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Turbulence of one-dimensional weakly nonlinear dispersive waves

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ABSTRACT. The turbulence of weakly nonlinear dispersive waves is studied by numerically integrating a three-parameter one-dimensional model equation. In particular the validity of weak turbulence theory is assessed. The predicted power-law solutions are explicitly determined and then compared with the numerical results. For both signs of nonlinearity, it is shown that the weakly turbulent regime is strongly influenced by the presence of coherent structures. These are wave collapses and quasisolitons.

1. Introduction

The weak turbulence theory developed by Zakharov [**8**] is a tool for obtaining the shape of frequency spectra in problems dealing with weakly nonlinear dispersive waves. The applications of this theory range from water waves in hydrodynamics to ion-acoustic waves in plasma physics. The weak turbulence theory is based on a hamiltonian formulation of the problem where only resonant interactions between weakly nonlinear waves are taken into account. It is then possible to derive approximate equations by performing perturbation expansions in terms of the nonlinearity parameter. Although the theory was developed more than thirty years ago, few proofs, either experimental or numerical, have been given to assess its validity (e.g. [**7**]). Recently, Majda et al. [**5**] proposed a one-dimensional model equation as a basis to check the validity of weak turbulence theory. Numerical computations on this model have been reported in [**1**], [**3**], [**5**] and [**9**]. In this paper we summarize the most important numerical results on this equation, which depends on three parameters, and show that the weakly turbulent regime is strongly influenced by the presence of coherent structures, namely wave collapses and quasisolitons.

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2. One-dimensional model equation

The following three-parameter nonlinear dispersive equation was proposed by Majda et al. [5]:

$$\mathrm{i}\,\frac{\partial \hat{\psi}_k}{\partial t} = \omega_k \hat{\psi}_k + \int T_{123k}\, \hat{\psi}_1 \hat{\psi}_2 \hat{\psi}_3^* \, \delta(k_1 + k_2 - k_3 - k)\, dk_1 dk_2 dk_3\,. \tag{2.1}$$

In equation (2.1), which has been written in Fourier space, $\hat{\psi}_k$ denotes the k-th component in the Fourier decomposition of the complex wave field $\psi(x,t)$ and $(*)$ stands for complex conjugation. Equation (2.1) depends on three parameters. The first parameter, α, is related to the linear frequency $\omega_k = |k|^\alpha$. The second parameter, β, is related to the interaction coefficient

$$T_{123k} = \lambda\, |k_1 k_2 k_3 k|^{\beta/4}\,. \tag{2.2}$$

The third parameter, λ, which also appears in the interaction coefficient (2.2) and is equal to ± 1, governs the balance between dispersive and nonlinear effects. One can use the terminology *focusing* for $\lambda = -1$ and *defocusing* for $\lambda = +1$. The system possesses two important first integrals, the Hamiltonian

$$H = \int \omega_k |\hat{\psi}_k|^2 dk + \frac{1}{2}\int T_{123k} \hat{\psi}_1 \hat{\psi}_2 \hat{\psi}_3^* \hat{\psi}_k^* \, \delta(k_1 + k_2 - k_3 - k)\, dk_1 dk_2 dk_3 dk$$

and the wave action (or number of particles)

$$N = \int |\hat{\psi}_k|^2 dk\,.$$

Equation (2.1) describes four-wave resonant interactions satisfying

$$k_1 + k_2 = k_3 + k \tag{2.3}$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_k\,. \tag{2.4}$$

It can be shown that when $\alpha < 1$ the system (2.3)-(2.4) has nontrivial solutions and that dominant interactions occur between four waves. In all computations the parameter α has been set equal to $1/2$. This case mimics gravity waves in deep water, whose dispersion relation is given by $\omega_k = (gk)^{1/2}$, where g is the acceleration due to gravity. Computations for $\lambda = +1$ were performed by Majda et al. [5]. Computations for $\lambda = \pm 1$ were recently performed by Cai et al. [1] and by Zakharov et al. [9].

