grad}_{\overline{u}} H) + (\mathrm{grad}_{\overline{u}} M)(i \, \mathrm{grad}_u H) \big) \, dX \ , \\ &= -i\varepsilon \int_{\mathbb{R}} \big(\overline{u} \, \mathrm{grad}_{\overline{u}} H - u \, \mathrm{grad}_u H \big) \, dX \ , \\ &= -i\varepsilon \int_{\mathbb{R}} \big(\overline{u} \, \mathrm{grad}_{\overline{u}} H - \overline{u} \, \mathrm{grad}_{\overline{u}} H \big) \, dX = 0 \, , \end{aligned}$$

after integrating by parts. This transformation preserves the symplectic map J_2 and the resulting simplification of (4.94) reads

$$\begin{aligned} \partial_t u &= -i\varepsilon^{-1}\mathrm{grad}_{\overline{u}} \big(H - \omega_0' I - (\omega_0 - k_0\omega_0') M \big) \, , \\ &= i\varepsilon^2 \left(\frac{1}{2}\omega_0'' \partial_X^2 u - k_0^3 |u|^2 u \right) \, , \end{aligned}$$

or equivalently

$$-i\partial_{\mathcal{T}}u = \frac{1}{2}\omega_0''\partial_X^2 u - k_0^3|u|^2 u\,, \tag{4.95}$$

where $\mathcal{T} = \varepsilon^2 t$ is the typical slow time scale in the modulation theory for deep-water waves. The subtraction of M from H reflects the property of phase (or gauge) invariance in this approximation, while the subtraction of I is equivalent to changing the coordinate system to a reference frame moving with the group velocity ω_0'. Because

$$\omega''(k) = -\frac{\omega(k)}{4k^2} = -\frac{1}{4}\sqrt{g}\,k^{-3/2}\,, \quad k > 0\,,$$

the coefficients of the dispersive and nonlinear terms in (4.95) are of the same sign, which implies that the NLS equation (4.95) is of focusing type and is subject to modulational instability as can be expected for two-dimensional gravity waves on deep water [90].

Using a similar method in the context of Hamiltonian perturbation theory, Guyenne and Părău [61] proposed an NLS equation for hydroelastic wave packets propagating in a floating ice sheet lying on deep water, and Craig et al. [35] derived a linear Schrödinger equation for small-amplitude surface waves that are modulated by larger-amplitude internal waves described as KdV solitons in two-layer flows.

The next-order correction to the cubic NLS equation, which is commonly referred to as Dysthe's equation, has also received much attention from the scientific community [53, 89]. It contains a nonlinear nonlocal term that represents effects from the wave-induced mean flow, one of them being a Doppler shift relative to k_0 which affects the prediction of modulational instability. Dysthe's original equation was not Hamiltonian, and it is only recently that Hamiltonian versions of it have been proposed [32, 34, 37, 58]. The next-order correction to (4.95) is given by

$$-i\partial_{\mathcal{T}}u = \frac{1}{2}\omega_0''\partial_X^2 u - k_0^3|u|^2 u - i\frac{\varepsilon}{6}\omega_0'''\partial_X^3 u + 3i\varepsilon k_0^2|u|^2\partial_X u + \varepsilon k_0^2 u|D_X||u|^2\,,$$

which is associated with the reduced Hamiltonian

$$\begin{aligned} &H - \omega_0' I - (\omega_0 - k_0\omega_0')M \\ &= \frac{\varepsilon^3}{2}\int_{\mathbb{R}} \Big(\omega_0''|\partial_X u|^2 + k_0^3|u|^4 + \frac{\varepsilon}{3}\omega_0'''\,\mathrm{Im}\big[(\overline{\partial_X u})(\partial_X^2 u)\big] \\ &\qquad + 3\varepsilon k_0^2|u|^2\,\mathrm{Im}(\overline{u}\partial_X u) - \varepsilon k_0^2|u|^2|D_X||u|^2\Big)dX\,. \end{aligned}$$

More details on its derivation can be found in [37].

Reconstruction of the Free Surface

Because the NLS equation (4.94) only describes the wave envelope, another step is required in order to reconstruct the actual shape of the free surface from information on the wave envelope. In modulation theory, this reconstruction is usually carried out perturbatively, based on an Ansatz similar to Stokes' expansion, by adding contributions from various harmonics of the wave spectrum.

Here the reconstruction procedure inverts the transformations associated with our modulational Ansatz and the third-order normal form that eliminates H_3. At any instant t, the conversion of η back to its original definition is governed by Burgers' equation

$$\partial_s \widetilde{\eta} + \widetilde{\eta} \partial_x \widetilde{\eta} = 0 \,, \tag{4.96}$$

for $s \in (-1, 0]$ with "initial" condition

$$\eta_I(x,t) = \eta(x,t)\big|_{s=-1} = \frac{\varepsilon}{\sqrt{2}} a^{-1}(D) \Big(u(\varepsilon x, t) e^{ik_0 x} + \overline{u}(\varepsilon x, t) e^{-ik_0 x} \Big) \,, \tag{4.97}$$

where u solves (4.94). The choice of this initial condition is dictated by the changes of variables (4.90) and (4.91), via inversion of Θ_1 and Θ_2. Equation (4.96) may be solved numerically and its solution at $s = 0$ is meant to represent the original physical variable η. In particular, the x-dependence in (4.96) and (4.97) can be easily handled by the fast Fourier transform, as discussed in a subsequent section. Note that η and $\widetilde{\eta}$ are directly related through (4.87). It is clear from (4.96) and (4.97) that the wave dynamics in this modulational regime generates higher-order contributions from lower (i.e. mean flow) and higher harmonics through nonlinear interactions. It is also worth pointing out that the integration time $s \in (-1, 0]$ is well within the existence time

$$s = O\left(\frac{1}{|\partial_x \widetilde{\eta}_I|} \right) = O(\varepsilon^{-1}) \,,$$

before a shock occurs, for Burgers' equation with smooth initial data $\widetilde{\eta}_i$. Using this non-perturbative procedure for surface reconstruction, it was found in [37] that predictions by the Hamiltonian Dysthe's equation compare very well with direct numerical simulations of the full equations (4.33)–(4.34).

