Applications of Integral Equation Methods to a Class of Fundamental Problems in Mechanics and Mathematical Physics

George C. Hsiao

Department of Mathematical Sciences
University of Delaware
Newark, Delaware USA
www.math.udel.edu/˜hsiao

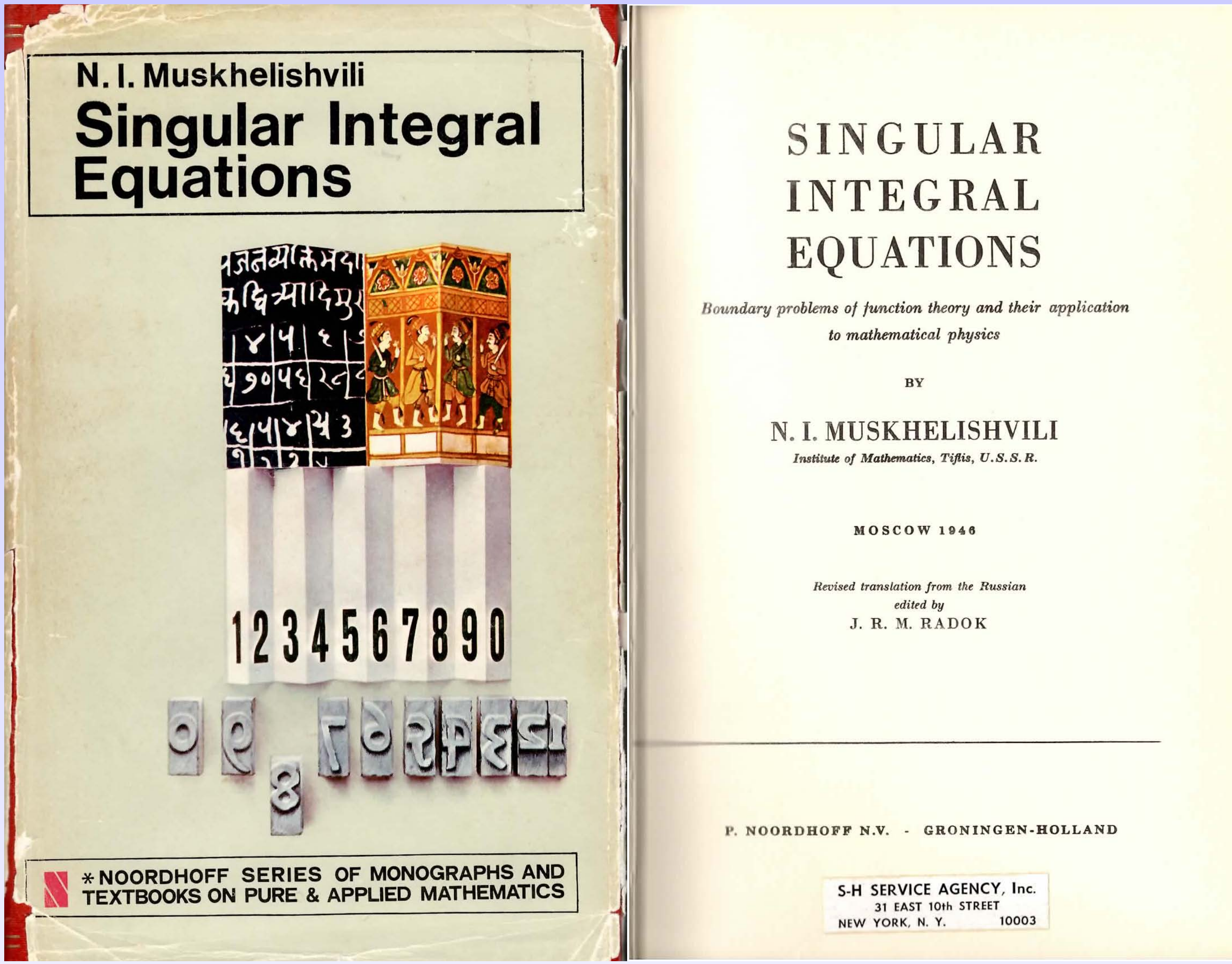
International Conference
"Continuum Mechanics and Related Problems of Analysis"
September 9-14 , 2011, Tbilisi, Georgia

Outline

1. Some Historical Remarks
2. Exterior Boundary Value Problems
3. Transmission Problems
4. Concluding Remarks

Some Historical Remarks

Muskhelishvili's monograph (1958 in English)



Simple layer potentials

Chapter 7. The Dirichlet Problem

§ 65. Solution of the Dirichlet problem by the potential of a simple layer. Fundamental problem of electrostatics.

