

Hamiltonian Modulation Theory for Water Waves on Arbitrary Depth

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ABSTRACT

We present a consistent and systematic Hamiltonian approach to nonlinear modulation of water waves on arbitrary depth, both in two and three dimensions. It is based on a reduction of the problem to a lower-dimensional system involving surface variables alone. Using techniques from homogenization theory and Hamiltonian perturbation theory for partial differential equations, together with an expansion of the Dirichlet–Neumann operator, we derive new Hamiltonian envelope models for surface gravity waves on finite and infinite depth. In particular, we derive a Hamiltonian version of Dysthe’s equation in the deep-water case. We analyze its Benjamin–Feir stability properties and test the results against numerical simulations. For this purpose, we introduce an efficient and accurate symplectic scheme for time integration, combined with a pseudospectral method for space discretization.

KEY WORDS: Nonlinear surface waves; modulation theory; Hamiltonian systems; Dysthe equation; Benjamin–Feir instability; symplectic integrators.

INTRODUCTION

Modulation theory is a well-established method to study the long-time evolution and stability of oscillatory solutions for nonlinear dispersive evolution equations describing wave phenomena. The usual modulational Ansatz is to anticipate a weakly nonlinear monochromatic form for solutions, and to derive equations describing the evolution of their envelope. In the case of surface gravity water waves, one typically finds the nonlinear Schrödinger (NLS) equation as a canonical model for the first nontrivial contribution, and the Dysthe equation at the next order (for deep water). It has been recognized that the addition of higher-order terms provides improvements on the stability properties of finite-amplitude waves, as compared to the NLS description.

One of the standard approaches to modulation theory is a direct perturbation method involving multiple scales in space and time. Whitham (1974) developed an alternate method of averaged Lagrangians with an associated transformation theory. Another approach was proposed by Zakharov and coworkers (1985) based on a Fourier mode coupling formalism and expansion in terms of a small parameter. Very recently, Gramstad and Trulsen (2011) derived a Hamiltonian form of the mod-

ified nonlinear Schrödinger equation (with exact linear dispersion) for gravity waves on arbitrary depth, starting from Krasitskii’s version of Zakharov’s equation.

In the present paper, we present a new systematic approach to the derivation of modulation equations for water waves, based on averaged Hamiltonians. Starting from the Hamiltonian formulation of the Euler equations for water waves, this method involves an expansion of the Dirichlet–Neumann operator together with techniques from homogenization theory and Hamiltonian perturbation theory. It is closest to the methods of Stiassnie (1984) and Zakharov et al. (1985), however what differentiates it is that we are careful to retain a certain point of view, namely that scaling transformations and changes of variables are considered to be canonical transformations, and the expansion in small parameter is performed directly in the expression of the Hamiltonian. As a result, the corresponding equations of motion automatically inherit the Hamiltonian character.

In the following, we describe our method in the general multi-dimensional setting, for water waves both on finite and infinite depth. As an application, we derive a Hamiltonian version of Dysthe’s equation. Stability analysis for Stokes waves and numerical simulations are also presented to illustrate the performance of this new model.

WATER WAVES ON FINITE DEPTH

Basic Governing Equations

We consider the motion of a free surface $\eta(x, t)$ on top of a fluid domain defined by $S(\eta) = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid -h < y < \eta(x, t)\}$ where x and y denote the horizontal and vertical coordinates respectively, and $n = 2$ or 3 is the space dimension. The quiescent water level is fixed at $y = 0$ and the impermeable bottom is located at constant depth $y = -h$. We assume the fluid is incompressible and inviscid, and the flow is irrotational, so that the fluid velocity can be expressed as $u = \nabla\varphi$, where φ represents the velocity potential. Under the above assumptions, the full boundary value problem for potential flow is given by

$$\Delta\varphi = 0 \quad \text{in } S(\eta), \quad (1a)$$

$$\partial_t\eta + \partial_x\varphi \cdot \partial_x\eta - \partial_y\varphi = 0 \quad \text{at } y = \eta(x, t), \quad (1b)$$

$$\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0 \quad \text{at } y = \eta(x, t), \quad (1c)$$

$$\partial_y\varphi = 0 \quad \text{at } y = -h, \quad (1d)$$

where g is the acceleration due to gravity, and $\nabla = (\partial_x, \partial_y)^\top$.

Hamiltonian Equations

It was noted by Zakharov (1968) that (1) can be reformulated as a Hamiltonian system in terms of the canonically conjugate variables η and $\xi = \varphi(x, \eta(x, t), t)$, thus allowing for a reduction in dimension of the water wave problem, from one posed inside the entire fluid domain to one posed at the free surface alone. Subsequently, Craig and Sulem (1993) showed that this surface reformulation can be made more explicit by introducing the Dirichlet–Neumann operator (DNO)

$$G(\eta)\xi = (-\partial_x \eta, 1)^\top \cdot \nabla \varphi|_{y=\eta}, \quad (2)$$

which takes Dirichlet data ξ at the free surface, solves the Laplace equation (1a) for φ with boundary condition (1d), and returns the corresponding Neumann data (i.e. the normal fluid velocity at the free surface). While this is a linear operator in ξ , it is nonlinear with explicit nonlocal dependence on η . In terms of ξ and $G(\eta)\xi$, the equations of motion take the canonical form

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix},$$

where the 2×2 matrix J represents the symplectic structure of the system. More specifically,

$$\partial_t \eta = G(\eta)\xi, \quad (3a)$$

$$\begin{aligned} \partial_t \xi = & -g\eta - \frac{1}{2(1+|\partial_x \eta|^2)} \left[|\partial_x \xi|^2 - (G(\eta)\xi)^2 \right. \\ & - 2(\partial_x \xi \cdot \partial_x \eta)G(\eta)\xi + |\partial_x \xi|^2 |\partial_x \eta|^2 \\ & \left. - (\partial_x \xi \cdot \partial_x \eta)^2 \right], \end{aligned} \quad (3b)$$

where H is the Hamiltonian of the system, given by

$$\begin{aligned} H = & \frac{1}{2} \int \int_{-h}^{\eta} |\nabla \varphi|^2 dy dx + \frac{1}{2} \int g \eta^2 dx, \\ = & \frac{1}{2} \int \xi G(\eta)\xi dx + \frac{1}{2} \int g \eta^2 dx. \end{aligned} \quad (4)$$

Dirichlet–Neumann Operator

It has been shown that, if η is sufficiently regular, then G is an analytic function of η (Craig et al. 1997), from which it follows that G can be written in terms of a convergent Taylor series

$$G(\eta) = \sum_{j=0}^{\infty} G_j(\eta), \quad (5)$$

and each term G_j in (5) can be determined recursively (Craig and Sulem 1993). For j odd,

$$\begin{aligned} G_j = & |D_x|^{j-1} D_x \frac{\eta^j}{j!} \cdot D_x - \sum_{l=2, \text{even}}^{j-1} |D_x|^l \frac{\eta^l}{l!} G_{j-l} \\ & - \sum_{l=1, \text{odd}}^j |D_x|^{l-1} G_0 \frac{\eta^l}{l!} G_{j-l}, \end{aligned} \quad (6a)$$

and, for $j > 0$ even,

$$\begin{aligned} G_j = & |D_x|^{j-2} G_0 D_x \frac{\eta^j}{j!} \cdot D_x - \sum_{l=2, \text{even}}^j |D_x|^l \frac{\eta^l}{l!} G_{j-l} \\ & - \sum_{l=1, \text{odd}}^{j-1} |D_x|^{l-1} G_0 \frac{\eta^l}{l!} G_{j-l}, \end{aligned} \quad (6b)$$

where $G_0 = |D_x| \tanh(h|D_x|)$ and $D_x = -i\partial_x$. In the limit of an infinitely deep fluid ($h \rightarrow +\infty$), G_0 reduces to $|D_x|$.

