On the Variational Formulation of a Transmision Problem for the Biot Equations *

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Abstract

In this paper, a variational formulation for the transmission problem of the fluid-bone interaction is formulated. The formulation is based on a modified Biot system of equations for the cancellous bone together with a boundary integral equation formulation of the pressure in the water. Existence and uniqueness for the weak solution of the interaction problem are established in appropriate Sobolev spaces.

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1 Introduction

This paper is concerned with a fluid-bone interaction problem. The physical situation can be simply described as follows: A cancellous bone specimen is

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placed in a water tank occupied a region extended to the infinity. We are interested in the harmonic motion of the frame and fluid within the bone and the scattered pressure in the water due to a point source (or many point sources) located ate point (or points) in the water. Mathematically, we can formulated the problem as a transmission problem for the Biot equations and the Helmoholz equation in the bone and water, respectively. As well known, the Biot-Stoll [**?**, **?**, **?**] model treats a poroplastic medium as an elastic frame with interspinal pore fluid. Cancellous bone is anisotropic, however, as pointed out by Williams [**?**], if the acoustic waves passing through it travel in the trabecular direction an isotropic model may be acceptable. For simplicity, we will simulate a two dimensional version of the experiments described in McKelvie and Palmer [**?**] and Hosokawa and Otani[**?**].

Because of not enough physical interface conditions, in order to formulate a well posed problem, one must modify the standard Biot equations (see (**??**) and (**??**) below) for the displacements fields for the frame and fluid within the bone. For the computational purpose, we will then reduce the Helmholtz equation to a boundary integral equation on the interface face of bone and water. This leads a nonlocal boundary value problem of which the variational formulation is formulated. Our main results concerning the existence and uniqueness results of the variational solution are given in Section 5.

The rest of paper is organized as follows. In the next section, we begin with the basic equations of the standard Biot model for a poroelastic material such as cancellous bones. Section 3 contain the modified Biot equations and the formulation of the nonlocal boundary value problem of the bone–fluid interaction problem. The variational formulation of the problem is given in Section 4. For interested readers, we include an appendix containing the relevant Biot-Stoll parameters appeared in the Biot model equations.

2 The Biot model for a poroelastic material

The motion of the frame and fluid within the bone are tracked by position vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{U} = (U_1, U_2)$. The constitutive equations used by Biot are those of a linear elastic material with terms added to account for the interaction of the frame and interstitial fluid

$$\begin{aligned}\sigma_{x_1x_1} &= 2\mu e_{x_1x_1} + \lambda e + Q\epsilon,\\ \sigma_{x_2x_2} &= 2\mu e_{x_2x_2} + \lambda e + Q\epsilon,\\ \sigma_{x_1x_2} &= \mu e_{x_1x_2}, \quad \sigma_{x_2x_1} = \mu e_{x_2x_1},\\ s &= Qe + R\epsilon,\end{aligned} \tag{2.1}$$

where the solid and fluid dilatations are given by

$$e = \nabla\cdot\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \epsilon = \nabla\cdot\mathbf{U} = \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2}. \tag{2.2}$$

Here the parameter μ is the measured complex frame shear modulus, and λ, Q and R are calculated from measured or estimated parameters given in Table 1 in Appendix. The strains are defined by

$$e_{x_1x_1} = \frac{\partial u_1}{\partial x_1}, \quad e_{x_1x_2} = e_{x_2x_1} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}, \quad e_{x_2x_2} = \frac{\partial u_2}{\partial x_2}. \tag{2.3}$$

Equations (**??**), (**??**) and (**??**) and an argument based upon Lagrangian dynamics are shown in [**?**, **?**] to lead to the following equations of motion for the displacements $\mathbf{u}, \mathbf{U}$ and dilatations e, ϵ

$$\begin{aligned}\mu\nabla^2\mathbf{u} + \nabla\Big((\lambda+\mu)e + Q\epsilon\Big) &= \frac{\partial^2}{\partial t^2}(\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}) + b\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}),\\ \nabla\Big(Qe + R\epsilon\Big) &= \frac{\partial^2}{\partial t^2}(\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}) - b\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}).\end{aligned} \tag{2.4}$$

Here ρ_{11} and ρ_{22} are density parameters for the solid and fluid, ρ_{12} is a density coupling parameter, and b is a dissipation parameter (see Appendix). If the poroelastic material is assumed to oscillate harmonically in time: $\mathbf{u}(x,y,t) = \mathbf{u}(x,y)e^{i\omega t}$, $\mathbf{U}(x,y,t) = \mathbf{U}(x,y)e^{i\omega t}$, then by substituting these representations into (**??**) gives

$$\begin{aligned}\mu\nabla^2\mathbf{u} + \nabla[(\lambda+\mu)e + Q\epsilon] + p_{11}\mathbf{u} + p_{12}\mathbf{U} &= 0,\\ \nabla[Qe + R\epsilon] + p_{12}\mathbf{u} + p_{22}\mathbf{U} &= 0,\end{aligned} \tag{2.5}$$

where

$$p_{11} := \omega^2\rho_{11} - i\omega b, \quad p_{12} := \omega^2\rho_{12} + i\omega b, \quad p_{22} := \omega^2\rho_{22} - i\omega b. \tag{2.6}$$

In the sequel, we refer equation (**??**) as the **standard Biot equations**.