As an alternative to the direct numerical solution of (4.96), a perturbative analytical expression for the surface elevation can be obtained in a manner consistent with the Hamiltonian framework, via the Taylor expansion near $s = -1$, i.e.

$$\eta\big|_{s=0} = \eta\big|_{s=-1} + \partial_s \eta\big|_{s=-1} + \frac{1}{2} \partial_s^2 \eta\big|_{s=-1} + \dots \,, \tag{4.98}$$

where

$$\partial_s \eta = \{K_3, \eta\} = \mathrm{grad}_\xi K_3 = \frac{1}{2} |D| \left(-i \, \mathrm{sgn}(D) \eta \right)^2 \,,$$

$$\partial_s^2 \eta = \partial_s \{K_3, \eta\} = \{K_3, \{K_3, \eta\}\} = \left\{ K_3, \mathrm{grad}_\xi K_3 \right\} \,,$$

and η at $s = -1$ is given by (4.97). By analogy with Stokes' expansion for near-monochromatic waves, the first, second, and third terms on the right-hand side of (4.98) include contributions from the first, second, and third harmonics, respectively.

4.6 Initial Value Problems

This section is devoted to the local and global existence theory for the initial value problem. We start by giving some historical facts and continue with a review of significant progress that took place in the last fifteen years. We will consider both the two- and three-dimensional settings. There are many different formulations of the water wave problem, involving in particular Lagrangian coordinates, Eulerian coordinates or complex coordinates that we will also discuss. Unless stated otherwise, the water wave equations will refer to (4.33) and (4.34), with possibly surface tension if we find it suitable for the discussion. In this case, the coefficient of surface tension will be denoted by σ. Moreover, we will use the notation (η_0, ξ_0) to refer to the initial conditions of (η, ξ).

4.6.1 Local Well-Posedness

A major difficulty is that Eqs. (4.33)–(4.34) form a quasilinear system of equations. We note that the gravity term containing g is of lower order. One can ask: why does the sign of g play such an important role in the well-posedness of the initial value problem? To answer this question, we turn to the linearized system about the state at rest $(\eta, \xi) = 0$. Setting $(\zeta, \psi) := (\delta\eta, \delta\xi)$, we have

$$\begin{pmatrix} \partial_t \zeta \\ \partial_t \psi \end{pmatrix} = \begin{pmatrix} 0 & G_0(D) \\ -g & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ \psi \end{pmatrix} := \mathcal{L} \begin{pmatrix} \zeta \\ \psi \end{pmatrix}.$$

The principal symbol of $\mathcal{L}$ is

$$\mathcal{L}_1 := \begin{pmatrix} 0 & G_0(k) \\ 0 & 0 \end{pmatrix},$$

which has multiple eigenvalues $\lambda(k) = 0$, hence it is not strictly hyperbolic. This fact renders the initial value problem very sensitive to perturbation, even by lower-order terms.

When $g < 0$, the full symbol

$$\mathcal{L}(k) = \mathcal{L}_1(k) + \mathcal{L}_0(k) = \begin{pmatrix} 0 & G_0(k) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -g & 0 \end{pmatrix}$$

has complex eigenvalues, while for $g > 0$ they remain real. When $g = 0$, the fundamental solution is given by the oscillatory kernel

$$\exp(t\mathcal{L}_1(k)) = \begin{pmatrix} 1 & tG_0(k) \\ 0 & 1 \end{pmatrix}.$$

This evolution operator has one derivative loss. Its action on $(\eta_0, \xi_0) \in H^r \times H^s$ is $(\eta(x,t), \xi(x,t)) = (\eta_0 + tG_0(D)\xi_0, \xi_0)$ where the first component is the sum of an element of H^r and an element of H^{s-1}.

Set $\omega_k = \sqrt{g|k|\tanh(h|k|)}$ based on the linear dispersion relation. When $g > 0$,

$$\exp(t\mathcal{L}(k)) = \begin{pmatrix} \cos(t\omega_k) & \frac{\omega_k}{g}\sin(t\omega_k) \\ -\frac{g}{\omega_k}\sin(t\omega_k) & \cos(t\omega_k) \end{pmatrix}. \tag{4.99}$$

This evolution operator maps $H^r \times H^{r+1/2} \longrightarrow H^r \times H^{r+1/2}$. However, when $g < 0$, the parametrix takes the form

$$\exp(t\mathcal{L}(k)) = \begin{pmatrix} \cosh(t\omega_k) & \frac{\omega_k}{g}\sinh(t\omega_k) \\ \frac{g}{\omega_k}\sinh(t\omega_k) & \cosh(t\omega_k) \end{pmatrix},$$

which has the property that it is unbounded as a map from any H^r to any H^{-s}. The linearized equations are thus ill-posed for $g < 0$.

One method to address the lack of strict hyperbolicity is to work with data (η, ξ) in the space of analytic functions C^ω_α, where C^ω_α is the space of functions $f(x)$ bounded and analytic in a complex neighborhood of $\mathbb{R}^{d-1}$ of width α. The parametrix $\exp(t\mathcal{L}(k))$ is bounded from C^ω_ρ to C^ω_σ for $\sigma < \rho$, as well as on certain Gevrey class scales. This was the approach adopted by Ovsyannikov [85] and Kano and Nishida [74] who proved local well-posedness of the two-dimensional water wave problem for analytic data, and they also provided a rigorous justification of the shallow-water scaling limit.

Theorem 4.6.1. *Consider the water wave equations scaled in the shallow-water regime $X = \varepsilon x$. Given initial data $(\eta_0, \xi_0) \in B_\alpha$ and $\varepsilon < \varepsilon_0$ sufficiently small, then there exists a time interval $[-T, +T]$ independent of ε, and an analytic solution $(\eta_\varepsilon(t, X), \xi_\varepsilon(t, X)) \in C^\omega_{\alpha(t)}$. Furthermore as $\varepsilon \to 0$, this solution converges to a solution of the shallow-water equations.*

The proof of this theorem uses the Nirenberg–Nishida abstract version of the Cauchy–Kovalevskaya theorem. The distinction between analytic and Sobolev initial data is not simply a technicality. Nalimov, in a fundamental paper [82], proved local well-posedness of the initial value problem for the problem of two-dimensional water waves in an infinitely deep basin, for an initial surface displacement and initial velocity contained in a sufficiently small ball in an appropriate Sobolev space. His result is local in time, giving a time interval over which solutions remain bounded within the initial Sobolev space. This result was later extended by Yosihara [100] to the case $0 < h < +\infty$. Nalimov's original theorem uses Lagrangian variables as there is a key but subtle cancellation in a subprincipal term that allows one to overcome the problem of multiple characteristics in energy estimates. Lagrangian coordinates are also useful to allow for overturning wave profiles. They were used in particular by Craig [24], Wu [96], and Schneider and Wayne [87].