Retaining terms of up to second order in η , i.e.

$$\begin{aligned} G_1(\eta) &= D_x \eta \cdot D_x - G_0 \eta G_0, \\ G_2(\eta) &= -\frac{1}{2} \left(|D_x|^2 \eta^2 G_0 + G_0 \eta^2 |D_x|^2 - 2G_0 \eta G_0 \eta G_0 \right), \end{aligned}$$

is sufficient for the purposes of the present study as this includes all the contributions relevant to four-wave interactions.

This formulation of the water wave problem is convenient for the modulational approach advocated here as well as in a number of other settings. This includes long-wave perturbation calculations for waves in single- and double-layer fluids (Craig et al. 2005a,b; 2011), as well as direct numerical simulations of surface water waves both on uniform and variable depth (Craig and Sulem 1993, Guyenne and Nicholls 2007, Xu and Guyenne 2009).

Canonical Transformations and Modulational Ansatz

Our Hamiltonian approach consists in making changes of variables which contain a small parameter, directly in the Hamiltonian (4), and then expanding and truncating it at a desired order. As a result, the corresponding equations of motion are automatically Hamiltonian, modulo changes in their symplectic structure as well. Details are given below.

The first step makes the canonical transformation to normal modes $(\eta, \xi) \rightarrow (z, \bar{z}, \tilde{\eta}, \tilde{\xi})$ defined by

$$\eta = \frac{1}{\sqrt{2}} a^{-1}(D_x)(z + \bar{z}) + \tilde{\eta}, \quad \tilde{\eta} = P_0 \eta, \quad (7)$$

$$\xi = \frac{1}{\sqrt{2}i} a(D_x)(z - \bar{z}) + \tilde{\xi}, \quad \tilde{\xi} = P_0 \xi, \quad (8)$$

where

$$a(D_x) = \sqrt[4]{\frac{g}{G_0}},$$

and $(\tilde{\eta}, \tilde{\xi})$ are the zeroth Fourier modes accounting for the mean flow. The symbol $\bar{\cdot}$ stands for complex conjugation, and P_0 is the projection that associates to (η, ξ) their zeroth-frequency modes. The presence of the mean fields $(\tilde{\eta}, \tilde{\xi})$ is due to the fact that $a^{-1}(0) = 0$ so that the simple change of variables $(\eta, \xi) \rightarrow (z, \bar{z})$ is not invertible. Conversely, these new variables can be expressed as $(z, \bar{z}, \tilde{\eta}, \tilde{\xi})^\top = A_1(\eta, \xi)^\top$ in terms of (η, ξ) and the 4×2 matrix

$$A_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} a(D_x)(\mathbb{I} - P_0) & ia^{-1}(D_x)(\mathbb{I} - P_0) \\ a(D_x)(\mathbb{I} - P_0) & -ia^{-1}(D_x)(\mathbb{I} - P_0) \\ \sqrt{2}P_0 & 0 \\ 0 & \sqrt{2}P_0 \end{pmatrix}.$$

The corresponding equations of motion read

$$\begin{aligned} \partial_t \begin{pmatrix} z \\ \bar{z} \\ \tilde{\eta} \\ \tilde{\xi} \end{pmatrix} &= J_1 \begin{pmatrix} \delta_z H \\ \delta_{\bar{z}} H \\ \delta_{\tilde{\eta}} H \\ \delta_{\tilde{\xi}} H \end{pmatrix}, \\ &= \begin{pmatrix} 0 & -i(\mathbb{I} - P_0) & 0 & 0 \\ i(\mathbb{I} - P_0) & 0 & 0 & 0 \\ 0 & 0 & 0 & P_0 \\ 0 & 0 & -P_0 & 0 \end{pmatrix} \begin{pmatrix} \delta_z H \\ \delta_{\bar{z}} H \\ \delta_{\tilde{\eta}} H \\ \delta_{\tilde{\xi}} H \end{pmatrix}, \end{aligned}$$

where \mathbb{I} denotes the identity operator and $J_1 = A_1 J A_1^\top$.

The next step introduces the modulational Ansatz

$$\begin{aligned} z &= \varepsilon u(X, t) e^{ik_0 \cdot x}, & \bar{z} &= \varepsilon \bar{u}(X, t) e^{-ik_0 \cdot x}, \\ \tilde{\eta} &= \varepsilon^\alpha \tilde{\eta}_1(X, t), & \tilde{\xi} &= \varepsilon^\beta \tilde{\xi}_1(X, t), \end{aligned} \quad (9)$$

meaning that we look for solutions in the form of monochromatic waves of carrier wavenumber k_0 and with slowly varying complex envelope u depending on the long spatial scale $X = \varepsilon x$. The wave amplitudes are assumed to be small as measured by the parameter $\varepsilon \sim |k_0| a_0 \sim |k_0| h \ll 1$, where a_0 is a characteristic wave amplitude. The exponents $\alpha \geq 1$ and $\beta \geq 1$ are to be determined by the subsequent asymptotic procedure. In matrix form, these new variables are given by

$$\begin{aligned} \begin{pmatrix} u \\ \bar{u} \\ \tilde{\eta}_1 \\ \tilde{\xi}_1 \end{pmatrix} &= A_2 \begin{pmatrix} z \\ \bar{z} \\ \tilde{\eta} \\ \tilde{\xi} \end{pmatrix}, \\ &= \begin{pmatrix} \varepsilon^{-1} e^{-ik_0 \cdot x} & 0 & 0 & 0 \\ 0 & \varepsilon^{-1} e^{ik_0 \cdot x} & 0 & 0 \\ 0 & 0 & \varepsilon^{-\alpha} & 0 \\ 0 & 0 & 0 & \varepsilon^{-\beta} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \\ \tilde{\eta} \\ \tilde{\xi} \end{pmatrix}, \end{aligned}$$

and the equations of motion are transformed into

$$\partial_t \begin{pmatrix} u \\ \bar{u} \\ \tilde{\eta}_1 \\ \tilde{\xi}_1 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H \\ \delta_{\bar{u}} H \\ \delta_{\tilde{\eta}_1} H \\ \delta_{\tilde{\xi}_1} H \end{pmatrix}, \quad (11)$$

with

$$\begin{aligned} J_2 &= \varepsilon^{n-1} A_2 J_1 A_2^\top, \\ &= \begin{pmatrix} 0 & -i\varepsilon^{n-3} & 0 & 0 \\ i\varepsilon^{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{n-1-\alpha-\beta} \\ 0 & 0 & -\varepsilon^{n-1-\alpha-\beta} & 0 \end{pmatrix}. \end{aligned}$$

Further details on these canonical transformations can be found in Craig et al. (2005a, 2010).