3 The transmission problem

To formulate the transmission problem, let bone specimen be occupied the region denoted by Ω^b and the exterior water region by Ω^w. In Ω^w the governing equation can be reduced to the two-dimensional non-homogeneous Helmoholz equation for fluid pressure p subject to the linearized Navier -Stokes equation for compressible fluid flow for the pressure p and fluid displacement $\mathbf{U}^w := (U_1^w, U_2^w)$. That is, we require that p and $\mathbf{U}^w$ satisfying the equations

$$-(\nabla^2 p + k_0^2 p) = f \quad \text{in} \quad \Omega^w, \tag{3.1}$$

$$\nabla p \, - \, \rho^w \omega^2 \mathbf{U}^w = \mathbf{f} \quad \text{in} \quad \Omega^w, \tag{3.2}$$

where $\mathbf{f}$ is a given function with compact support, and $f = -div\,\mathbf{f}$; see the Appendix of [**?**].

In the bone specimen Ω^b, however, in order to formulate a well-posed boundary value problem, one must modify the standard form of the Biot equation (**??**), since there are not enough transmission conditions for the components of displacements fields $\mathbf{u}$ and $\mathbf{U}$ for the frame and fluid within the bone. The main idea here is to replace the unknowns $\mathbf{U}$ form the fluid displacement fields of the fluid within the bone specimen Ω^b by a single known stress s in (**??**) in the equations (**??**). To see this, we first express ϵ and $\mathbf{U}$ in terms of s from (**??**) and (**??**),

$$\epsilon = \frac{1}{R}(s - Q\,e), \quad \mathbf{U} = -\frac{1}{p_{22}}(\nabla s + p_{12}\mathbf{u}). \tag{3.3}$$

By taking the divergence of the second equation of (**??**), we obtain

$$\nabla^2 s + p_{12}\,e + p_{22}\,\epsilon = 0,$$

which reduces to

$$\nabla^2\,s + \frac{p_{22}}{R}\,s + (p_{12} - \frac{p_{22}Q}{R})\,e = 0, \tag{3.4}$$

by making use of (**??**). Similarly, the first equation of (**??**)can be written in the form:

$$\mu\nabla^2\mathbf{u} + \nabla\Big[(\lambda + \mu - \frac{Q^2}{R})\,e + (\frac{Q}{R} - \frac{p_{12}}{p_{22}})\,s\Big] + (p_{11} - \frac{p_{12}^2}{p_{22}})\,\mathbf{u} = \mathbf{0}. \tag{3.5}$$

Equations (**??**) and (**??**) then form the **modified Biot equations** for $\mathbf{u}$ and s in the bone specimen Ω^b. We are now in a position to formulate the transmission problem for the bone-fluid interaction:

Definition 1 (The non-homogeneous transition problem ($\mathbf{TP}_f$)) *The problem consists of finding the triplet* $(\mathbf{u}, s, p)$ *such that*

$$(E_b)\ \ \mu\nabla^2\mathbf{u}+\nabla\Big((\lambda+\mu-\frac{Q^2}{R})\,e+(\frac{Q}{R}-\frac{p_{12}}{p_{22}})\,s\Big)+(p_{11}-\frac{p_{12}^2}{p_{22}})\,\mathbf{u}=\mathbf{0}\quad in\quad \Omega^b,$$

$$(E_s)\qquad \nabla^2\,s+\frac{p_{22}}{R}\,s+(p_{12}-\frac{p_{22}Q}{R})\,e=0\quad in\quad \Omega^b$$

$$(E_p),\qquad -\left(\triangle p+k_0^2 p\right)=f\quad \text{in}\quad \Omega^w,\quad f:=-div\,\mathbf{f}$$

having compact support in Ω^w*, together with the transmission conditions*

$$(B_1)\qquad \big(\underline{\underline{\sigma}}(\mathbf{u})+Q\ \text{div}\ \mathbf{U}+s\big)\,\mathbf{n}=-p\ \mathbf{n}\quad \text{on}\quad \Gamma=\partial\Omega^b$$