Let us derive the system of equations for two-dimensional water waves written in Lagrangian coordinates from the basic Euler's equations. Let $\mathbf{u} = (u, v)$ be the two-dimensional fluid velocity. Then Euler's equations (4.1)–(4.2) together with (4.3) read

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(\mathcal{P} + gy)\,, \tag{4.100}$$

$$\partial_x u + \partial_y v = 0\,, \tag{4.101}$$

$$\partial_x v - \partial_y v = 0\,, \tag{4.102}$$

in a fluid region $\Omega = \{-\infty < x < +\infty, -h < y < \eta(x,t)\}$. At the bottom $\{y = -h\}$, $v = 0$ while at the interface $\Gamma = \{(x, \eta(x,t))\}$, the pressure $\mathcal{P}$ is constant and the vector (t, u, v) is tangent to the surface $\{(t, x, \eta(x,t)), t \in \mathbb{R}, x \in \mathbb{R}\}$. For simplicity, the fluid density ρ has been absorbed into the definition of $\mathcal{P}$.

The Lagrangian coordinates of the free surface are taken in the form $(x + X(x,t), Y(x,t))$, where we consider motion for which (X, Y) are bounded localized perturbations of the free surface $\{(x, 0)\}$ for the fluid at rest. To describe the motion of the free surface, we take the point $(x + X, Y)$ to be the coordinate of a Lagrangian particle on the free surface. Writing $\mathbf{X} = (X, Y)^\top$, the acceleration of such a Lagrangian particle is given by

$$\partial_t \mathbf{X} = \mathbf{u}(x + X, Y, t)\,.$$

From Euler's equations, we get

$$\partial_t^2 \mathbf{X} = \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(\mathcal{P} + gy)\,, \tag{4.103}$$

$$\partial_x \mathcal{P}(x + X, Y) = \nabla\mathcal{P}\cdot\begin{pmatrix} 1 + \partial_x X \\ \partial_x Y \end{pmatrix} = 0\,.$$

Equations (4.101) and (4.102) are the Cauchy–Riemann equations for the analytic function $f(z)$ where $f = u - iv$ and $z = x + iy$. Together with the bottom boundary condition $v(x, -h) = 0$, there is a singular integral operator on the top surface, recovering boundary values of v from boundary values of u and under the condition that $(u - iv)(z) \to 0$ as $z \to \infty$, there exists an operator K depending on Γ, or equivalently on $\mathbf{X}$ such that

$$v = K(\mathbf{X})u\,. \tag{4.104}$$

Using (4.103) to eliminate the pressure from (4.100) and recalling that $\partial_t \mathbf{X} = \mathbf{u}$, the equations of the free surface read

$$(1 + \partial_x X)\partial_t^2 X + \partial_x Y(g + \partial_t^2 Y) = 0\,,$$

$$\partial_t Y = K(\mathbf{X})\partial_t X = 0\,.$$

It is in this setting that Craig [24] proved that, given $\mathbf{X}(x, 0)$ for the initial value problem in Sobolev spaces H^s $(s \geq 3)$, such that the initial free surface is represented by a simple chord-arc curve and is sufficiently small, then there exists $T > 0$ and a solution over the time interval $[-T, +T]$.

The smallness assumption was later removed by Wu [96] in the case of infinite depth $(h \to +\infty)$. Wu [97] then extended this result to the three-dimensional problem, again for $h \to +\infty$, using quaternionic coordinates. The case of finite depth in two and three dimensions, possibly with a smooth non-flat bottom $y = \beta(x)$, was studied by Lannes [78]. The system under consideration is the water wave problem in Eulerian coordinates as given by (4.33)–(4.34). In this case, the DNO depends on both η and β. Conditions are that $\beta \in C^\infty$, varies "slowly" and the initial free surface η_0 satisfies

$$\min\{\eta_0 - \beta, -\beta\} \geq h_0\,,$$

for a certain $h_0 > 0$. More precisely, given initial data $(\eta_0, \xi_0) \in H^r$ (r sufficiently large), there exists $T > 0$ and a unique solution (η, ξ) such that

$$(\eta, \xi) \in C^1([0, +T]\,, H^r(\mathbb{R}^{d-1}) \times H^r(\mathbb{R}^{d-1}))\,.$$

The condition that the bottom varies slowly ensures that the Taylor sign condition

$$-\nabla\mathcal{P} \cdot \mathbf{n} \geq c_0 > 0$$

is satisfied on the free surface Γ, where $\mathbf{n}$ is the outward normal unit vector to Γ.

The Taylor sign condition is a central concept in the theory of water waves. It is known that the water wave motion can be subject to the Taylor instability when surface tension is neglected and the Taylor sign condition fails [54, 92]. Beale et al. [9] showed that the linearization of the water wave equations around a given solution is well-posed, provided this exact solution satisfies the generalized Taylor sign condition. A central element of Wu's analysis is that the Taylor criterion holds in two and three dimensions. This criterion expresses that the surface is not accelerating into the fluid region more rapidly than the normal acceleration due to gravity. We explain formally below why the Taylor condition automatically holds for uniform or infinite water depth.

We start from (4.103) written as

$$\partial_t^2\mathbf{X} - \mathbf{g} = -\nabla\mathcal{P}\,,$$

and take the scalar product with $\mathbf{n}$,

$$\mathbf{n} \cdot \left(\partial_t^2\mathbf{X} - \mathbf{g}\right) = -\partial_n\mathcal{P}\,. \tag{4.105}$$

Assume that a smooth solution exists for $(-T, +T)$ and fix a time t in this time interval. Denote by

$$\mathfrak{a}(x,t) = \mathbf{n} \cdot \left(\partial_t^2\mathbf{X} - \mathbf{g}\right). \tag{4.106}$$

Assuming that $\mathbf{u}$ and its derivatives vanish at infinity, hence so does $\partial_t^2\mathbf{X}$, and since $g > 0$, there exist constants c_1 and R such that $\mathfrak{a}(x,t) \geq c_1$ for $|x| \geq R$. On the other hand, by virtue of (4.100) and the fact that $\mathbf{u} = \nabla\varphi$, the pressure $\mathcal{P}$ satisfies the elliptic problem

$$\begin{aligned} -\Delta\mathcal{P} &= \frac{1}{2}\Delta(|\nabla\varphi|^2) \geq 0\,, \quad \text{in} \quad \Omega\,, \\ \mathcal{P} &= 0\,, \quad \text{on} \quad \Gamma\,. \end{aligned}$$

The function $\mathcal{P}$ is a sub-harmonic function. It reaches its minimum on the boundary of the domain, and at such a point, the outward normal derivative is strictly negative. Thus it cannot reach its minimum at $y = -h$ where $\partial_n \mathcal{P} = 0$. Its minimum is therefore reached on the free surface where $\mathcal{P}$ vanishes identically. Thus, $\mathcal{P}$ is positive in the fluid domain. Moreover, any point of the free surface being a minimum for $\mathcal{P}$, one has $\partial_n \mathcal{P} < 0$ everywhere on Γ, i.e. $\partial_n \mathcal{P}(x, \eta(x,t)) < 0$ for all $x \in \mathbb{R}^{d-1}$. By a continuity argument, there exists $c_2 > 0$ such that $-\partial_n \mathcal{P}(x, \eta(x,t)) > c_2$ for all x in the ball $|x| \leq R$. Choosing $c_0 = \min(c_1, c_2)$, we have $\mathfrak{a}(x,t) \geq c_0 > 0$ for all $x \in \mathbb{R}^{d-1}$.