3. Kolmogorov-type spectra

For a weak nonlinearity, Zakharov's theory [10] leads to a kinetic equation for the two-point correlation function $n_k = \langle |\hat{\psi}_k|^2 \rangle$:

$$\begin{aligned}\frac{\partial n_k}{\partial t} &= 4\pi \int |T_{123k}|^2 \,(n_1 n_2 n_3 + n_1 n_2 n_k - n_1 n_3 n_k - n_2 n_3 n_k) \\ &\quad \times\ \delta(\omega_1 + \omega_2 - \omega_3 - \omega_k)\, \delta(k_1 + k_2 - k_3 - k)\, dk_1 dk_2 dk_3\,.\end{aligned}$$

The two main hypotheses for deriving the kinetic equation are the assumptions of gaussianity and of random phases. The stationary Kolmogorov-type solutions are given by

$$n_k = a_1\, |Q|^{1/3}\, k^{-2\beta/3 - 1 + \alpha/3} \tag{3.1}$$

$$n_k = a_2\, |P|^{1/3}\, k^{-2\beta/3 - 1} \tag{3.2}$$

TABLE 1. Slope and flux sign for the Kolmogorov-type solutions (3.1)-(3.2). The dispersion parameter α is equal to 1/2.

β	-1	$-3/4$	$-1/2$	$-1/4$	0	$+3$
power of k in (3.1)	$-1/6$	$-1/3$	$-1/2$	$-2/3$	$-5/6$	$-17/6$
sign of Q	$+$	$+$	0	$-$	$-$	$-$
power of k in (3.2)	$-1/3$	$-1/2$	$-2/3$	$-5/6$	-1	-3
sign of P	$-$	0	$+$	$+$	$+$	$+$

and are associated respectively with a particle flux Q and an energy flux P. The coefficients a_1 and a_2 denote the dimensionless Kolmogorov constants. It is important to emphasize that these solutions do not depend on the sign of nonlinearity λ. Such solutions can be written for all values of β and $\alpha < 1$. But there is a physical argument which plays a crucial role in deciding the realizability of the Kolmogorov-type spectra. Suppose that pumping is performed at some frequencies ω_k around ω_f and that damping operates at frequencies ω_k near zero as well as at frequencies ω_k much larger than ω_f. Weak turbulence theory then states that the energy is expected to flow from ω_f to higher ω_k's (direct cascade with $P > 0$) while the particles mainly head for lower ω_k's (inverse cascade with $Q < 0$). Accordingly, we need to evaluate the fluxes in order to select, among the rich family of power laws (3.1) and (3.2), those which are likely to result from numerical simulations of equation (2.1) with damping and forcing. Only the cases

$$\beta < -3/2 \quad \text{and} \quad \beta > 2\alpha - 3/2$$

i.e.

$$\beta < -3/2 \quad \text{and} \quad \beta > -1/2 \quad \text{if} \quad \alpha = 1/2$$

are relevant because they correspond to a particle flux towards large scales ($Q < 0$) and to an energy flux towards small scales ($P > 0$). The signs of the fluxes are shown in Table 1 for $\alpha = 1/2$ [**9**]. Computations are performed in the range $\beta > -1/2$, which includes the case of simple cubic nonlinearity ($\beta = 0$) and the case of gravity waves ($\beta = 3$).

4. Solitons, collapses and quasisolitons

The numerical results presented below show that the weakly turbulent regime is strongly influenced by the presence of coherent structures. These are solitons, quasisolitons or collapses. The existence of solitons depends on the parameter λ. Looking for soliton solutions of (2.1) of the form

$$\hat{\psi}_k(t) = e^{i(\Omega - kV)t}\hat{\phi}_k$$

with Ω and V constant leads to

$$\hat{\phi}_k = -\frac{1}{\Omega - kV + \omega_k}\int T_{123k}\,\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3^*\,\delta(k_1 + k_2 - k_3 - k)\,dk_1dk_2dk_3\,. \tag{4.1}$$