with vanishing of the tangent frame stress $\sigma_{12}=\sigma_{21}=0$*, where* $\underline{\underline{\sigma}}(\mathbf{u})$ *and* $\underline{\underline{\varepsilon}}(\mathbf{u})$ *denote the stress and strain tensors, and div* $\mathbf{U}$ *the fluid dilatation*

$$\underline{\underline{\sigma}}(\mathbf{u})=\lambda\text{div}\mathbf{u}+2\mu\underline{\underline{\varepsilon}}(\mathbf{u}),\quad \underline{\underline{\varepsilon}}(\mathbf{u})=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)$$
$$\text{div}\ \mathbf{U}=\frac{1}{R}\left(s-Q\ e\right),$$

$$(B_2)\ \ \rho^w\omega^2\left[1-\beta\left(1+\frac{p_{12}}{p_{22}}\right)\right]\mathbf{u}\cdot\mathbf{n}-\frac{\beta\rho^w\omega^2}{p_{22}}\frac{\partial s}{\partial n}=\left(\frac{\partial p}{\partial n}-\mathbf{n}\cdot\mathbf{f}\right)\quad \text{on}\quad \Gamma$$

$$(B_3).\qquad s=-\beta p\quad \text{on}\quad \Gamma.$$

In addition we assume that the Sommerfeld radiation condition holds for p*.*

In the formulation, transmission condition (B_1) and (B_2) represents respectively, the continuity of the flux and continuity of the aggregate pressure , while condition (B_3) expresses the continuity of pore pressure.

For the uniqueness proof, we now introduce the traction-free solution of the bone as in fluid-structure interaction problem [**?**].

Definition 2 (Traction free problem) *The problem for* $(\mathbf{u}, s)$ *in* Ω^b *consists of the partial differential equations* (E_b) *and* (E_s) *together with the homogeneous boundary conditions*

$$(B_1)_0 \qquad \Big(\underline{\underline{\sigma}}(\mathbf{u}) + Q \text{ div } \mathbf{U} + s\Big)\, \mathbf{n} = 0 \quad \text{on} \quad \Gamma,$$

$$(B_2)_0 \qquad \rho^w \omega^2 \left[1 - \beta\left(1 + \frac{p_{12}}{p_{22}}\right)\right] \mathbf{u}\cdot\mathbf{n} - \frac{\beta\rho^w\omega^2}{p_{22}}\frac{\partial s}{\partial n} = \frac{\partial p}{\partial n} \quad \text{on} \quad \Gamma,$$

$$(B_3)_0 \qquad s = 0 \quad \text{on} \quad \Gamma.$$

are called traction free problem for $(\mathbf{u}, s)$*, and the corresponding non-trivial solutions are referred to as the traction free solutions.*

For the variational formulation, we now reduce the partial differential equation (E_p) for p, namely

$$-\left(\triangle p + k_0^2 p\right) = f \quad \text{in} \quad \Omega^w, \tag{3.6}$$

to a boundary integral equation for p on Γ. We use the indirect approach for the reduction of partial differential equation by seeking a solution p in the form of a simple-layer potential

$$p = -\mathbf{S}\phi + p_f \quad \text{in} \quad \Omega^w, \tag{3.7}$$

where ϕ is an unknown density function and $\mathbf{S}\phi$ is the simple layer potential

$$\mathbf{S}\phi(x) := \int_\Gamma \frac{i}{4} H_0^{(1)}(k_0|x-y|)\phi(y)ds_y, \quad x \in \Omega^w, \tag{3.8}$$

where $-\frac{i}{4}H_0^{(1)}(k_0|x-y|)$ denotes the fundamental solution of the Helmholtz operator $\Delta + k_0$, and p_f,

$$p_f(x) := \frac{i}{4} \int\limits_{\text{supp}f} H_0^{(1)}\left(k_0|\mathbf{x}-\mathbf{y}|\right) f(\mathbf{y})\, d\mathbf{y}, \quad x \in \Omega^w,$$

is a particular solution of (**??**), which is known. Hence, if $p|_\Gamma$ is known , applying the trace operator γ_0 to (**??**), we then obtain a boundary integral equation for the known density ϕ

$$p(\mathbf{x})|_\Gamma = -\mathbf{V}\phi + \gamma_0 p_p, \tag{3.9}$$

where $\mathbf{V} = \gamma_0 \mathbf{S}$ is the simple layer boundary integral operator. Then from the transmission condition (B_3), we obtain the boundary integral equation