Finally, let us discuss the condition imposed on the bottom variations. It is expressed as

$$\Pi_\beta(\mathbf{u}_0|_{y=\beta(x)}) \leq \frac{g}{\sqrt{1 + |\partial_x \beta|^2}}\,,$$

where $\mathbf{u}_0$ is the initial velocity associated with ξ_0, given by $\mathbf{u}_0 = \nabla \varphi_0$, where φ_0 is the velocity potential obtained by solving Laplace's equation in the fluid domain $\{(x,y) : x \in \mathbb{R}^{d-1}, \beta(x) < y < \eta_0(x)\}$ with Dirichlet boundary conditions ξ_0 at the free surface $\{y = \eta_0(x)\}$ and homogeneous Neumann boundary conditions at the bottom. The symbol Π_β denotes the second fundamental form associated with the bottom $\{y = \beta(x)\}$. From a mathematical viewpoint, the Taylor criterion is crucial because the quasilinear system is not strictly hyperbolic and requires a Lévy condition on the subprincipal symbol to be well-posed; this is indeed the role played here by the Taylor criterion.

A detailed study of the DNO is key to obtain energy estimates. We know that it is linear but depends nonlinearly on the parameterization of the free surface. This dependence is smooth, and even analytic [20, 39]. The derivative of $G(\eta)$ with respect to the surface elevation η, also called its shape derivative, is central for the study of the linearized water wave system around a reference state. Introducing the horizontal and vertical components of the fluid velocity on the free surface,

$$u = \partial_x \xi - (\partial_x \eta) v\,, \quad v = \frac{G(\eta)\xi + \partial_x \eta \cdot \partial_x \xi}{1 + |\partial_x \eta|^2}\,,$$

the derivative of $G(\eta)$ with respect to η is the operator acting on $\delta\eta = \zeta$ (see Sect. 4.3.2, [78])

$$\operatorname{grad}_\eta G(\eta)\xi \,:\, \zeta \longmapsto \operatorname{grad}_\eta G(\eta)\xi \cdot \zeta = -G(\eta)(v\,\zeta) - \partial_x \cdot (u\,\zeta)\,. \tag{4.107}$$

We refer to Alazard and Métivier [6] and Alazard et al. [2, 3] for a thorough study of the DNO with use of paradifferential calculus. Another key element to obtain a priori energy estimates is the choice of appropriate variables, referred to as the "good" unknown of Ahlinac. We formally present below the derivation of these variables and how it relates to the Taylor criterion. We refer to the above papers for a rigorous analysis. Write the water wave system (4.33)–(4.34) as

$$\partial_t U + \mathcal{F}(U) = 0\,,$$

with $U = (\eta, \xi)^\top$. Let $\underline{U} = (\underline{\eta}, \underline{\xi})^\top$ be a reference state. We denote by $\underline{u}$, $\underline{v}$ the horizontal and vertical components of the fluid velocity on the free surface associated with $\underline{U}$. The linearized water wave operator near a reference state is

$$\underline{\mathcal{L}} = \partial_t + \mathrm{grad}_{\underline{U}} \mathcal{F} .$$

From the explicit form of $\mathcal{F}(U)$ as given by (4.33)–(4.34), we get

$$\mathrm{grad}_{\underline{U}} \mathcal{F} = \begin{pmatrix} -\mathrm{grad}_{\underline{\eta}} G(\cdot)\underline{\xi} & -G(\underline{\eta}) \\ -\underline{v}\, \mathrm{grad}_{\underline{\eta}} G(\cdot)\underline{\xi} - \underline{v}\, \underline{u} \cdot \partial_x + g & \underline{u} \cdot \partial_x - \underline{v}\, G(\underline{\eta}) \end{pmatrix} .$$

Using (4.107),

$$\underline{\mathcal{L}} = \partial_t + \begin{pmatrix} -G(\underline{\eta})(\underline{v}\, \cdot) + \partial_x \cdot (\,\cdot\, \underline{u}) & -G(\underline{\eta})\, \cdot \\ \underline{v}\, G(\underline{\eta})(\underline{v}\, \cdot) + (g + \underline{v}\, \partial_x \cdot \underline{u}) & \underline{u} \cdot \partial_x \, \cdot - \underline{v}\, G(\underline{\eta})\, \cdot \end{pmatrix} .$$

We now introduce the important change of variables.

Lemma 4.6.2. *Let $\underline{U}$ be a reference state. If $U = (\eta, \xi)^\top$ satisfies the system $\underline{\mathcal{L}} U = K$, then $V := (\eta, \xi - \underline{v}\, \eta)^\top$ satisfies $\underline{\mathcal{M}} V = \underline{\mathcal{H}}$, where*

$$\underline{\mathcal{M}} := \partial_t + \begin{pmatrix} \partial_x \cdot (\cdot\, \underline{u}) & -G(\underline{\eta})\, \cdot \\ \underline{\mathfrak{a}} & \underline{u} \cdot \partial_x \end{pmatrix} , \quad \underline{\mathcal{H}} := \begin{pmatrix} K_1 \\ K_2 - \underline{v} K_1 \end{pmatrix} ,$$

where $\underline{\mathfrak{a}} := g + \partial_t \underline{v} + \underline{u} \cdot \partial_x \underline{v}$.

The coefficient $\underline{\mathfrak{a}}$ appearing in the operator $\underline{\mathcal{M}}$ identifies with the quantity defined in (4.106) in the context of Lagrangian coordinates. The condition $\underline{\mathfrak{a}} \geq c_0 > 0$ is the Taylor criterion. This condition, imposed on the subprincipal symbol of $\underline{\mathcal{M}}$, will ensure that the Cauchy problem

$$\underline{\mathcal{M}} V = \underline{\mathcal{H}} , \quad V|_{t=0} = V_0$$

is well-posed in appropriate spaces.