Expansion of the Hamiltonian

Inserting the successive changes of variables (7)–(8) and (9)–(10) in up to $O(\eta^2)$ contributions of the Hamiltonian, we obtain

$$H = H^{(2)} + H^{(3)} + H^{(4)} + \dots,$$

where

$$\begin{aligned} H^{(2)} &= \frac{1}{2} \int (\xi G_0 \xi + g \eta^2) dx, \\ \varepsilon^{n-3} H^{(2)} &= \int \bar{u} \omega(k_0 + \varepsilon D_X) u dx \\ &\quad + \frac{1}{2} \varepsilon^{2\beta} h \int \tilde{\xi}_1 |D_X|^2 \tilde{\xi}_1 dx + \frac{1}{2} g \varepsilon^{2\alpha-2} \int \tilde{\eta}_1^2 dx \\ &\quad + \dots, \end{aligned} \quad (12)$$

$$\begin{aligned} H^{(3)} &= \frac{1}{2} \int \xi G_1(\eta) \xi dx, \\ \varepsilon^{n-3} H^{(3)} &= \varepsilon^{\beta+1} \int (\alpha_1(k_0) |u|^2 \tilde{\eta}_1 + i |u|^2 k_0 \cdot D_X \tilde{\xi}_1) dx \\ &\quad + \varepsilon^{\beta+2} \alpha_2^j(k_0) \int (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) \tilde{\eta}_1 dx \\ &\quad + \frac{1}{2} i \varepsilon^{\beta+2} \int (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) D_{X_j} \tilde{\xi}_1 dx \\ &\quad + \dots, \end{aligned} \quad (13)$$

with

$$\begin{aligned} \alpha_1(k_0) &= \frac{a^2(k_0)}{2} (|k_0|^2 - G_0^2(k_0)), \\ \alpha_2^j(k_0) &= \frac{a^2(k_0)}{2} \left(k_{0j} + |k_0|^2 \frac{\partial_{k_j} a}{a(k_0)} \right. \\ &\quad \left. - G_0(k_0) \partial_{k_j} G_0(k_0) - G_0^2(k_0) \frac{\partial_{k_j} a}{a(k_0)} \right), \end{aligned}$$

and

$$\begin{aligned} H^{(4)} &= \frac{1}{2} \int \xi G_2(\eta) \xi dx, \\ \varepsilon^{n-3} H^{(4)} &= \varepsilon^2 \alpha_3(k_0) \int |u|^4 dx \\ &\quad + \varepsilon^3 \alpha_4^j(k_0) \int |u|^2 (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) dx \\ &\quad + \dots, \end{aligned} \quad (14)$$

with

$$\begin{aligned} \alpha_3(k_0) &= \frac{1}{4} (G_0(k_0)^2 G_0(2k_0) - |k_0|^2 G_0(k_0)), \\ \alpha_4^j(k_0) &= \frac{3}{8} G_0(k_0) |k_0|^{-2} (G_0(2k_0) G_0(k_0) - |k_0|^2) k_{0j} \\ &\quad + \frac{h}{8} (1 - \tanh^2(h|k_0|)) (2G_0(2k_0) G_0(k_0) - |k_0|^2) k_{0j} \\ &\quad + \frac{h}{2} G_0(k_0)^2 (1 - \tanh^2(2h|k_0|)) k_{0j}. \end{aligned}$$

Note that $\omega(D_X) = (gG_0)^{1/2} = \sqrt{g|D_X| \tanh(h|D_X|)}$ represents the full linear dispersion relation in the finite-depth case. Balancing mean-flow terms in $H^{(3)}$ suggested that $\alpha = \beta + 1$. In the above equations and hereafter, the Einstein summation convention is used for repeated indices ($j, l, m = \{1, \dots, n-1\}$).

The Hamiltonian is then further renormalized by subtracting multiples of the conserved wave action

$$\varepsilon^{n-3} M = \int |u|^2 dx,$$

and of the conserved impulse (or momentum)

$$\begin{aligned} I &= \int \eta \partial_x \xi dx, \\ \varepsilon^{n-3} I &= \int [k_0 |u|^2 + \frac{1}{2} \varepsilon (\bar{u} D_X u + u \overline{D_X u}) + i \varepsilon^{2\beta} \tilde{\eta}_1 D_X \tilde{\xi}_1] dx, \end{aligned}$$

and by eliminating non-resonant terms, in light of the scale separation lemma of Craig et al. (2005b) which implies that fast oscillations essentially homogenize to zero and do not contribute to the effective Hamiltonian. As a result, the renormalized form reads

$$\begin{aligned} \hat{H} &= H - \partial_k \omega(k_0) \cdot I - (\omega(k_0) - k_0 \cdot \partial_k \omega(k_0)) M, \\ \varepsilon^{n-3} \hat{H} &= \varepsilon^2 \int \left[\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \bar{u} D_{X_j X_l}^2 u + \alpha_3(k_0) |u|^4 \right. \\ &\quad + \varepsilon^{2\beta-2} \left(\frac{h}{2} \tilde{\xi}_1 |D_X|^2 \tilde{\xi}_1 + \frac{g}{2} \tilde{\eta}_1^2 + i \partial_{k_j} \omega(k_0) \tilde{\eta}_1 D_{X_j} \tilde{\xi}_1 \right) \\ &\quad + \alpha_1(k_0) \varepsilon^{\beta-1} \tilde{\eta}_1 |u|^2 + i \varepsilon^{\beta-1} |u|^2 k_0 \cdot D_X \tilde{\xi}_1 \\ &\quad + \frac{\varepsilon}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \bar{u} D_{X_j X_l X_m}^3 u \\ &\quad \left. + \varepsilon (\alpha_2^j(k_0) \varepsilon^{\beta-1} \tilde{\eta}_1 + \frac{i}{2} \varepsilon^{\beta-1} D_{X_j} \tilde{\xi}_1 + \alpha_4^j(k_0) |u|^2) \right. \\ &\quad \left. \times (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) + O(\varepsilon^2) \right] dx, \end{aligned} \quad (15)$$

after expanding $\omega(k_0 + \varepsilon D_X)$ to third order in ε . The conservation of M (and its subtraction from H) reflects the fact that our approximation of the water wave problem is phase-invariant. The subtraction of I is equivalent to saying that one transforms the system into a reference coordinate frame moving with the group velocity $\partial_k \omega(k_0)$.