$$(E_{pb}) \qquad \mathbf{V}\phi - \frac{1}{\beta}s = \gamma_0 p_f.$$

Definition 3 (Nonlocal boundary value problem) *The transmission problem $\mathbf{TP}_f$ is termed a nonlocal boundary value problem for the triple $(\mathbf{u}, s, \phi)$ if the triple $(\mathbf{u}, s, \phi)$ satisfies equations (E_b), (E_s), and the boundary integral equation (E_{3p}) together with the transmission conditions*

$$(B_{1b}), \qquad \Big(\underline{\underline{\sigma}}(\mathbf{u}) + Q \text{ div } \mathbf{U} + s\Big)\, \mathbf{n} = -p\, \mathbf{n} \quad \text{on} \quad \Gamma = \partial\Omega^b$$

with

$$p = -\mathbf{V}\phi + \gamma_0 p_f.$$

(*Here again $\sigma_{12} = \sigma_{21} = 0$, where $\underline{\underline{\sigma}}(\mathbf{u})$ denotes the stress tensor, and div $\mathbf{U}$ the fluid dilatation*)

$$(B_{2b}) \;\; \rho^w\omega^2 \left[1 - \beta\left(1 + \frac{p_{12}}{p_{22}}\right)\right] \mathbf{u}\cdot\mathbf{n} - \frac{\beta\rho^w\omega^2}{p_{22}}\frac{\partial s}{\partial n} \;=\; \left(\frac{\partial p}{\partial n} - \mathbf{n}\cdot\mathbf{f}\right) \quad \text{on} \quad \Gamma$$

with

$$\frac{\partial p}{\partial n} = \frac{1}{2}\Big(\phi - \mathbf{K}'\phi\Big) + \frac{\partial p_f}{\partial n}.$$

The boundary integral operator $\mathbf{K}'$ in (B_{2b}) is defined by

$$\mathbf{K}'\phi(x) := \frac{i}{4}\int_\Gamma \frac{\partial}{\partial n_x} H_0^{(1)}(k_0|x-y|)\phi(y)ds_y, \;\; x \in \Gamma.$$

We note that condition (B_{2b}) can be explicitly written in terms of ϕ

$$\frac{\partial s}{\partial n} = \frac{p_{22}}{\beta}\Big\{ \left[1 - \beta\left(1 + \frac{p_{12}}{p_{22}}\right)\right] \mathbf{u}\cdot\mathbf{n} - \frac{1}{\rho^w\omega^2}\Big(\frac{1}{2}\phi - \mathbf{K}'\phi\Big)\Big\} + \frac{p_{22}}{\beta\rho^w\omega^2}(\mathbf{n}\cdot\mathbf{f} - \frac{\partial}{\partial n}p_f),$$

which will be needed for the variational formulation in the next section.

4 The Variational formulation

In this section, we will consider the variational formulation of the nonlocal boundary value problem. As usual, multiplying (E_b) by the conjugate of the test function $\mathbf{v}$ and integrating by parts, we obtain

$$\int_{\Omega^b} \left\{ \left[\left(\lambda - \frac{Q^2}{R}\right) (\operatorname{div} \mathbf{u}) (\operatorname{div} \overline{\mathbf{v}}) + 2\mu \,\underline{\underline{\varepsilon}}(\mathbf{u}) : \overline{\underline{\underline{\varepsilon}}(\mathbf{v})} \right] + \left(\frac{Q}{R} - \frac{p_{12}}{p_{22}}\right) s(\operatorname{div} \overline{\mathbf{v}}) - \left(p_{11} - \frac{p_{12}^2}{p_{22}}\right) \mathbf{u} \cdot \overline{\mathbf{v}} \right\} d\mathbf{x}$$
$$- \int_{\Gamma} \left[\left(\lambda - \frac{Q^2}{R}\right) \operatorname{div} \mathbf{u} + 2\mu \,\underline{\underline{\varepsilon}}(\mathbf{u}) + \left(\frac{Q}{R} - \frac{p_{12}}{p_{22}}\right) s \right] \mathbf{n} \cdot \overline{\mathbf{v}} \, ds \;=\; 0. \qquad (4.1)$$