In [3], Alazard et al. reduced the required regularity on the initial conditions. Their local existence result involves assumptions which, in view of Sobolev embedding require that the initial free surface be only of class $C^{3/2+\epsilon}$ for some $\epsilon > 0$ and consequently, has unbounded curvature, while the initial velocity has only Lipschitz regularity. The initial free surface and the trace of the initial velocity there are assumed to be in $H^{s+1/2}(\mathbb{R}^{d-1}) \times H^s(\mathbb{R}^{d-1})$, $s > 1 + (d-1)/2$. The only assumption on the domain is that it contains a fixed strip below the free surface, allowing the bottom to be very rough in the sense that it has no regularity assumptions. Their analysis introduces new techniques and new tools, including paradifferential calculus and a microlocal description of the DNO. De Poyferré and Nguyen [46] extended these low regularity results to gravity-capillary waves. We refer to the recent review of Ionescu and Pusateri [71] for an introduction to paradifferential calculus and its use for the study of properties of the DNO.

In conclusion, the local well-posedness theory is well understood in a variety of different physical settings. Christodoulou and Lindblad [19] addressed the equations of motion for a fluid domain with a free surface, but with no body forces such as gravity coming into play. The problem is posed in arbitrary space dimensions, and the fluid, while being incompressible, is not required to satisfy any irrotationality condition. The content of this paper is a series of a priori estimates for the initial value problem, essentially under the sole hypothesis that $\mathbf{n} \cdot \nabla \mathcal{P} < 0$ on the free surface (the Taylor criterion discussed above). In addition, this paper adopts a geometrical point of view, estimating quantities such as the second fundamental form and the velocity of the free surface.

Coutand and Shkoller [22] extended local well-posedness results to three-dimensional incompressible free-surface Euler's equations with vorticity and surface tension, including two-fluid systems. Shatah and Zeng [88] derived estimates to the free-boundary value problem for Euler's equations with surface tension, and without surface tension provided the Rayleigh–Taylor sign condition holds. They proved that as surface tension tends to zero, when the Rayleigh–Taylor condition is satisfied, solutions converge to Euler's flow with zero surface tension. Iguchi [69] addressed the validity of KdV approximation in the presence of surface tension as well as effects of the bottom on this long-wave approximation. A thorough study of scaling regimes related to the shallow-water regime and their validity is given in Lannes' monograph [79]. Like in many other quasilinear problems, large initial data can lead to finite-time singularities. Existence of smooth initial data for which a singularity in the form of overturning waves (splash or splat singularity) develops in finite time was shown by Castro et al. [17] for the two-dimensional water wave problem, and by Coutand and Shkoller [23] in the three-dimensional case.

4.6.2 Recent Results on Global Well-Posedness for Small Data

In the last ten years, there has been considerable amount of work and several milestone articles on global well-posedness and long-time behavior of solutions for water waves in two and three dimensions, given small and smooth initial data. There are different frameworks depending on whether one considers the problem in the whole space or in a periodic setting. We start with the problem in the whole space where initial data are assumed to decay at infinity and the channel is assumed to have infinite depth ($h \to +\infty$). The classical mechanism to establish global regularity for quasilinear equations has two main components: establish a priori estimates of high-order energy functionals such as Sobolev norms and weighted norms, and prove dispersion and decay of the solution over time.

We first discuss the case of three-dimensional gravity waves $(g, \sigma) = (1, 0)$. Global well-posedness for small initial data was established independently by Wu [98] and by Germain et al. [56], with different assumptions on the initial conditions. In [98], the smallness condition on the surface elevation is measured based on its steepness. This does not require the wave amplitude to be small. The free surface $(x(\alpha,\beta,t), y(\alpha,\beta,t), (\alpha,\beta) \in \mathbb{R}^2)$ is parametrized in Lagrangian coordinates. The analysis is based on a key change of variables

$$\theta = (I - K)Z\,,$$

where K is the Hilbert transform associated with the fluid domain, see (4.104). The new quantity satisfies an evolution equation that does not contain quadratic terms, only nonlinear terms that are cubic and of higher order which, combined with dispersive estimates, leads to a control of the growth in time. As $t \to +\infty$, the L^∞ norm of the wave steepness, the acceleration of the free surface and the derivative of the fluid velocity there decay like t^{-1}.

The analysis in [56] provides global well-posedness and scattering. It is performed on the system (4.33)–(4.34) in Eulerian coordinates and is based on the method of space-time resonances, first introduced by those authors in the context of three-dimensional NLS equations with quadratic nonlinearities. From the canonical variables (η, ξ), introduce

$$U := \eta + i\,|D|^{1/2}\xi\,,$$

and its associated linear profile

$$u = e^{it|D|^{1/2}}U\,.$$

Theorem 4.6.3. *Assume initial conditions $\eta_0, \xi_0 : \mathbb{R}^2 \to \mathbb{R}$ are small and smooth enough such that*

$$\|U_0\|_{H^{n+1}} + \||\partial_x|^{3/4}(x\dot{U}_0)\|_{L^2} + \sup_{s\geq 0}\langle s\rangle\|e^{-is|\partial_x|}U_0\|_{W^{4,\infty}} \leq \varepsilon\,,$$

for n sufficiently large and ε sufficiently small, with $\langle s\rangle = \sqrt{1+s^2}$. Then there is a unique solution $U \in C^1([0,\infty), H^n(\mathbb{R}^2))$ to the initial value problem (4.33)–(4.34). In addition, the solution satisfies the long-time estimate

$$\langle t\rangle^{-\delta}\|U_0\|_{H^n} + \langle t\rangle^{-\delta}\||\partial_x|^{3/4}(x\dot{u})\|_{L^2} + \langle t\rangle\|U_0\|_{W^{4,\infty}}\|U(t)\|_{L^2} \lesssim \varepsilon\,,$$

for all $t \geq 0$, where δ is a small constant.

In a subsequent paper, Germain et al. [57] considered three-dimensional pure capillary waves $(g,\sigma) = (0,1)$. They proved global well-posedness and scattering under smallness conditions on the data. The case of three-dimensional gravity-capillary waves was addressed by Deng et al. [47]. A description of some of the main ideas and techniques involved in these results can be found in the review paper [71].

We turn to the two-dimensional water wave problem in the pure gravity case $(g,\sigma) = (1,0)$. To extend a local solution for longer times, one needs to use dispersive effects of the equations, which are weaker in two dimensions than in three dimensions. Global regularity for small data was proved by three groups of authors: Alazard and Delort [4, 5], Ionescu and Pusateri [70], and Ifrim and Tataru [68], with somewhat different assumptions on the initial data. The main idea behind the proof of [4] is the use of paradifferential calculus to obtain delicate L^2 and L^∞ estimates. In [70], the authors identify a suitable nonlinear logarithmic correction and show decay of solutions with modified scattering. In [68], the problem is formulated in position-velocity potential holomorphic coordinates. These time-dependent coordinates are defined by a conformal map in the lower complex half-plane. Global well-posedness of two-dimensional pure capillary waves $(g, \sigma) = (0, 1)$ was proved in [72].