Hamiltonian Fourth-Order System

Balancing mean-flow terms of orders $O(\varepsilon^{2\beta-2})$ and $O(\varepsilon^{\beta-1})$ in (15) suggests we choose $\beta = 1$ (and thus $\alpha = 2$). In terms of the slow time $\tau = \varepsilon^2 t$, the corresponding equations of motion are

$$\begin{aligned} i\partial_\tau u &= \varepsilon^{n-5} \delta_{\bar{H}} \hat{H}, \\ &= -\frac{1}{2} \partial_{k_j k_l}^2 \omega \partial_{X_j X_l}^2 u + 2\alpha_3 |u|^2 u + \left(\alpha_1 \tilde{\eta}_1 + k_0 \cdot \partial_X \tilde{\xi}_1 \right) u \\ &\quad + i\varepsilon \left[\frac{1}{6} \partial_{k_j k_l k_m}^3 \omega \partial_{X_j X_l X_m}^3 u - \alpha_2^j \left(\tilde{\eta}_1 \partial_{X_j} u + \partial_{X_j} (u \tilde{\eta}_1) \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\partial_{X_j} \tilde{\xi}_1 \partial_{X_j} u + \partial_{X_j} (u \partial_{X_j} \tilde{\xi}_1) \right) - 4\alpha_4^j |u|^2 \partial_{X_j} u \right] \\ &\quad + O(\varepsilon^2), \end{aligned} \quad (16)$$

$$\begin{aligned} \varepsilon \partial_\tau \tilde{\eta}_1 &= \varepsilon^{n-5} \delta_{\tilde{\xi}_1} \hat{H}, \\ &= h |D_X|^2 \tilde{\xi}_1 - k_0 \cdot \partial_X |u|^2 - \partial_{k_j} \omega \partial_{X_j} \tilde{\eta}_1 \\ &\quad - \frac{i}{2} \varepsilon \partial_{X_j} \left(u \partial_{X_j} \bar{u} - \bar{u} \partial_{X_j} u \right) + O(\varepsilon^2), \end{aligned} \quad (17)$$

$$\begin{aligned} \varepsilon \partial_\tau \tilde{\xi}_1 &= -\varepsilon^{n-5} \delta_{\tilde{\eta}_1} \hat{H}, \\ &= -\left(g \tilde{\eta}_1 + \alpha_1 |u|^2 + \partial_{k_j} \omega \partial_{X_j} \tilde{\xi}_1 \right) \\ &\quad - i\varepsilon \alpha_2^j \left(u \partial_{X_j} \bar{u} - \bar{u} \partial_{X_j} u \right) + O(\varepsilon^2). \end{aligned} \quad (18)$$

For notational convenience, we have dropped the dependence of the coefficients α_i and ω on k_0 . At lowest order, this system reduces to the NLS equation in the case $n = 2$, and to a Davey–Stewartson system in the case $n = 3$.

When the $O(\varepsilon^2)$ terms are neglected, system (17)–(18) may be viewed as a Hamiltonian version of the fourth-order system derived by Brinch-Nielsen and Jonsson (1986) for gravity waves on finite depth, and the corresponding Hamiltonian takes the form

$$\begin{aligned} H &= \int \left[\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j} \bar{u} \partial_{X_l} u + \alpha_3(k_0) |u|^4 \right. \\ &\quad \left. + \frac{h}{2} \tilde{\xi}_1 |D_X|^2 \tilde{\xi}_1 + \frac{g}{2} \tilde{\eta}_1^2 + \partial_{k_j} \omega(k_0) \tilde{\eta}_1 \partial_{X_j} \tilde{\xi}_1 + \alpha_1(k_0) \tilde{\eta}_1 |u|^2 \right. \\ &\quad \left. + |u|^2 k_0 \cdot \partial_X \tilde{\xi}_1 + \frac{\varepsilon}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \Im \left(\partial_{X_j} \bar{u} \partial_{X_l X_m}^2 u \right) \right. \\ &\quad \left. + 2\varepsilon \left(\alpha_2^j(k_0) \tilde{\eta}_1 + \frac{1}{2} \partial_{X_j} \tilde{\xi}_1 + \alpha_4^j(k_0) |u|^2 \right) \Im \left(\bar{u} \partial_{X_j} u \right) \right] dX, \end{aligned}$$

where \Im stands for the imaginary part.

WATER WAVES ON INFINITE DEPTH

Hamiltonian Dysthe Equation

In the infinite-depth case, it is possible to derive a simpler fourth-order system than in the finite-depth case, yielding a closed-form Hamiltonian equation for the wave envelope at the order of approximation being considered.

The fundamental technical difference with the previous situation is that $G_0 = |D_X|$. As a consequence, we have

$$\int \tilde{\xi}_1 G_0 \tilde{\xi}_1 dX = \varepsilon \int \tilde{\xi}_1 |D_X| \tilde{\xi}_1 dX,$$

together with

$$\alpha_1(k_0) = 0, \quad \alpha_3(k_0) = \frac{|k_0|^3}{4}, \quad \alpha_4^j(k_0) = \frac{3}{8} |k_0| k_{0j},$$

so the resulting renormalized Hamiltonian is

$$\begin{aligned} \varepsilon^{n-3} \hat{H} &= \varepsilon^2 \int \left[\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \bar{u} D_{X_j X_l}^2 u + \alpha_3(k_0) |u|^4 \right. \\ &\quad \left. + \frac{1}{2} \varepsilon^{2\beta-3} \tilde{\xi}_1 |D_X| \tilde{\xi}_1 + \varepsilon^{2\beta-2} \left(\frac{g}{2} \tilde{\eta}_1^2 + i \partial_{k_j} \omega(k_0) \tilde{\eta}_1 D_{X_j} \tilde{\xi}_1 \right) \right. \\ &\quad \left. + i \varepsilon^{\beta-1} |u|^2 k_0 \cdot D_X \tilde{\xi}_1 + \frac{\varepsilon}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \bar{u} D_{X_j X_l X_m}^3 u \right. \\ &\quad \left. + \varepsilon \left(\alpha_2^j(k_0) \varepsilon^{\beta-1} \tilde{\eta}_1 + \frac{i}{2} \varepsilon^{\beta-1} D_{X_j} \tilde{\xi}_1 + \alpha_4^j(k_0) |u|^2 \right) \right. \\ &\quad \left. \times \left(\bar{u} D_{X_j} u + u \overline{D_{X_j} u} \right) + O(\varepsilon^2) \right] dX. \end{aligned} \quad (19)$$