We define the sesquilinear bilinear form

$$a(\mathbf{u}, \mathbf{v}) \;:=\; \int_{\Omega^b} \left[\left(\lambda - \frac{Q^2}{R}\right) (\operatorname{div} \mathbf{u}) (\operatorname{div} \overline{\mathbf{v}}) + 2\mu \,\underline{\underline{\varepsilon}}(\mathbf{u}) : \underline{\underline{\varepsilon}}(\overline{\mathbf{v}}) \right] \, d\mathbf{x}, \qquad (4.2)$$

and by rewriting the boundary term in (**??**), we see that

$$a(\mathbf{u}, \mathbf{v}) \;+\; \int_{\Omega^b} \left(\frac{Q}{R} - \frac{p_{12}}{p_{22}}\right) s(\operatorname{div} \overline{\mathbf{v}}) d\mathbf{x} \;-\; \int_{\Omega^b} \left(p_{11} - \frac{p_{12}^2}{p_{22}}\right) \mathbf{u} \cdot \overline{\mathbf{v}} \, d\mathbf{x} + \int_{\Gamma} \left(1 + \frac{p_{12}}{p_{22}}\right) s \; \mathbf{n} \cdot \overline{\mathbf{v}} \, ds_\Gamma$$
$$- \int_{\Gamma} \left(\lambda \operatorname{div} \mathbf{u} + 2\mu \,\underline{\underline{\varepsilon}}(\mathbf{u}) + Q \Big[\frac{1}{R}(s - Q \, div \, \mathbf{u}) \Big] + s \right) \mathbf{n} \cdot \overline{\mathbf{v}} \, ds_\Gamma \;=\; 0$$

Hence, the above equation with the transmission condition (E_{1b}) leads to the variational equation of equation (E_b):

$$a(\mathbf{u}, \mathbf{v}) \;+\; \int_{\Omega^b} \left(\frac{Q}{R} - \frac{p_{12}}{p_{22}}\right) s(\operatorname{div} \overline{\mathbf{v}}) d\mathbf{x} \;-\; \int_{\Omega^b} \left(p_{11} - \frac{p_{12}^2}{p_{22}}\right) \mathbf{u} \cdot \overline{\mathbf{v}} \, d\mathbf{x} - \Big[1 - \beta(1 + \frac{p_{12}}{p_{22}}) \Big] < \mathbf{V}\phi\mathbf{n}, \overline{\mathbf{v}} >_\Gamma \qquad (4.3)$$
$$= - \Big[1 - \beta(1 + \frac{p_{12}}{p_{22}}) \Big] < \gamma_0 p_f, \overline{\mathbf{v}} >_\Gamma, \quad \forall \, \mathbf{v} \in \left(H^1(\Omega^b)\right)^2.$$

We repeat this process for the s equation by multiplying equation (E_s) by the test function τ and integrating by parts yields

$$\int\limits_{\Omega^b} \nabla s \cdot \nabla \overline{\tau} \, d\mathbf{x} - \int\limits_{\Omega^b} \frac{p_{22}}{R} s \, \overline{\tau} \, d\mathbf{x} - \int\limits_{\Omega^b} \left(p_{12} - \frac{p_{22} Q}{R} \right) (div \ \mathbf{u}) \, \overline{\tau} \, d\mathbf{x} - \int\limits_{\Gamma} \frac{\partial s}{\partial n} \overline{\tau} \, ds \ = \ 0.$$

By introducing the sesquilinear form

$$b(s,\tau) \ = \int\limits_{\Omega^b} \nabla s \cdot \nabla \overline{\tau} \, d\mathbf{x} \tag{4.4}$$

and using the condition (B_{2b}), the variational form of the s equation now may be written as

$$\begin{aligned} b(s,\tau) \ + \ p_{22} \int\limits_{\Omega^b} \left(\frac{Q}{R} - \frac{p_{12}}{p_{22}} - \right) div \ (\mathbf{u}) \, \overline{\tau} \, d\mathbf{x} - \int\limits_{\Omega^b} \frac{p_{22}}{R} s \, \overline{\tau} \, d\mathbf{x} \\ - \frac{p_{22}}{\beta} \left[1 - \beta \left(1 + \frac{p_{12}}{p_{22}} \right) \right] < \mathbf{u} \cdot \mathbf{n}, \overline{\tau} >_{\Gamma} + \frac{p_{22}}{\beta \rho^w \omega^2} < \Big(\frac{1}{2} \phi - \mathbf{K}' \phi \Big), \overline{\tau} >_{\Gamma} \\ = \frac{p_{22}}{\beta \rho^w \omega^2} < (\mathbf{n} \cdot \mathbf{f} - \frac{\partial}{\partial n} p_f), \overline{\tau} >_{\Gamma}, \quad \forall \, \tau \in H^1(\Omega^b). \end{aligned} \tag{4.5}$$

Finally, we multiply the boundary integral equation (E_{pb}) by the test function ψ, and integrate it. This yields the variational equation for (E_{pb})

$$\frac{p_{22}}{2\rho^w \omega^2} \left\langle \mathbf{V}\phi, \overline{\psi} \right\rangle - \frac{p_{22}}{2\rho^w \omega^2 \beta} \left\langle s, \overline{\psi} \right\rangle \ = \ \frac{p_{22}}{2\rho^w \omega^2} \left\langle \gamma_0 p_p, \overline{\psi} \right\rangle, \quad \forall \, \psi \in H^{-1/2}(\Gamma). \tag{4.6}$$