4.6.3 Water Waves in a Periodic Geometry

We now consider the case of spatially periodic solutions. A major difficulty in this analysis is that, unlike many results on the real line with decaying Cauchy data, one cannot make use of dispersive properties of the linear flow. Normal forms methods have been used successfully. In Sect. 4.4, we presented the formal setting of Birkhoff normal form transformations. As mentioned earlier, in dimension $d = 2$ and for infinite depth, Dyachenko and Zakharov [51] and Craig and Wolfork [44] showed at a formal level that cubic terms and non-resonant quartic terms can be removed by appropriate canonical transformations, and four-wave resonances have a special integrable form. In [13], Berti et al. provided a rigorous setting for the reduction of these equations to Birkhoff normal form up to degree four. As a consequence, they proved that for an initial free surface in the form of a periodic perturbation to the state at rest, with size $O(\varepsilon)$ in a Sobolev space of sufficient regularity, solutions of the water wave problem remain smooth and small up to a time of order $O(\varepsilon^{-3})$. This work is an impressive tour de force where the authors overcome several major difficulties, including the presence of small divisors arising from near-resonances, which may cause loss of derivatives in addition to loss due to the form of the nonlinearity.

The case of periodic gravity-capillary waves was addressed by Berti and Delort [11]. This is a situation where for exceptional values of the physical parameters (g, σ, h), three-wave resonances associated with Wilton ripples may occur. Excluding this exceptional parameter subset of zero measure, these authors proved that any solution of the Cauchy problem for gravity-capillary waves with spatially periodic, smooth initial data of small size $O(\varepsilon)$, is almost globally defined in time on Sobolev spaces, i.e. it exists on a time interval of length ε^{-N} for any N. Furthermore, exploiting the fact that there are finitely many three-wave resonances, Berti et al. [12] proved that for all values of (g, σ, h), initial data of size $O(\varepsilon)$ in a sufficiently smooth Sobolev space lead to a solution that remains in an $O(\varepsilon)$ ball of the same Sobolev space, up to times of order $O(\varepsilon^{-2})$.

The problem of long-time existence for multi-dimensional periodic gravity-capillary waves was addressed by Ionescu and Pusateri [73]. The authors showed that for initial data of sufficiently small size $O(\varepsilon)$, smooth solutions exist up to times of order $O(\varepsilon^{-5/2})$ for almost all values of σ. This is the first result proving long-time existence in a periodic domain of dimension greater than one.

4.7 Numerical Simulation of Surface Gravity Waves

Numerical simulation has also been an important tool for research on the water wave problem. The corresponding literature is vast and the reader is referred to [48] for a recent review. As an illustration, we describe two different numerical methods for directly solving the full Euler's equations in the potential-flow formulation. The first method computes nonlinear wave solutions that are steadily progressing, with an emphasis on solitary waves on water of finite depth. It is based on an integral reformulation of the boundary value problem (4.8)–(4.11) in a moving reference frame. The second method considers the general time evolution problem by solving the Hamiltonian equations (4.33) and (4.34). We will present an application to the head-on collision of two solitary waves on shallow water, including a comparison with laboratory measurements. We refer to Constantin's monograph [21] for an extensive review on traveling water waves.

4.7.1 Tanaka's Method for Solitary Waves

In this context, the fluid domain is two-dimensional ($d = 2$) with uniform finite depth h. Solitary waves are computed by a modified version of Tanaka's method [91] as proposed by Craig et al. [28]. It is based on an integral formulation for the complex velocity potential, in a reference frame moving with wave speed c. The dimensionless crest velocity q_c (normalized by c) fully defines the wave field in this setting.

More specifically, the complex velocity potential $W = \varphi + i\psi$ such that $\varphi = 0$ at the crest and $\psi = 0$ at the flat bottom is introduced. The fluid region is mapped onto the uniform strip $0 < \psi < 1$, $-\infty < \varphi < +\infty$ of the W plane, with $\psi = 1$ corresponding to the free surface. Defining $\mathcal{Q} = \ln(dW/dz)$ where $z = x + iy$, this quantity is an analytic function of z and W, with vanishing boundary conditions at infinity. It can be expressed as $\mathcal{Q} = \ln q - i\theta$ where q is the velocity magnitude and θ is the angle between the velocity and the x-axis. Bernoulli's condition (4.10) on Γ and the no-flux condition (4.11) on Γ_b then read

$$\frac{dq^3}{d\varphi} = -\frac{3}{\mathrm{Fr}^2}\sin\theta\,, \quad \text{for} \quad \psi = 1\,, \tag{4.108}$$

and

$$\theta = 0\,, \quad \text{for} \quad \psi = 0\,, \tag{4.109}$$

respectively, with $\mathrm{Fr} = c/\sqrt{gh}$ denoting the Froude number.

The problem of finding solitary wave solutions of (4.8)–(4.11) is thus transformed into the problem of finding a complex-valued function $\mathcal{Q}$ that is analytic with respect to W within the unit strip $0 < \psi < 1$, that decays at infinity, and satisfies the two boundary conditions (4.108) and (4.109). This can be done by iteration as follows:

1. Fix an initial guess for $0 < q_c < 1$ and $\beta(\varphi) = \ln q(\varphi)$, such that $\beta(0) = \ln q_c$ and $\beta(\pm\infty) = 0$.
2. Compute the singular integral
$$- \theta(\varphi) = \mathrm{PV} \int_{-\infty}^{+\infty} \frac{\beta(\varphi')}{2\sinh\left(\frac{\pi(\varphi'-\varphi)}{2}\right)}\, d\varphi' \,, \tag{4.110}$$
for $\theta(\varphi)$.
3. Integrate
$$1 - q_c^2 = -\frac{3}{\mathrm{Fr}^2} \int_0^{+\infty} \sin\theta(\varphi)\, d\varphi \,, \tag{4.111}$$
to find Fr^2 from $\theta(\varphi)$.
4. Evaluate
$$q^3(\varphi) - q_c^3 = -\frac{3}{\mathrm{Fr}^2} \int_0^{\varphi} \sin\theta(\varphi')\, d\varphi' \,, \tag{4.112}$$
to find $q^3(\varphi)$ from $\theta(\varphi)$ and Fr^2.
5. Determine new $\beta(\varphi) = \ln q(\varphi)$.
6. Repeat steps 2–5 until convergence is achieved for Fr^2.

The wave profile and velocity potential are reconstructed from the free surface velocity. For the computation of steep solitary waves, the variable transformation
$$\varphi = \alpha\gamma + \gamma^n$$
is introduced, where α is a positive real number and n is a positive odd integer. Lagrangian interpolation and trapezoidal rule are used to evaluate numerically the integrals in (4.110)–(4.112). Typically, for $\alpha = 0.01$, $n = 5$, wave height $a_0/h = 0.4$, and a tolerance of 10^{-10} on Fr^2, it is found that 60 iterations are required to achieve convergence. Figure 4.2 illustrates several profiles of solitary waves computed by this numerical algorithm.