For $\alpha < 1$, the condition $\Omega - kV + |k|^\alpha \neq 0, \forall k \in \mathbb{R}$, implies that the propagating speed V is zero. Rewriting equation (4.1) in variational form:

$$\delta(H + \Omega N) = 0$$

one can conclude that 'stationary' solitons can exist only if $\lambda = -1$. In that case an equilibrium between nonlinear and dispersive effects is possible. As for nonlinear Schrödinger-type equations, the linear stability criterion for solitons is given by $\partial N/\partial\Omega > 0$ [**4**]. In our case this gives

$$\beta < \alpha - 1$$

i.e.

$$\beta < -1/2 \quad \text{if} \quad \alpha = 1/2\,.$$

Therefore the solitons are unstable in the regime of interest.

In view of this result, it is natural to look at the formation of collapses. They are typically described by self-similar solutions of the form

$$\hat{\psi}_k(t) = (t_0 - t)^{p+i\epsilon}\,\chi(\xi)$$

where

$$\xi = k\,(t_0 - t)^{1/\alpha}, \quad p = \frac{\beta - \alpha + 2}{2\alpha}, \quad \epsilon = \text{arbitrary constant}\,.$$

An analysis of the convergence of the Hamiltonian and of the wave action integral as $t \to t_0$ shows that necessary conditions for collapses to exist when $\alpha = 1/2$ are $\beta > -1/2$ for $\lambda = -1$, which coincides with the soliton instability criterion, and $\beta > 0$ for $\lambda = +1$. In spectral space, the self-similar solution behaves at $t = t_0$ like

$$n_k \simeq k^{-\beta+\alpha-2} \tag{4.2}$$

which is analogous to Phillips spectrum for deep water gravity waves [**6**].

In the case $\lambda = +1$, quasisolitons can exist. These are approximate solutions of equation (4.1) which look like envelope solitons. In the limit of a narrow spectrum centered at $k = k_m$, such as $\Omega - k_m V + k_m^\alpha \neq 0$, these quasisolitons are given by

$$\psi(x,t) \simeq \phi(x - Vt)\,e^{i\Omega t + ik_m(x-Vt)} \tag{4.3}$$

with ϕ, Ω and V given by

$$\phi(\xi) = \sqrt{\frac{\alpha(1-\alpha)}{k_m^{\beta-\alpha+2}}}\,\frac{\kappa}{\cosh(\kappa\,\xi)}\,, \qquad \kappa = |k - k_m| \ll k_m$$

$$\Omega = -(1-\alpha)\,k_m^\alpha - \frac{1}{2}\alpha\,(1-\alpha)\,k_m^{\alpha-2}\kappa^2\,, \qquad V = \alpha\,k_m^{\alpha-1}\,.$$

When κ/k_m is small, the quasisolitons look almost like true solitons and can persist for a long time. They can play an important role in weak turbulence. When κ/k_m is large, the quasisolitons can become unstable and develop into wave collapse.

5. Numerical results

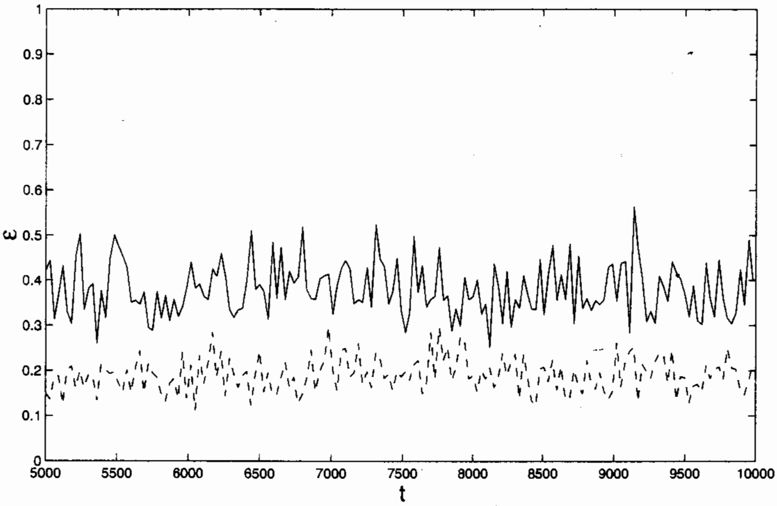
The numerical computations are performed by adding to equation (2.1) a source term in a narrow spectral band as well as a damping term containing a wave action sink at large scales and an energy sink at small scales:

$$\begin{aligned} \mathrm{i}\,\frac{\partial\hat{\psi}_k}{\partial t} \;&=\; \omega_k\hat{\psi}_k + \int T_{123k}\,\hat{\psi}_1\hat{\psi}_2\hat{\psi}_3^*\,\delta(k_1 + k_2 - k_3 - k)\,dk_1dk_2dk_3 \\ &\quad + \mathrm{i}\,(F_k + D_k)\,\hat{\psi}_k \end{aligned} \tag{5.1}$$

with

$$F_k = \sum_j f_j\delta(k - k_j) \quad \text{and} \quad D_k = -\nu^-\,|k|^{-d^-} - \nu^+\,|k|^{d^+}\,.$$

FIGURE 1. Level of nonlinearity as a function of time. The parameters are $\alpha = 1/2, \beta = 0$ and $\lambda = +1$ (solid line); $\lambda = -1$ (dashed line).



A pseudospectral code with 2048 modes is used to integrate equation (5.1). Details can be found in [**9**].

5.1. Numerical results for $\beta = 0, \lambda = \pm 1$. The study is restricted to the direct cascade. Typical initial conditions are given by random noise. Simulations are run until a quasi-steady regime is established which is characterized by small fluctuations of the energy and the number of particles around some mean value. Then time averaging begins and continues for a length of time which significantly exceeds the characteristic time scale of the slowest harmonic from the inertial range (free of the source and the sink). In turn, the time-step of the integration has to provide, at least, accurate enough resolution of the fastest harmonic in the system. As our experiments show, one has to use an even smaller time-step than defined by the last condition: the presence of fast nonlinear events in the system requires the use of a time-step $\Delta t = 0.005$, which is 40 times smaller than the smallest linear frequency period. Time averaging with such a small time step leads to a computationally time-consuming procedure despite the one-dimensionality of the problem. Figure 1 shows the time evolution of the average nonlinearity ϵ, which is defined as the ratio of the nonlinear part to the linear part of the Hamiltonian, each part being calculated over the whole field. Of course, this definition does not really make sense when external forces are applied but it provides a relatively good estimation of the level of nonlinearity once the system reaches the steady state. The mean values of ϵ are 0.4 when $\lambda = +1$ and 0.2 when $\lambda = -1$. They are relatively small. Thus, the condition of small nonlinearity required by the theory holds for both systems. However the theory cannot explain the difference in the values of ϵ, since the same forcing is imposed in both systems.

The difference between the focusing and the defocusing cases is even more obvious when one looks at the dissipation rates of particles and quadratic energy

TABLE 2. $\alpha = 1/2, \beta = 0$. Time-averaged values of the wave action, quadratic energy and corresponding fluxes in the stationary state.

λ	N	E	Q^-	Q^+	P^-	P^+
+1	3	19	0.1957	0.0090	0.276	0.258
−1	1	9	0.0098	0.0478	0.014	1.430

for small wavenumbers:

$$Q^- = 2\int_{k<k_f} \nu^- |k|^{-d^-} |\hat{\psi}_k|^2\, dk\,, \quad P^- = 2\int_{k<k_f} \nu^- |k|^{-d^-} \omega_k |\hat{\psi}_k|^2\, dk$$

and for large wavenumbers

$$Q^+ = 2\int_{k>k_f} \nu^+ |k|^{d^+} |\hat{\psi}_k|^2\, dk\,, \quad P^+ = 2\int_{k>k_f} \nu^+ |k|^{d^+} \omega_k |\hat{\psi}_k|^2\, dk$$

where k_f is the characteristic wavenumber of forcing. Their time-averaged values in the stationary state are collected in Table 2.