The presence of the mean-flow term of order $O(\varepsilon^{2\beta-3})$ in (19) suggests that $\beta = 2$ (and thus $\alpha = 3$). With this scaling, (19) reduces to

$$\begin{aligned} \varepsilon^{n-3} \hat{H} &= \varepsilon^2 \int \left[\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \bar{u} D_{X_j X_l}^2 u + \frac{|k_0|^3}{4} |u|^4 \right. \\ &\quad \left. + \varepsilon \left(\frac{1}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \bar{u} D_{X_j X_l X_m}^3 u + \frac{1}{2} \tilde{\xi}_1 |D_X| \tilde{\xi}_1 \right. \right. \\ &\quad \left. \left. + i |u|^2 k_0 \cdot D_X \tilde{\xi}_1 + \frac{3}{8} |k_0| k_{0j} |u|^2 \left(\bar{u} D_{X_j} u + u \overline{D_{X_j} u} \right) \right) \right. \\ &\quad \left. + O(\varepsilon^2) \right] dX, \end{aligned} \quad (20)$$

and the corresponding equations of motion read

$$\begin{aligned} i\partial_\tau u &= \frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) D_{X_j X_l}^2 u + \frac{|k_0|^3}{2} |u|^2 u \\ &\quad + \varepsilon \left(\frac{1}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) D_{X_j X_l X_m}^3 u + i k_0 \cdot u D_X \tilde{\xi}_1 \right. \\ &\quad \left. + \frac{3}{2} |k_0| |u|^2 k_0 \cdot D_X u \right) + O(\varepsilon^2), \end{aligned} \quad (21)$$

$$\varepsilon^2 \partial_\tau \tilde{\eta}_1 = |D_X| \tilde{\xi}_1 - i k_0 \cdot D_X |u|^2 + O(\varepsilon), \quad (22)$$

$$\varepsilon^2 \partial_\tau \tilde{\xi}_1 = O(\varepsilon). \quad (23)$$

Since the mean field $\tilde{\xi}_1$ only appears at order $O(\varepsilon)$ in (21), we can solve (22) for $\tilde{\xi}_1$ to leading order, yielding

$$\tilde{\xi}_1 = i |D_X|^{-1} k_0 \cdot D_X |u|^2 + O(\varepsilon), \quad (24)$$

and plug this expression in (21). The envelope equation (21) then takes the closed form

$$\begin{aligned} 2i\partial_\tau u &= -\partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j X_l}^2 u + |k_0|^3 |u|^2 u \\ &\quad + \varepsilon \left(\frac{i}{3} \partial_{k_j k_l k_m}^3 \omega(k_0) \partial_{X_j X_l X_m}^3 u - 3i |k_0| |u|^2 k_0 \cdot \partial_X u \right. \\ &\quad \left. + 2u k_{0j} k_{0l} |D_X|^{-1} \partial_{X_j X_l}^2 |u|^2 \right). \end{aligned} \quad (25)$$

We note the presence of the nonlocal operator $|D_X|^{-1} = (-\Delta)^{-1/2}$ in (25), which accounts for the mean-flow effects. It can be checked that (25) has the symplectic structure

$$\partial_\tau u = -i \delta_{\bar{H}} H,$$

with the closed-form Hamiltonian

$$\begin{aligned} H &= \frac{1}{2} \int \left[\partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j} u \partial_{X_l} \bar{u} + \frac{|k_0|^3}{2} |u|^4 \right. \\ &\quad \left. + \varepsilon \left(\frac{1}{3} \partial_{k_j k_l k_m}^3 \omega(k_0) \Im \left(\partial_{X_j} \bar{u} \partial_{X_l X_m}^2 u \right) - k_{0j} k_{0l} \left(\partial_{X_j} |u|^2 \right) |D_X|^{-1} \partial_{X_l} |u|^2 \right. \right. \\ &\quad \left. \left. + \frac{3}{2} |k_0| k_{0j} |u|^2 \Im \left(\bar{u} \partial_{X_j} u \right) \right) \right] dX, \end{aligned} \quad (26)$$

which is obtained by substituting $\tilde{\xi}_1$ with (24) in (20). Eq. (25) may be viewed as a Hamiltonian version of Dysthe's equation (1979) for gravity waves on deep water.

Relation with Existing Dysthe Equations

Introducing the new variable

$$\psi = \frac{1}{\sqrt{2}} \left(\frac{|k_0|}{g} \right)^{1/4} \left(1 + \frac{\varepsilon}{4|k_0|^2} k_0 \cdot D_X \right) u,$$

which is a first-order approximation of the free-surface envelope as given by (7), and inserting it in (25), we find up to order $O(\varepsilon)$

$$\begin{aligned} 2i\partial_\tau \psi &= -\partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j X_l}^2 \psi + 2g^{1/2} |k_0|^{5/2} |\psi|^2 \psi \\ &+ \varepsilon \left(\frac{i}{3} \partial_{k_j k_l k_m}^3 \omega(k_0) \partial_{X_j X_l X_m}^3 \psi - 6ig^{1/2} |k_0|^{1/2} |\psi|^2 k_0 \cdot \partial_X \psi \right. \\ &- ig^{1/2} |k_0|^{1/2} \psi^2 k_0 \cdot \partial_X \bar{\psi} \\ &\left. + 4g^{1/2} |k_0|^{-1/2} \psi k_{0j} k_{0l} |D_X|^{-1} \partial_{X_j X_l}^2 |\psi|^2 \right). \end{aligned} \quad (27)$$

This equation contains all the usual Dysthe terms including the additional high-order nonlinear term $\psi^2 k_0 \cdot \partial_X \bar{\psi}$ as can be seen e.g. in Dysthe (1979), Stiassnie (1984) and Lo and Mei (1985). The Hamiltonian structure is however lost in the transformation to the ψ -variable. Moreover, besides the general form of (27), we should not expect to obtain precisely the same numerical coefficients as in previous work, since the various approaches do not use precisely the same physical variables (in particular the choices of the velocity potential and of the wave envelope).

Benjamin–Feir Stability Analysis

It is of interest to investigate the Benjamin–Feir stability of a uniform wavetrain (i.e. a Stokes wave) in the framework of the Hamiltonian model (25). For this purpose, we consider the general three-dimensional case $n = 3$ such that $x = (x_1, x_2)^\top \in \mathbb{R}^2$, and we assume that k_0 is aligned in the x_1 -direction.