From the solution of the modified Dirichlet problem, obtained in the last section, it is not difficult to arrive by a method, entirely analogous to that used in § 68, at the solution of the classical Dirichlet problem. In order to avoid repetition this will not be done here, but instead the Dirichlet problem will be solved directly by two methods: with the help of the modified potential of a simple layer and by means of the ordinary potential of a simple layer; as a by-product the solution of the so-called fundamental problem of electrostatics will be obtained.

- Fichera, G., Linear elliptic equations of higher order in two independent variables and singular integral equations, with applications to anisotropic inhomogeneous elasticity. In *Partial Differential Equations and Continuum Mechanics*, Langer, R. E. (ed). The University of Wisconsin Press: Wisconsin, 1961; 55-80.

Fichera's method:

It is a general method in terms of *simple–layer potential* for treating Dirichlet problems for a large class of elliptic equations of higher order with variable coefficients in the plane.

A model problem in $\mathbb{R}^2$

$$-\Delta u = 0 \quad \text{in} \quad \Omega \quad (\text{ or } \Omega^c := \mathbb{R}^2 \setminus \overline{\Omega})$$

$$u|_\Gamma = f \quad \text{on} \quad \Gamma := \partial\Omega$$

$$(+\mathit{Cond.\ at}\ \infty)$$

$$u(x) = \int_\Gamma E(x,y)\sigma(y)\, ds_y \ , \quad x \in \mathbb{R}^2 \setminus \Gamma$$

$$\boxed{V\sigma := \int_\Gamma E(x,y)\sigma(y)\, ds_y = f \quad \text{on} \quad \Gamma}$$

$$E(x,y) = -\frac{1}{2\pi}\, log\, |x-y|,$$

$$-\Delta\, E(x,y) = \delta(|x-y|)$$

A modified Fichera's method

$$\int_\Gamma \frac{\partial E}{\partial s_x}(x,y)\sigma(y)\, ds_y = \frac{\partial}{\partial s_x} f(x) \quad \text{on} \quad \Gamma$$

$$u(x) = \int_\Gamma E(x,y)\sigma(y) ds_y + \omega\ , \quad x \in \mathbb{R}^2 \backslash \Gamma$$

$$\boxed{\begin{array}{c} V\sigma + \omega = f \quad \text{on} \quad \Gamma \\ \int_\Gamma \sigma\, ds = A \end{array}} \qquad \text{(MBIE)}$$

Theorem [Hsiao and MacCamy 1973]

Given $(f, A) \in C^{1,\alpha}(\Gamma) \times \mathbb{R}$, (MBIE) has a unique solution pairs $(\sigma, \omega) \in C^{0,\alpha}(\Gamma) \times \mathbb{R}$.

A special class of BVPs

$$\Delta^m u - s\Delta^{m-1} u = 0 \quad \text{in} \quad \Omega \text{ (or } \Omega^c\text{)},$$
$$\frac{\partial^{m-1} u}{\partial x_1^{m-k} \partial x_2^{k-1}} = f_k \quad \text{on} \quad \Gamma,$$
$$(+\text{ Conds at } \infty), \quad 1 \leq k \leq m, \quad m = 1 \text{ or } 2.$$

- Hsiao, G. C. and R. C. MacCamy, Solution of boundary value problems by integral equations of the first kind. *SIAM Rev.* 1973; **15** : 687-705.
- Hsiao, G. C. and W. F. Wendland, A finite element method for some integral equations of the first kind. *J. Math. Anal. Appl.* 1977; **58** : 449-481.

A fundamental result $(m = 1, s = 0)$

Theorem [Hsiao and Wendland 1977].