The stationary isotropic spectra of turbulence are displayed in Figures 2 and 3. Again the results depend on the value of λ. For both cases the theoretical spectrum provides a higher level of turbulence than the observed one. In the focusing case ($\lambda = -1$) this difference is almost of one order of magnitude but the slope fits the predicted value -1 well. For $\lambda = +1$, the observed spectrum almost coincides with the weak turbulence one at low frequencies and then decays faster at higher wavenumbers. In this range, the slope is close to $-5/4$ as found in [**5**]. Note that a new derivation of the Majda et al.'s spectrum is proposed in [**9**].

Comparison of the turbulence levels and fluxes of particles Q^+ for both signs of nonlinearity leads to a paradoxal result. At $\lambda = -1$ the total number of particles is three times less than at $\lambda = +1$, while the dissipation rate of particles is higher by one order of magnitude. It can be explained only by the presence in this case of a much more powerful mechanism of nonlinear interactions, which provides very fast wave particles transport to high frequencies. In our opinion, this mechanism is wave collapse. Sporadic collapsing events developing on top of the weak turbulence background could send most of particles to high wavenumbers without violation of energy conservation, because in each self-similar collapse structure the amount of total energy is zero. Such a collapsing event is shown in Figure 4. Note that the contribution of collapses to the high-frequency spectrum is weak because they produce a Phillips-type spectrum which decays very fast as $k \to +\infty$. In our case, equation (4.2) becomes

$$n_k \simeq k^{-3/2}\,.$$

Hence, only the weakly turbulent component k^{-1} survives at large wavenumbers. The coexistence of wave collapse and weak turbulence was also observed in [**2**] for the nonlinear Schrödinger equation.

At $\lambda = +1$ the picture of turbulence matches the weak turbulence prediction both quantitatively and qualitatively. Meanwhile, the spectrum at high k's is steeper than the theoretical one. So far we cannot give a consistent explanation of this fact. We can just guess that it is somehow connected with quasisolitons.

FIGURE 2. $\beta = 0, \lambda = -1$. Stationary and isotropic spectra n_k vs. wavenumber. We compare the computed spectrum with the predicted one of Kolmogorov-type $n_k = c\,k^{-1}$ with $c = a_2 P^{1/3}$ (straight line).

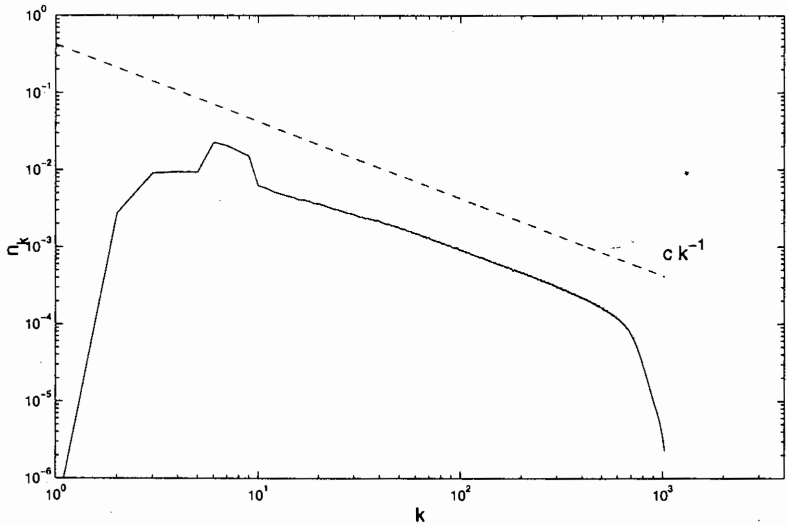


FIGURE 3. $\beta = 0, \lambda = +1$. Stationary and isotropic spectra n_k vs. wavenumber. We compare the computed spectrum with the predicted one of Kolmogorov-type $n_k = c\,k^{-1}$ with $c = a_2 P^{1/3}$ (straight line).