We first observe that (25) admits exact uniform solutions

$$u_0(\tau) = A_0 e^{-\frac{i}{2} k_0^3 A_0^2 \tau},$$

where A_0 is a real constant. Inserting in (25) a perturbed solution of the form

$$u(X, \tau) = u_0(\tau) \left[1 + B(X, \tau) \right],$$

where

$$B(X, \tau) = B_1 e^{\Omega\tau + i(\lambda X_1 + \mu X_2)} + B_2 e^{\bar{\Omega}\tau - i(\lambda X_1 + \mu X_2)},$$

is a plane-wave perturbation with sideband wavenumbers (λ, μ) and constant complex amplitudes (B_1, B_2) , and retaining linear terms in (B_1, B_2) only, we find that the condition $\Re(\Omega) \neq 0$ for instability yields

$$A_0^2 \sqrt{gk_0} \left(\frac{\lambda^2}{2} - \mu^2 \right) \left(k_0 - 2\varepsilon \frac{\lambda^2}{\sqrt{\lambda^2 + \mu^2}} \right) - \frac{g}{4k_0^3} \left(\frac{\lambda^2}{2} - \mu^2 \right)^2 > 0, \quad (28)$$

where \Re denotes the real part. Eq. (28) indicates that the linear dispersive terms as well as the nonlocal mean-flow term play an important role on the growth of sideband perturbations. In particular, the mean flow causes a small ‘Doppler shift’ relative to the carrier wavenumber k_0 , which is consistent with previous observations e.g. by Dysthe (1979). In the present mathematical framework, the mean-flow contribution in

(28) as represented by the term proportional to $(\lambda^2 + \mu^2)^{-1/2}$ is clearly identifiable, given the fact that this is the Fourier symbol of the nonlocal operator $|D_X|^{-1}$ that defines $\tilde{\xi}_1$ in (24).

Fig. 1 shows the instability region enclosed by the zero-contour level of condition (28) for $\varepsilon = 1$, $k_0 = 1$ and $A_0 = 0.05, 0.1, 0.15$. For comparison, the instability region for the NLS equation in the case $A_0 = 0.15$ is also presented. Unlike the latter for which the instability region extends to infinity, that of (25) remains bounded near the origin, which prevents energy from leaking to higher wavenumbers. The larger A_0 (or equivalently the larger ε), the larger the instability region. Overall, we observe strong similarities with results from previous work (e.g. Trulsen and Dysthe 1996, Trulsen et al. 2000). A stability analysis for Stokes waves on finite depth can be found in Gramstad and Trulsen (2011) for their Hamiltonian model.

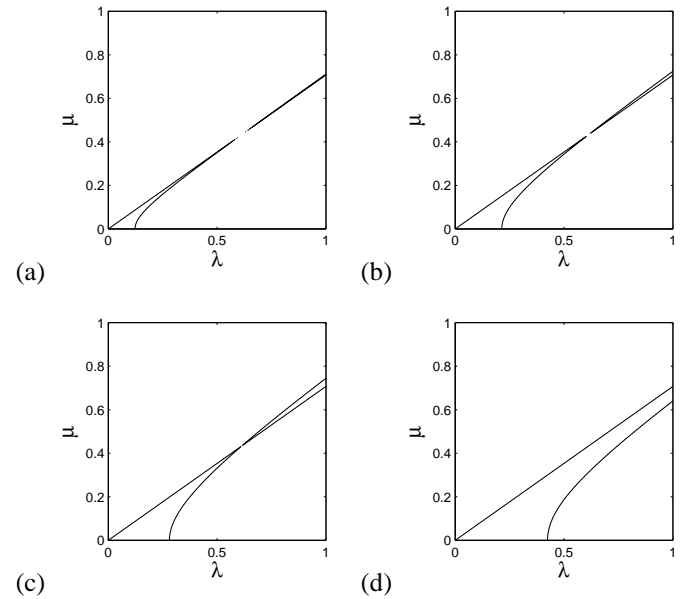


Figure 1: Instability regions for the Hamiltonian Dysthe equation (25) for (a) $A_0 = 0.05$, (b) $A_0 = 0.1$ and (c) $A_0 = 0.15$. For comparison, the instability region for the NLS equation is shown in panel (d) for $A_0 = 0.15$. The other parameters are $\varepsilon = 1$ and $k_0 = 1$.

NUMERICAL SIMULATIONS

In this section, we concentrate on the infinite-depth case and present two-dimensional ($n = 2$) numerical simulations to illustrate the performance of the Hamiltonian Dysthe equation (25) with regards to its conservative and Benjamin–Feir stability properties. For this purpose, all variables are non-dimensionalized according to Stokes wave theory in deep water, i.e. lengths are multiplied by k_0 and times multiplied by $\omega(k_0) = (gk_0)^{1/2}$ so that $g = 1$ and $k_0 = 1$. We also take this opportunity to introduce an efficient and accurate symplectic scheme for the time integration of (25), motivated by the fact that it is a Hamiltonian partial differential equation. Details are given below.

Two-dimensional Model

In the two-dimensional case ($n = 2$), Eq. (25) reduces to

$$\begin{aligned} \partial_\tau u &= -\frac{i}{8} \frac{\sqrt{g}}{k_0^{3/2}} \partial_X^2 u - \frac{i}{2} k_0^3 |u|^2 u + \frac{\varepsilon}{16} \frac{\sqrt{g}}{k_0^{5/2}} \partial_X^3 u \\ &\quad - \frac{3}{2} \varepsilon k_0^2 |u|^2 \partial_X u + i \varepsilon k_0^2 |D_X| |u|^2, \end{aligned} \quad (29)$$

whose Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2} \int \left[\frac{k_0^3}{2} |u|^4 - \frac{1}{4} \frac{\sqrt{g}}{k_0^{3/2}} |\partial_X u|^2 + \frac{\varepsilon}{8} \frac{\sqrt{g}}{k_0^{5/2}} \Im \left(\partial_X \bar{u} \partial_X^2 u \right) \right. \\ &\quad \left. + \frac{3}{2} \varepsilon k_0^2 |u|^2 \Im \left(\bar{u} \partial_X u \right) - \varepsilon k_0^2 |u|^2 |D_X| |u|^2 \right] dX. \end{aligned} \quad (30)$$

Note that $X = X_1 \in \mathbb{R}$ and $k_0 > 0$.

Numerical Methods

We assume periodic boundary conditions in the X -direction, and use a pseudospectral method for space discretization. More specifically, the envelope u is represented by a truncated Fourier series with N modes (which also corresponds to the number of grid points in the physical X -space). Applications of spatial derivatives and nonlocal operators are performed in Fourier space, while nonlinear products are evaluated in physical space at a discrete set of equally spaced points. For example, application of the nonlocal operator $|D_X|$ in physical space is equivalent to multiplication by $|\lambda|$ in Fourier space. All operations are executed efficiently using the fast Fourier transform \mathcal{F} , and aliasing errors are removed by zero-padding in Fourier space.