4.7.2 High-Order Spectral Method

Not only is the lower-dimensional formulation (4.33)–(4.34) of the full equations for nonlinear water waves, in combination with the series form (4.41) of the DNO, convenient for asymptotic modeling and rigorous mathematical analysis, it also lends itself well to direct numerical simulation by using a high-order spectral method. This constitutes a unique feature of the approach developed by Craig and Sulem [40]. Details on this numerical scheme are provided below.

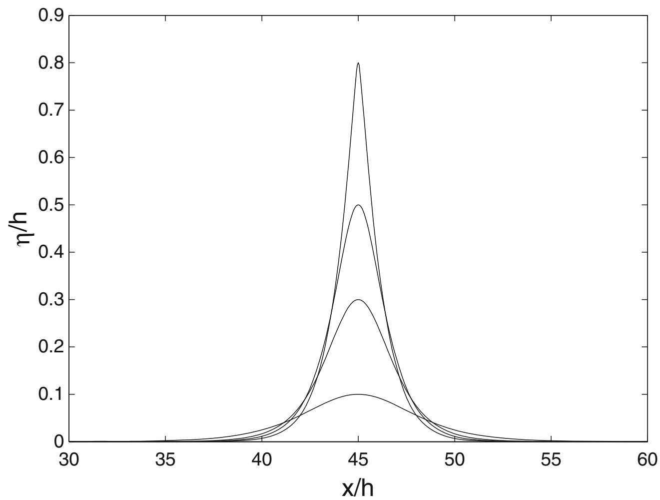

Figure 4.2: Solitary waves of height $a_0/h = 0.1,\ 0.3,\ 0.5,\ 0.8$ computed by the modified Tanaka's method

Space Discretization

Assuming periodic boundary conditions in x, a pseudo-spectral method based on the fast Fourier transform (FFT) is used for space discretization. This is a natural choice for computing the DNO since each term in (4.41) involves concatenations of Fourier multipliers with powers of η. Accordingly, both functions η and ξ are expanded in truncated Fourier series. The spatial derivatives and Fourier multipliers are evaluated in the Fourier space, while the nonlinear products are calculated in the physical space on a regular grid of N collocation points. For example, if we wish to apply the zeroth-order operator $G_0(D)$ (or any other related Fourier multiplier) to a function ξ in the physical space, we proceed in the following way

$$G_0(D)\xi = \mathcal{F}^{-1}\left\{|k|\tanh(h|k|)\xi_k\right\} ,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the direct and inverse Fourier transforms, respectively, (as computed by the FFT), and $\xi_k = \mathcal{F}(\xi)$ represents the Fourier coefficient associated with ξ.

In practice, the DNO series (4.41) is also truncated up to a finite number of terms M but, by analyticity, a small number of them (typically $M < 10 \ll N$) is sufficient to achieve fast convergence and highly accurate results [38, 60, 62, 83]. As pointed out earlier, the adjoint recursion formulas (4.45) and (4.46) for the DNO are computationally efficient because they allow us to store and reuse the G_j's as vector operations on ξ. This results in fast calculations and the computational cost for evaluating (4.41) is estimated to be $O(M^2 N \log N)$ operations via the FFT. Unlike boundary integral methods for such Laplace problems [64], there is no need to assemble and solve any dense matrix system at every time step. Moreover, the computer implementation is insensitive to the spatial dimension d and water depth h.

Time Integration

Time integration of (4.33) and (4.34) is performed in the Fourier space, so that the linear terms can be solved exactly by the integrating factor technique [40]. For this purpose, we separate the linear and nonlinear parts, and write these equations as

$$\partial_t v = \mathcal{L} v + \mathcal{N}(v)\,, \tag{4.113}$$

where the linear part $\mathcal{L}v$ is defined by

$$\mathcal{L} v = \begin{pmatrix} 0 & G_0(D) \\ -g & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix},$$

and the nonlinear part $\mathcal{N}(v)$ takes the form

$$\mathcal{N}(v) = \begin{pmatrix} \big(G(\eta) - G_0(D)\big)\xi \\ -\frac{|\partial_x \xi|^2}{2} + \frac{\left(G(\eta)\xi + \partial_x \eta \cdot \partial_x \xi\right)^2}{2(1+|\partial_x \eta|^2)} \end{pmatrix}.$$

The change of variables $v_k(t) = \Phi_k(t) w_k(t)$ in the Fourier space reduces (4.113) to

$$\partial_t w_k = \Phi_k^{-1} \mathcal{N}_k\big(\Phi_k w_k\big)\,,$$

via the integrating factor $\Phi_k(t) = \exp(t\mathcal{L}(k))$ as given by (4.99). Note that

$$\Phi_0(t) = \begin{pmatrix} 1 & 0 \\ -gt & 1 \end{pmatrix},$$

for $k = 0$. This integrating factor is a semigroup and satisfies the property $\Phi_k^{-1}(t) = \Phi_k(-t)$. It coincides with the fundamental matrix that determines the general linear solution of (4.113). The resulting system for $w_k(t)$ only contains nonlinear terms and can be solved numerically in time using various schemes such as the classical fourth-order Runge–Kutta method, Adams–Bashford/Moulton predictor-corrector method, or a symplectic Gauss–Legendre Runge–Kutta method, with constant step Δt [61, 65, 67, 99]. After converting back to v_k, this scheme reads

$$v_k^{n+1} = \Phi_k(\Delta t) v_k^n + \frac{\Delta t}{6} \Phi_k(\Delta t)\big(f_1 + 2f_2 + 2f_3 + f_4\big)\,,$$

where

$$\begin{aligned} f_1 &= \mathcal{N}_k \left(v_k^n \right) , \\ f_2 &= \Phi_k \left(-\frac{\Delta t}{2} \right) \mathcal{N}_k \left[\Phi_k \left(\frac{\Delta t}{2} \right) \left(v_k^n + \frac{\Delta t}{2} f_1 \right) \right] , \\ f_3 &= \Phi_k \left(-\frac{\Delta t}{2} \right) \mathcal{N}_k \left[\Phi_k \left(\frac{\Delta t}{2} \right) \left(v_k^n + \frac{\Delta t}{2} f_2 \right) \right] , \\ f_4 &= \Phi_k(-\Delta t) \, \mathcal{N}_k \Big[\Phi_k(\Delta t) \big(v_k^n + \Delta t \, f_3 \big) \Big] , \end{aligned}$$

for the solution at time $t_{n+1} = t_n + \Delta t$.