Under the assumption:

$$\max_{x,\, y \,\in\, \Gamma} |x - y| < 1,$$

the simple-layer boundary integral operator

$$V\sigma := \int_\Gamma E(x, y)\sigma(y)\, ds_y, \quad x \in \Gamma$$

satisfies the inequalities

$$\gamma_1 \, ||\sigma||^2_{-1/2} \leqslant \gamma_2 \, ||V\sigma||^2_{1/2} \leqslant \langle \sigma, V\sigma \rangle \leqslant \gamma_3 \, ||\sigma||^2_{-1/2}$$

for all $\sigma \in H^{-1/2}(\Gamma)$*, where* γ_i*'s are constants.*

A weak solution

Given $f \in H^{1/2}(\Gamma)$, $\sigma \in H^{-1/2}(\Gamma)$ is said to be a weak solution of the boundary integral equation

$$V\sigma = f \quad \text{on} \quad \Gamma,$$

if it satisfies the variational form

$$\boxed{\langle \chi, V\sigma \rangle = \langle \chi, f \rangle \quad \forall\, \chi \in H^{-1/2}(\Gamma)}$$

Here $H^{-1/2}(\Gamma) =$ dual of $H^{1/2}(\Gamma)$ is the energy space for the operator V.

Remarks for the general Γ

$$
\begin{aligned}
V\sigma(x) &:= -\frac{1}{2\,\pi}\int_\Gamma \log|x-y|\;\sigma(y)ds_y \\
&= -\frac{1}{2\,\pi}\int_\Gamma \log\left(\frac{|x-y|}{2d}\right)\sigma(y)ds_y \\
&- \quad c\int_\Gamma \sigma(y)ds_y
\end{aligned}
$$

with $c = \frac{1}{2\pi}\,\log(2d)$ and $d = \max_{x,\,y\in\Gamma}|x-y|$.

Gårding's inequalities

$$V\sigma(x) := \int_\Gamma E(x,y)\sigma(y)ds_y, \quad x \in \Gamma$$

$$\langle \sigma, V\sigma \rangle \geqq c_0 \, ||\sigma||^2_{H^{-1/2}(\Gamma)} - c_1 \, ||\sigma||^2_{H^{-(1/2+\epsilon)}(\Gamma)}$$

$$\begin{aligned} \left\langle \begin{pmatrix} \sigma \\ \omega \end{pmatrix}, \mathcal{B} \begin{pmatrix} \sigma \\ \omega \end{pmatrix} \right\rangle \; &\geq \; c_0 \left\{ ||\sigma||^2_{H^{-1/2}(\Gamma)} + |\omega|^2 \right\} \\ &- \; c_1 \left\{ ||\sigma||^2_{H^{-(1/2+\epsilon)}(\Gamma)} + |\omega|^2 \right\} \end{aligned}$$

for all $(\sigma, \omega) \in H^{-1/2}(\Gamma) \times \mathbb{R}$; $\epsilon > 0$, a constant.

Boundary operator $\mathcal{B}$ for the modified system

$$\mathcal{B}\begin{pmatrix}\sigma\\ \omega\end{pmatrix} := \begin{pmatrix} V & 1 \\ \int_\Gamma \cdot \, ds & 0 \end{pmatrix}\begin{pmatrix}\sigma\\ \omega\end{pmatrix} = \begin{pmatrix} V\sigma + \omega \\ \int_\Gamma \sigma\, ds \end{pmatrix} = \begin{pmatrix} f \\ A \end{pmatrix}$$

$$\left\langle \begin{pmatrix}\chi\\ \kappa\end{pmatrix}, \mathcal{B}\begin{pmatrix}\sigma\\ \omega\end{pmatrix} \right\rangle = \langle \chi, V\sigma\rangle + \langle \chi, 1\rangle\omega + \kappa\langle \sigma, 1\rangle$$

for $(\sigma, \omega) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ and for all $(\chi, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$

Theorem [Hsiao and Wendland 1977]