Collecting (**??**), (**??**), and (**??**), we have the variational formulation for the nonlocal boundary value problem:

Definition 4 (Variational formulation) *Given* $\mathbf{f}$*, find the triple* $(\mathbf{u}, s, \phi) \in \big(H^1(\Omega^b)\big)^2 \times H^1(\Omega^b) \times H^{-1/2}(\Gamma)$ *such that*

$$\mathcal{A}\Big(\mathbf{u}, s, \phi), (\mathbf{v}, \tau, \psi)\Big) = \boldsymbol{\ell_f} \ (\mathbf{v}, \tau, \phi) \tag{4.7}$$

for all $(\mathbf{v},\tau,\psi) \in \left(H^1(\Omega^b)\right)^2 \times H^1(\Omega^b) \times H^{-1/2}(\Gamma)$, *where* $\mathcal{A}$ *and* $\boldsymbol{\ell_f}$ *are respectively the sesqulinear form and linear functional defined by*

$$\begin{aligned}
\mathcal{A}\Big(\mathbf{u},s,\phi),(\mathbf{v},\tau,\psi)\Big) &:= a(\mathbf{u},\mathbf{v}) \,+\, b(s,\tau) \,+\, \frac{p_{22}}{2\rho^w\omega^2}\left\langle \mathbf{V}\phi,\overline{\psi}\right\rangle_\Gamma \\
&+ \left(\frac{Q}{R}-\frac{p_{12}}{p_{22}}\right)\Big[\int_{\Omega^b} s(\,\mathrm{div}\,\overline{\mathbf{v}})d\mathbf{x} \,+\, p_{22}\int_{\Omega^b} div\,(\mathbf{u})\,\overline{\tau}\,d\mathbf{x}\Big] \\
&- \left(p_{11}-\frac{p_{12}^2}{p_{22}}\right)\int_{\Omega^b} \mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x} - \frac{p_{22}}{R}\int_{\Omega^b} s\,\overline{\tau}\,d\mathbf{x} \qquad (4.8)\\
&-\Big[1-\beta(1+\frac{p_{12}}{p_{22}})\Big]\Big\{ < \mathbf{V}\phi\mathbf{n},\overline{\mathbf{v}} >_\Gamma + \frac{p_{22}}{\beta} < \mathbf{u}\cdot\mathbf{n},\overline{\tau} >_\Gamma \Big\} \\
&+ \frac{p_{22}}{\beta\rho^w\omega^2}\Big[< \Big(\frac{1}{2}\phi - \mathbf{K}'\phi\Big),\overline{\tau} >_\Gamma - \frac{1}{2}\left\langle s,\overline{\psi}\right\rangle_\Gamma \Big]
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\ell_f}\;(\mathbf{v},\tau,\phi) &:= -\Big[1-\beta(1+\frac{p_{12}}{p_{22}})\Big] < \gamma_0 p_f,\overline{\mathbf{v}} >_\Gamma + \frac{p_{22}}{\beta\rho^w\omega^2} < (\mathbf{n}\cdot\mathbf{f} - \frac{\partial}{\partial n}p_f)\overline{\tau} >_\Gamma \\
&+ \frac{p_{22}}{2\rho^w\omega^2}\left\langle \gamma_0 p_p,\overline{\psi}\right\rangle \qquad (4.9)
\end{aligned}$$

5 Existence and uniqueness

From the definition of the sesqulinear form $\mathcal{A}(\cdot,\cdot)$ in (**??**), it is not difficult to see that $\mathcal{A}(\cdot,\cdot)$ satisfies a Gårding's inequality. Setting $(\mathbf{v},\tau,\psi) = (\mathbf{u},s,\phi)$, we see that