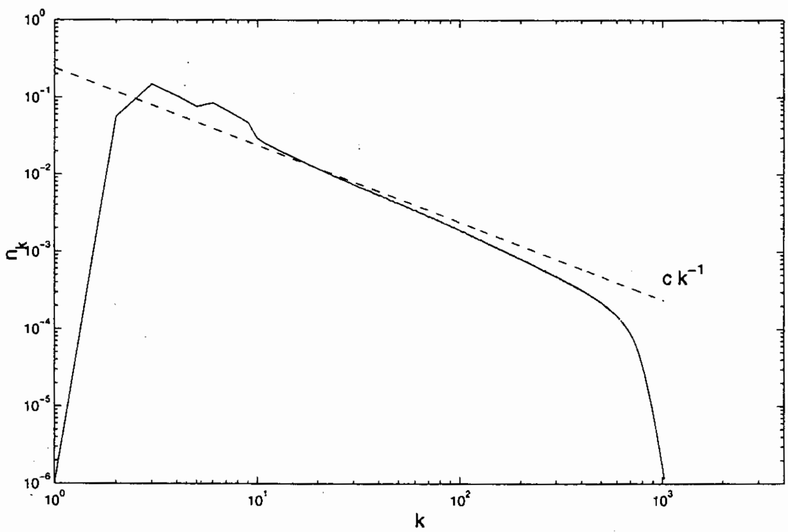
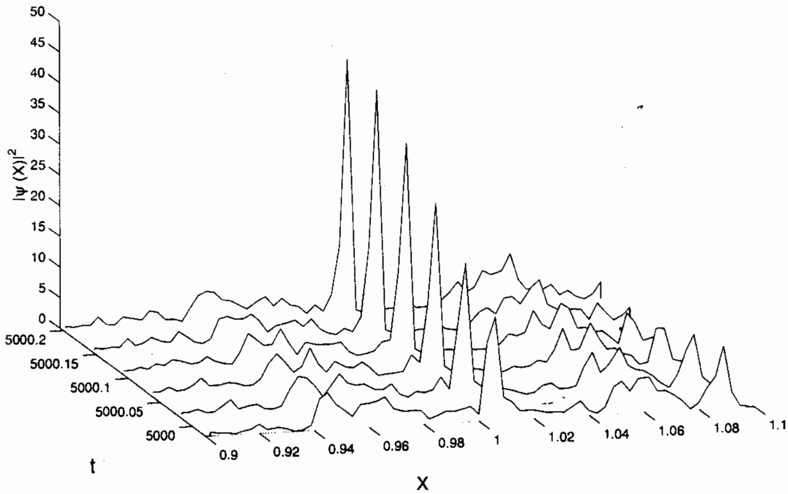


FIGURE 4. $\beta = 0, \lambda = -1$. Evolution towards collapse at $x \simeq 1$ between $t = 4999.980$ and $t = 5000.205$.



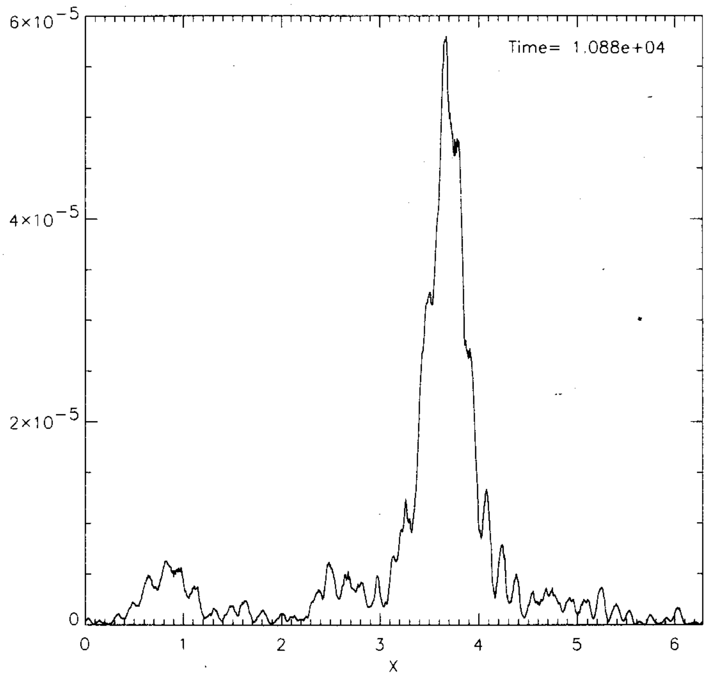
5.2. Numerical results for $\beta = 3, \lambda = +1$. Computations were performed with $\beta = 3$ because this case is analogous to gravity water waves. Moreover the strength of the interactions is larger than in the case $\beta = 0$.