Time integration of (29) is carried out in Fourier space, which allows the linear terms to be solved exactly by the integrating factor technique. The nonlinear terms are integrated in time using a symplectic fourth-order (2-stage) Gauss–Legendre Runge–Kutta scheme (Xu and Guyenne 2009). Applied to (29), this scheme reads

$$\begin{aligned} \hat{u}^{n+1} &= e^{\mathcal{L} \Delta \tau} \hat{u}^n + \Delta \tau e^{\mathcal{L} \Delta \tau} \left[b_1 e^{-c_1 \mathcal{L} \Delta \tau} \mathcal{N} \left(e^{c_1 \mathcal{L} \Delta \tau} \hat{u}^{(1)} \right) \right. \\ &\quad \left. + b_2 e^{-c_2 \mathcal{L} \Delta \tau} \mathcal{N} \left(e^{c_2 \mathcal{L} \Delta \tau} \hat{u}^{(2)} \right) \right], \end{aligned} \quad (31)$$

$$\begin{aligned} \hat{u}^{(1)} &= \hat{u}^n + \Delta \tau a_{11} e^{-c_1 \mathcal{L} \Delta \tau} \mathcal{N} \left(e^{c_1 \mathcal{L} \Delta \tau} \hat{u}^{(1)} \right) \\ &\quad + \Delta \tau a_{12} e^{-c_2 \mathcal{L} \Delta \tau} \mathcal{N} \left(e^{c_2 \mathcal{L} \Delta \tau} \hat{u}^{(2)} \right), \end{aligned} \quad (32)$$

$$\begin{aligned} \hat{u}^{(2)} &= \hat{u}^n + \Delta \tau a_{21} e^{-c_1 \mathcal{L} \Delta \tau} \mathcal{N} \left(e^{c_1 \mathcal{L} \Delta \tau} \hat{u}^{(1)} \right) \\ &\quad + \Delta \tau a_{22} e^{-c_2 \mathcal{L} \Delta \tau} \mathcal{N} \left(e^{c_2 \mathcal{L} \Delta \tau} \hat{u}^{(2)} \right), \end{aligned} \quad (33)$$

for the solution $\hat{u}^{n+1} = \mathcal{F}(u)^{n+1}$ at time $\tau_{n+1} = \tau_n + \Delta \tau$, where $\Delta \tau$ is the constant time step. The values of the coefficients are

$$a_{11} = a_{22} = \frac{1}{4}, \quad a_{12} = \frac{1}{4} + \frac{\sqrt{3}}{6}, \quad a_{21} = \frac{1}{4} - \frac{\sqrt{3}}{6},$$

$$b_1 = b_2 = \frac{1}{2}, \quad c_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}, \quad c_2 = \frac{1}{2} - \frac{\sqrt{3}}{6}.$$

The operator

$$\mathcal{L} = \frac{i}{8} \frac{\sqrt{g}}{k_0^{3/2}} \lambda^2 - i \frac{\varepsilon}{16} \frac{\sqrt{g}}{k_0^{5/2}} \lambda^3,$$

denotes the Fourier multiplier associated with the linear terms in (29), while \mathcal{N} contains the nonlinear terms. At each time step, the nonlinear system (32)–(33) for $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ is solved by fixed point iteration. We have typically observed in our applications that 3 iterations are needed to achieve convergence given a tolerance of 10^{-8} on the relative error.

Numerical Results

Having the test on Benjamin–Feir instability in mind, we specify as initial condition

$$u(X, 0) = A_0 \left[1 + a_p \cos(\lambda_p X) \right].$$

For the purpose of our numerical illustrations, we restrict our attention to the case $A_0 = 0.15$, $\varepsilon = 1$, $a_p = 0.01$ and $\lambda_p = 0.2$. The value $\lambda_p = 0.2$ corresponds to the most unstable disturbance as indicated in Fig. 2. Note that, because of our choice of non-dimensionalization, choosing $A_0 \ll 1$ while fixing $\varepsilon = 1$ is similar to choosing $\varepsilon \ll 1$ while fixing $A_0 = 1$.

Fig. 3 shows the time evolution of the relative errors

$$\frac{\Delta M}{M_0} = \left| \frac{M - M_0}{M_0} \right|, \quad \frac{\Delta H}{H_0} = \left| \frac{H - H_0}{H_0} \right|,$$

on wave action M and Hamiltonian H respectively, up to $\tau = 2500$ (M_0 and H_0 are the initial values at $\tau = 0$). We used a domain length $L = 20\pi$, spatial resolution $N = 256$ and time step $\Delta \tau = 10^{-3}$. Overall, both M and H are very well conserved, although their errors tend to gradually grow in time. This is likely due to the development of the Benjamin–Feir instability combined with the accumulation of numerical errors.

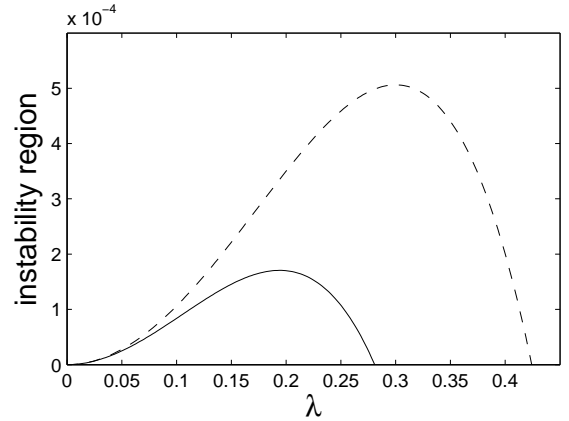


Figure 2: Instability region as given by condition (28) for the Hamiltonian Dysthe equation (29) for $A_0 = 0.15$ and $\varepsilon = 1$ (solid line). For comparison, the instability region for the NLS equation is also presented (dashed line).

Fig. 4 depicts the time evolution of the normalized amplitudes for the fundamental and sidebands, $|\hat{u}(0)|$ and $|\hat{u}(\pm \lambda_p)|$. The general pattern is somewhat irregular, although we discern some recurring exchange of energy between the fundamental and sidebands, with the lower sideband being more excited than the upper one. This moderately irregular behavior may be attributed to the triggering of higher-order instabilities by the Benjamin–Feir instability, and to their subsequent mutual interaction, as suggested by Su and Green (1984). We have also run computations with smaller values of A_0 which show nearer-recurring behaviors but extending over longer periods of time.

Finally, Fig. 5 presents snapshots of the envelope magnitude $|u|$ together with the corresponding (rescaled) free-surface elevation $\varepsilon^{-1} \eta$. Using the transformation (7), the latter quantity can be easily computed

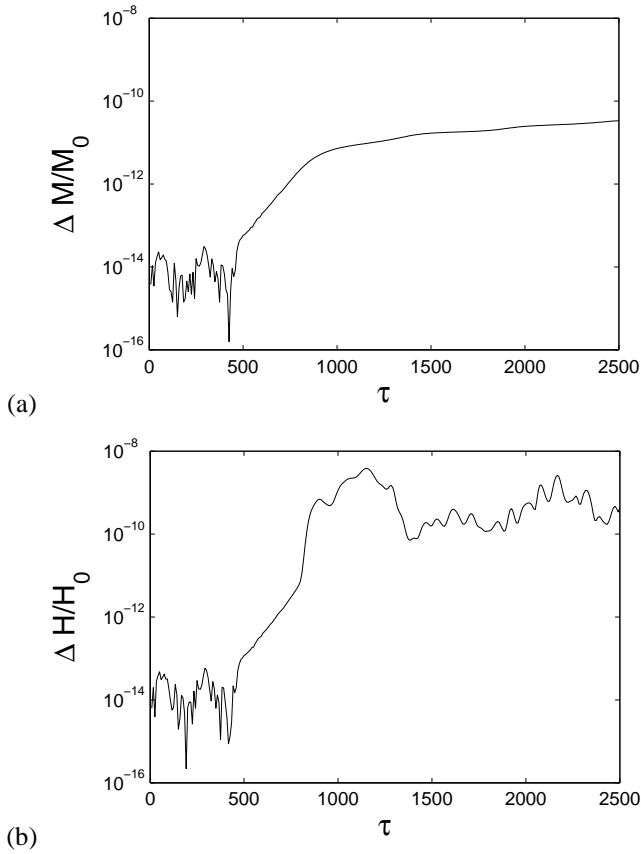


Figure 3: Relative errors on (a) wave action M and (b) Hamiltonian H as a function of time for $A_0 = 0.15$, $\varepsilon = 1$, $a_p = 0.01$ and $\lambda_p = 0.2$.