4.7.3 Collision of Solitary Waves

As an application, we consider the asymmetric head-on collision of two solitary waves of unequal amplitudes moving in opposite directions. Our numerical simulation is tested against a wavetank experiment carried out in the Pritchard Fluid Mechanics laboratory at Penn State University. In this experimental setup, a first localized waveform is generated by a wavemaker, reflects off of the far end of the tank, and then interacts with a second wave generated by the wavemaker. The water surface level is measured in a spatial window around the region of collision at regular intervals of time. These two waveforms are generated to be profiles of a KdV soliton (hence they are not strictly traveling wave solutions of Euler's equations but only close). In addition, the reflected wave may well deviate further from an exact solitary wave profile due to interaction with the wall, and experience a slight attenuation of amplitude due to its longer travel distance in the wavetank. Hence the interaction has a degree of asymmetry and it is not strictly between exact solitary waves. Figure 4.3 records the experimental measurements at eight times during this collision, within a window located in the middle of the wavetank. The wave moving from right to left is coming directly from the wavemaker, while the one moving from left to right has reflected from the end wall of the tank. The resulting measurements are compared with two numerically generated traces, which are superimposed on this figure. The first is a numerical simulation of (4.33)–(4.34) using the above numerical methods. Initial data for this simulation are given by KdV soliton profiles, matching those being generated by the wavemaker in the tank. The second is a linear superposition of two pure KdV solitons, centered on the two solitary-like waves present in Fig. 4.3a and adjusted to their amplitudes. Translating at constant (and opposing) velocities, they act as a reference for the amplitude and phase shift of the actual solutions that are undergoing the interaction.

Details of this collision in the experiment are relatively well represented in the numerical simulation, which in all frames predicts the measured wave profile with small error, and which reproduces the peak locations and their amplitudes

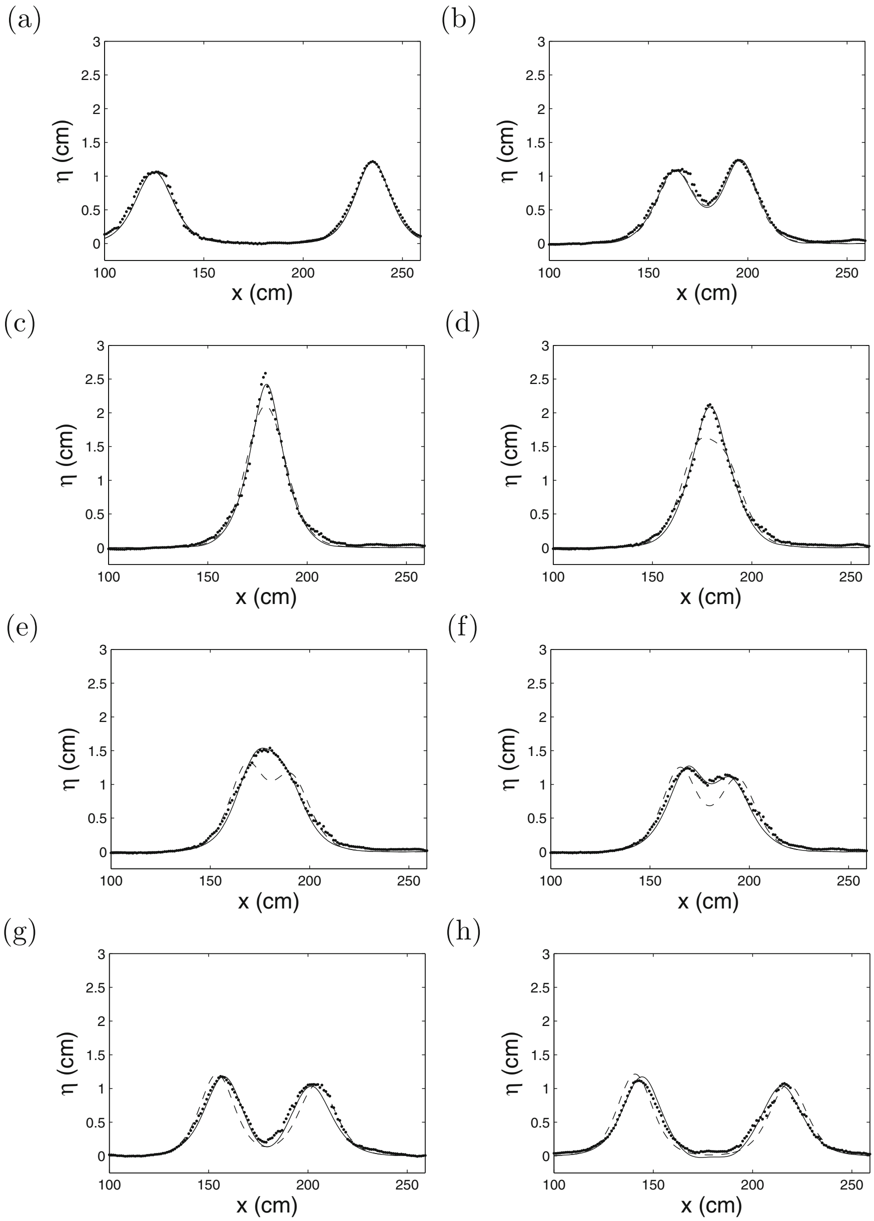

Figure 4.3: Asymmetric head-on collision of two solitary waves of height $a_1 = 1.217$ cm and $a_2 = 1.063$ cm at (**a**) $t = 18.29993$ s, (**b**) $t = 18.80067$ s, (**c**) $t = 19.05257$ s, (**d**) $t = 19.10173$ s, (**e**) $t = 19.15088$ s, (**f**) $t = 19.19389$ s, (**g**) $t = 19.32905$ s, (**h**) $t = 19.50109$ s. The water depth is $h = 5$ cm. Numerical results (solid line), experimental results (dots), sum of two KdV solitons (dashed line)

very well. Two exceptions are that the numerical solution slightly undershoots the measured wave amplitude at the point of largest run-up (Fig. 4.3c), and the peak centers in the numerical solution are slightly delayed behind the experimental measurements after the interaction (Fig. 4.3h). Both clearly differ from the superposition of KdV solitons. Some of the discrepancy between the experimental solution and numerical simulation can be attributed to the fact that neither is starting from an exact solitary wave. Furthermore, neither a trailing residual nor any changes in amplitude due to the inelastic nature of the interaction can be picked out from the experimental uncertainties of the wavetank measurements. Numerical parameters specified for this simulation are $\Delta t = 0.01$, $N = 1024$, and $M = 8$. A more detailed account of results on head-on and overtaking collisions of solitary waves can be found in [28], including a precise quantification of the residual wave and phase shift arising from the interaction.

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