Equation (5.1) was again integrated numerically over long times. The system is first separated into several soliton-like structures and low-amplitude quasi-linear waves. Processes of mutual interactions slowly redistribute the number of waves in a way leading to the growth of initially bigger quasisolitons and the decay of initially smaller quasisolitons. The final state then consists of one big quasisoliton moving in a sea of small quasilinear waves as shown in Figure 5. The shape of the quasisoliton is well described by the formula (4.3). The reader is referred to our paper [**9**] for more detail on quasisolitons. This phenomenon is similar to the 'droplet' effect observed in the non-integrable nonlinear Schrödinger equation [**11**]. The soliton solution turns out to be a statistical attractor for the system: long time evolution leads to the condensation of the number of particles into a single soliton which minimizes the Hamiltonian.

6. Conclusions

In conclusion, the numerical results show a discrepancy with the theory, which is mainly due to the presence of localised coherent structures, collapses in the focusing case ($\lambda = -1$) and quasisolitons in the defocusing case ($\lambda = +1$). In other words, both mechanisms, weak turbulence and coherent structures, are present and lead to a complex mixed picture. The discrepancy between numerics and theory may also be due to the sparsity of resonances in one dimension and the numerical discretization. Four-wave interactions are not as efficient and localised structures become dominant. Therefore equation (2.1) is not such a good model to assess the validity of weak turbulence theory.

FIGURE 5. $\beta = 3, \lambda = +1$. Snapshot of a quasisoliton at $x \simeq 3.7$ and $t = 10880$.



References

[1] D. Cai, A. J. Majda, D. W. McLaughlin, E. G. Tabak, *Spectral bifurcations in dispersive wave turbulence*, Proc. Nat. Acad. Sci. **96** (1999), 14216–14221.

[2] S. Dyachenko, A. C. Newell, A. Pushkarev, V. E. Zakharov, *Optical turbulence: weak turbulence, condensates and collapsing filaments in the nonlinear Schrödinger equation*, Physica D **57** (1992), 96–160.

[3] P. Guyenne, V. E. Zakharov, A. Pushkarev, F. Dias, *Turbulence d'ondes dans des modèles unidimensionnels*, C. R. Acad. Sci. Paris, Série IIb **328** (10) (2000), 757–762.

[4] E. A. Kuznetsov, A. M. Rubenchik, V. E. Zakharov, *Soliton stability in plasmas and hydrodynamics*, Phys. Rep. **142** (1986), 103–165.

[5] A. J. Majda, D. W. McLaughlin, E. G. Tabak, *A one-dimensional model for dispersive wave turbulence*, J. Nonlinear Sci. **6** (1997), 9–44.

[6] O. M. Phillips, *The dynamics of the upper ocean*, Cambridge Univ. Press, 1977.

[7] A. N. Pushkarev, V. E. Zakharov, *Turbulence of capillary waves*, Phys. Rev. Lett. **76** (1996), 3320–3323.

[8] V. E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of deep fluid*, J. Appl. Mech. Tech. Phys. **2** (1968), 190–194.

[9] V. E. Zakharov, P. Guyenne, A. Pushkarev, F. Dias, *Wave turbulence in one-dimensional models*, Physica D **152-153** (2001), 573–619.

[10] V. E. Zakharov, V. S. L'vov, G. Falkovich, *Kolmogorov Spectra of Turbulence I*, Springer-Verlag, 1992.

[11] V. E. Zakharov, A. N. Pushkarev, V. F. Shvets, V. V. Yankov, *Soliton turbulence*, JETP Lett. **48** (2) (1988), 83–87.

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