$$\begin{aligned}
\mathcal{A}\Big(\mathbf{u},s,\phi),(\mathbf{u},s,\phi)\Big) &:= a(\mathbf{u},\mathbf{u}) \,+\, b(s,s) \,+\, \frac{p_{22}}{2\rho^w\omega^2}\left\langle \mathbf{V}\phi,\overline{\phi}\right\rangle_\Gamma \\
&+ \left(\frac{Q}{R}-\frac{p_{12}}{p_{22}}\right)\Big[\int_{\Omega^b} s(\,\mathrm{div}\,\overline{\mathbf{u}})d\mathbf{x} \,+\, p_{22}\int_{\Omega^b} div\,(\mathbf{u})\,\overline{s}\,d\mathbf{x}\Big] \\
&- \left(p_{11}-\frac{p_{12}^2}{p_{22}}\right)\int_{\Omega^b} |\mathbf{u}|^2\,d\mathbf{x} - \frac{p_{22}}{R}\int_{\Omega^b} |s|^2\,d\mathbf{x} \\
&-\Big[1-\beta(1+\frac{p_{12}}{p_{22}})\Big]\Big\{ < \mathbf{V}\phi\mathbf{n},\overline{\mathbf{u}} >_\Gamma + \frac{p_{22}}{\beta} < \mathbf{u}\cdot\mathbf{n},\overline{\mathbf{u}} >_\Gamma \Big\} \\
&+ \frac{p_{22}}{\beta\rho^w\omega^2}\Big[< \Big(\frac{1}{2}\phi - \mathbf{K}'\phi\Big),\overline{s} >_\Gamma - \frac{1}{2}\left\langle s,\overline{\phi}\right\rangle_\Gamma \Big]
\end{aligned}$$

We can show that

$$Re\mathcal{A}\Big(\mathbf{u}, s, \phi), (\mathbf{u}, s, \phi)\Big) = a(\mathbf{u}, \mathbf{u}) \,+\, b(s, s) \,+\, \frac{p_{22}}{2\rho^w\omega^2}\left\langle \mathbf{V}\phi, \overline{\phi}\right\rangle_\Gamma$$
$$+\mathcal{C}\Big(\mathbf{u}, s, \phi), (\mathbf{u}, s, \phi)\Big),$$

where $\mathcal{C}$ is compact on $\left(H^1(\Omega^b)\right)^2 \times H^1(\Omega^b) \times H^{-1/2}(\Gamma)$. In fact we have

Theorem 5 *The sesqulinear form in (??) satisfies the Gårding's inequality in the form*

$$\begin{aligned} Re\mathcal{A}\left(\mathbf{u}, s, \varphi, \mathbf{u}, s, \phi\right) \;&\geq\; \alpha\left\{\|\mathbf{u}\|^2_{\left(H^1(\Omega^b)\right)^2} + \|s\|^2_{H^1(\Omega^b)} + \|s\|^2_{H^{-\frac{1}{2}}(\Gamma)}\right\} \\ &-\; \delta\left\{\|\mathbf{u}\|^2_{\left(H^{1-\epsilon}(\Omega^b)\right)^2} + \|s\|^2_{H^{1-\epsilon}(\Omega^b)} + \|s\|^2_{H^{-\frac{1}{2}-\epsilon}(\Gamma)}\right\}, \end{aligned}$$

where $\alpha > 0$ and $\delta \geq 0$ are constant and $\epsilon > 0$ is a small parameter.

As is well known, Gårding's inequality implies the validity of the Fredholm alternative. Hence uniqueness implies the existence. For this purpose, we now consider the the homogeneous transmission problem $\mathbf{TP}_f$ with $f = 0$, since the uniqueness of the solution of the variational equation (**??**) will be depending upon that of $\mathbf{TP}_f$.

Theorem 6 *If the triplet $(\mathbf{u}, s, p)$ is a classical solution of homogeneous transmission problem $\mathbf{TP}_0$ with $Im\, k_0 = 0$, then $p = 0$.*

Proof. The proof follows the standard uniqueness proof used for the scattering transition problem. A simple application of the divergence theorem together with the radiation condition leads to

$$\int\limits_{S_R}\left|\frac{\partial p}{\partial r} - ik_0 p\right|^2 ds \;=\; \int\limits_{S_R}\left(\left|\frac{\partial p}{\partial r}\right|^2 + |k_0\, p|^2\right) ds \;+\; 2k_0 Im\int\limits_{\partial\Omega^b} p\frac{\partial \bar{p}}{\partial n}\, ds \;=\; o(1)$$

as $R \to \infty$, where S_R is the surface of a ball of radius R, enclosing the body Ω^b. The main idea here is to show that $\int\limits_{\partial\Omega^b} p\frac{\partial\bar{p}}{\partial n}\, ds$ is real. Then

$$\int\limits_{S_R}\left(\left|\frac{\partial p}{\partial r}\right|^2 + |k_0\, p|^2\right) ds \;=\; o(1)$$

and from the Rellich-Vekua lemma, $p = 0$. To show this, we now compute $\int\limits_{\partial\Omega^b} p\frac{\partial\bar{p}}{\partial n}\,ds$ by using the variational forms for (E_b) and (E_s) in Section 4. A simple computation shows that by eliminating the common term

$$\int\limits_{\Gamma}\left[1-\beta\left(1+\frac{p_{12}}{p_{22}}\right)\right]p\mathbf{n}\cdot\mathbf{u}\,ds_\Gamma,$$