as

$$\eta(X, \tau) = \frac{\varepsilon}{\sqrt{2}} \left[\mathcal{F}^{-1} \left(\sqrt{\frac{|k_0 + \varepsilon\lambda|}{g}} \widehat{u} \right) e^{ik_0 X/\varepsilon} + \mathcal{F}^{-1} \left(\sqrt{\frac{|k_0 + \varepsilon\lambda|}{g}} \widehat{\bar{u}} \right) e^{-ik_0 X/\varepsilon} \right].$$

This expression neglects the mean field $\tilde{\eta}$ which does not contribute at the order of approximation considered here, but it includes exactly all the contributions from the higher sidebands. We see that the solution develops strong amplitude modulations as a result of the Benjamin–Feir instability. The development of the left-right asymmetry in the profile of $|u|$ is consistent with the asymmetric evolution observed between the lower and upper sidebands in Fig. 4. Moreover, the occurrence of smaller-scale oscillations in the profile of $|u|$ seems to support our previous observation from Fig. 4 that higher-order instabilities also come into play, triggered by the Benjamin–Feir instability. It is also interesting to note that, in general, the profile of $|u|$ does not exactly coincide with the actual shape of the free-surface elevation. This is explained in part by the fact that the relation (7) between u and η , as defined in our Hamiltonian approach, is not a simple relation of proportionality.

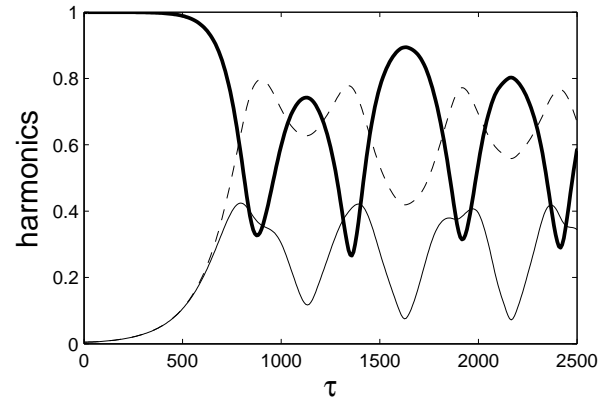


Figure 4: Normalized harmonics as a function of time for $A_0 = 0.15$, $\varepsilon = 1$, $a_p = 0.01$ and $\lambda_p = 0.2$. Fundamental $|\widehat{u}(0)|$ (thick solid line). Lower sideband $|\widehat{u}(-\lambda_p)|$ (dashed line). Upper sideband $|\widehat{u}(\lambda_p)|$ (thin solid line).

CONCLUSIONS

We have presented a systematic Hamiltonian approach to deriving envelope model equations for surface gravity waves on arbitrary depth, both in two and three dimensions. It is based on a surface reformulation of the water wave problem, and involves an expansion of the Dirichlet–Neumann operator which is performed directly in the expression of the Hamiltonian, together with transformations of the symplectic structure of the system. As an application, we have derived a Hamiltonian version of Dysthe’s equation in the deep-water case. We have then analyzed its Benjamin–Feir stability properties, and performed numerical simulations to illustrate its performance.

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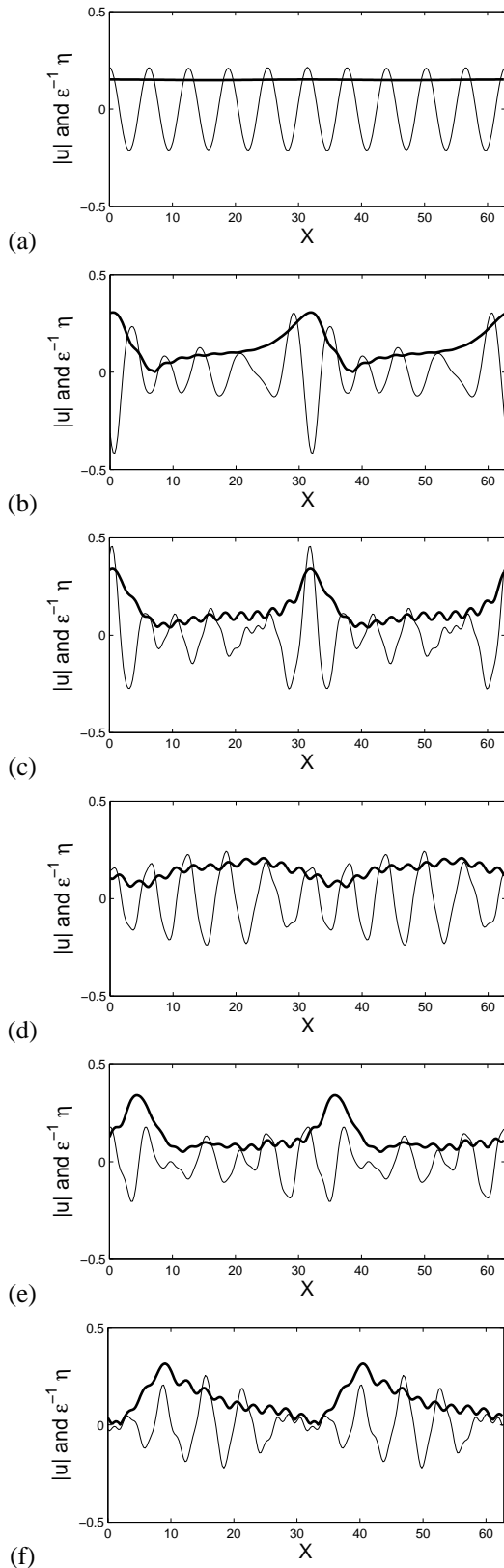


Figure 5: Snapshots of the envelope magnitude $|u|$ (thick solid line) and free-surface elevation $\varepsilon^{-1}\eta$ (thin solid line) at (a) $\tau = 0$, (b) 799.967, (c) 1324.946, (d) 1658.266, (e) 2408.236 and (f) 2500, for $A_0 = 0.15$, $\varepsilon = 1$, $a_p = 0.01$ and $\lambda_p = 0.2$.

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