we obtain

$$\frac{1}{\rho^w\omega^2}\int\limits_{\Gamma} p\overline{\frac{\partial p}{\partial n}}\,ds_\Gamma = \frac{1}{p_{22}}b(s,s)-\frac{1}{R}\int\limits_{\Omega^b}|s|^2\,d\mathbf{x}+\left(\frac{Q}{R}-\frac{p_{12}}{p_{22}}\right)\int\limits_{\Omega^b}\left(\overline{\operatorname{div}u}\right)s\,d\mathbf{x}-$$

$$\left\{a(,u,u)+\left(\frac{Q}{R}-\frac{p_{12}}{p_{22}}\right)\int\limits_{\Omega^b}\left(\overline{\operatorname{div}u}\right)s\,d\mathbf{x}-\left(p_{11}-\frac{p_{12}^2}{p_{22}}\right)\int\limits_{\Omega^b}|u|^2\,d\mathbf{x}\right\}, \quad (5.1)$$

which implies immediately that the term is real and hence

$$Im\int\limits_{\Gamma} p\frac{\partial\bar{p}}{\partial n}\,ds_\Gamma = 0,$$

as was desired. ■

We remark that Theorem **??** does not imply that the components $(\mathbf{u}, s)$ of the triple $(\mathbf{u}, s, p)$ considered in $\mathbf{T}P_0$ are trivial solutions, since they may be solutions of the traction free problem defined in Section 3. Hence in order to ensure the existence of a solution of the variational equation (**??**), we make the following assumptions.

Assumptions:

(I) There is no traction free solution.

(II) The square of the wave number, k_0^2, is not an eigenvalue of the Dirichlet problem for the negative Laplacian in Ω^b.

We remark that Assumption (II) is a guarantee for the invertbility of the simple-layer operator $\mathbf{V}$ (see [**?**, p. 30]). We now summarize our results in the following theorem.

Theorem 7 *Under Assumptions* (I) *and* (II) *, there exists an unique solution of the problem* $\mathbf{TP}_f$ *in* $\left(H^1(\Omega^b)\right)^2 \times H^1(\Omega^b) \times H^{-\frac{1}{2}}(\Gamma)$.

6 Appendix

6.1 The Biot-Stoll parameters

The Biot-Stoll parameters are calculated from the inputs of Table **??** using the formulas

$$\begin{aligned}\rho_{11} &= (1-\beta)\rho_r - \beta(\rho_f - m\beta)\\ \rho_{12} &= \beta(\rho_f - m\beta)\\ \rho_{22} &= m\beta^2\\ b &= \frac{F\left(a\sqrt{\omega\rho_f/\eta}\right)\beta^2\eta}{k}\end{aligned}$$

where

$$m = \frac{\alpha\rho_f}{\beta}$$

and the multiplicative factor $F(\zeta)$, which was introduced in [**?**] to correct for the invalidity of the assumption of Poiseuille flow at high frequencies, is given by

$$F(\zeta) = \frac{1}{4}\frac{\zeta T(\zeta)}{1 - 2T(\zeta)/i\zeta} \tag{6.2}$$

where T is defined in terms of Kelvin functions

$$T(\zeta) = \frac{\mathrm{ber}'(\zeta) + i\mathrm{bei}'(\zeta)}{\mathrm{ber}(\zeta) + i\mathrm{bei}(\zeta)}.$$

The parameter μ, the complex frame shear modulus is measured. The other parameters λ, R and Q occurring in the constitutive equations are calculated from the measured or estimated values of the parameters given in Table **??** using the formulas

$$\begin{aligned}\lambda &= K_b - \frac{2}{3}\mu + \frac{(K_r - K_b)^2 - 2\beta K_r(K_r - K_b) + \beta^2 K_r^2}{D - K_b} \qquad (6.3)\\ R &= \frac{\beta^2 K_r^2}{D - K_b}\\ Q &= \frac{\beta K_r\left((1-\beta)K_r - K_b\right)}{D - K_b}.\end{aligned}$$

Symbol	Parameter
ρ_f	Density of the pore fluid
ρ_r	Density of frame material
K_b	Complex frame bulk modulus
μ	Complex frame shear modulus
K_f	Fluid bulk modulus
K_r	Frame material bulk modulus
β	Porosity
η	Viscosity of pore fluid
k	Permeability
α	Structure constant
a	Pore size parameter

Table 1: Parameters in the Biot model

where

$$D = K_r(1 + \beta(K_r/K_f - 1)). \tag{6.4}$$

The bulk and shear moduli K_b and μ are often given imaginary parts to account for frame inelasticity.

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