Given $(f, A) \in H^{1/2}(\Gamma) \times \mathbb{R}$*, there exists a unique solution pair* $(\sigma, \omega) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ *of the system*

$$\boxed{\begin{aligned} \langle \chi, V\sigma \rangle + \langle \chi, 1 \rangle \omega &= \langle \chi, f \rangle, \\ \kappa \langle \sigma, 1 \rangle &= \kappa A \end{aligned}}$$

for all $(\chi, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$*.*

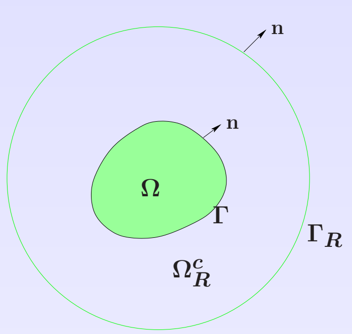
A closely related consequence

For $\sigma \in H^{-1/2}(\Gamma)$, let $u(x) := \int_\Gamma E(x,y)\sigma(y)ds_y, \quad x \in \mathbb{R}^2 \setminus \Gamma$.
Then $u \in H^1_{loc}(\mathbb{R}^2)$ satisfies

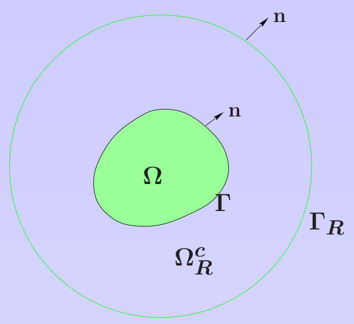
$$-\Delta u = 0, x \in \mathbb{R}^2 \setminus \Gamma, \quad [u]|_\Gamma = 0, \quad [\frac{\partial u}{\partial n}]|_\Gamma = \sigma$$

$$u - A\,log|x| = o(1)\ as\ |x| \to \infty,\ A = -\frac{1}{2\pi}\int_\Gamma \sigma ds.$$

$$([v]|_\Gamma := v^- - v^+)$$



$$\langle \sigma, V\sigma\rangle = \int_\Omega |\nabla u|^2 dx + \int_{\Omega_R^c} |\nabla u|^2 dx + \int_{\Gamma_R} \frac{\partial u}{\partial n} u\, ds$$
$$= a_\Omega(u,u) + a_{\Omega_R^c}(u,u) + \cdots$$



$$\langle \sigma, V\sigma \rangle = \int_\Omega |\nabla u|^2 dx + \int_{\Omega_R^c} |\nabla u|^2 dx + \int_{\Gamma_R} \frac{\partial u}{\partial n} u \, ds$$
$$= a_\Omega(u,u) + a_{\Omega_R^c}(u,u) + \cdots$$

Gårding's inequality for the bilinear $\langle \sigma, V\sigma \rangle$ for the boundary integral operator V on the boundary Γ is a consequence of the corresponding Gåriding's inequality of the bilinear form associated with a related transmission problem for the partial differential operator $-\Delta$ in the domain $\Omega \cup \Omega^c$

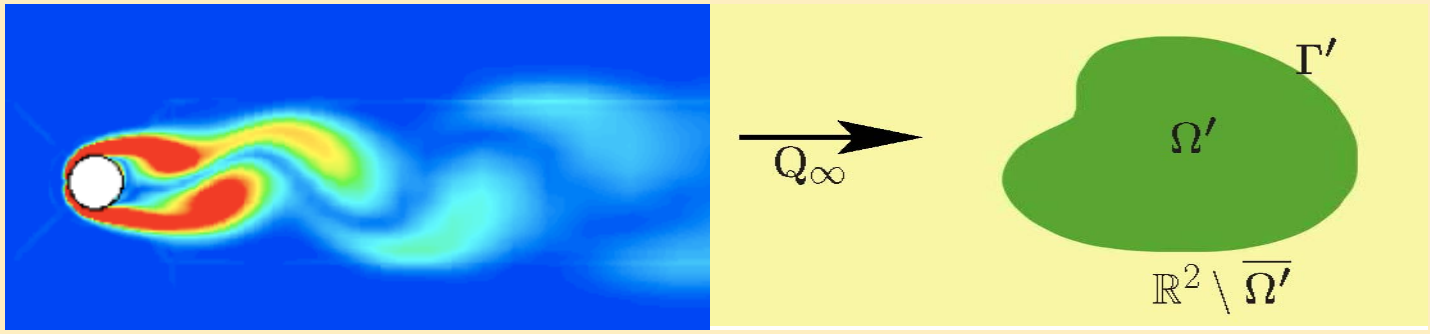
Generalization

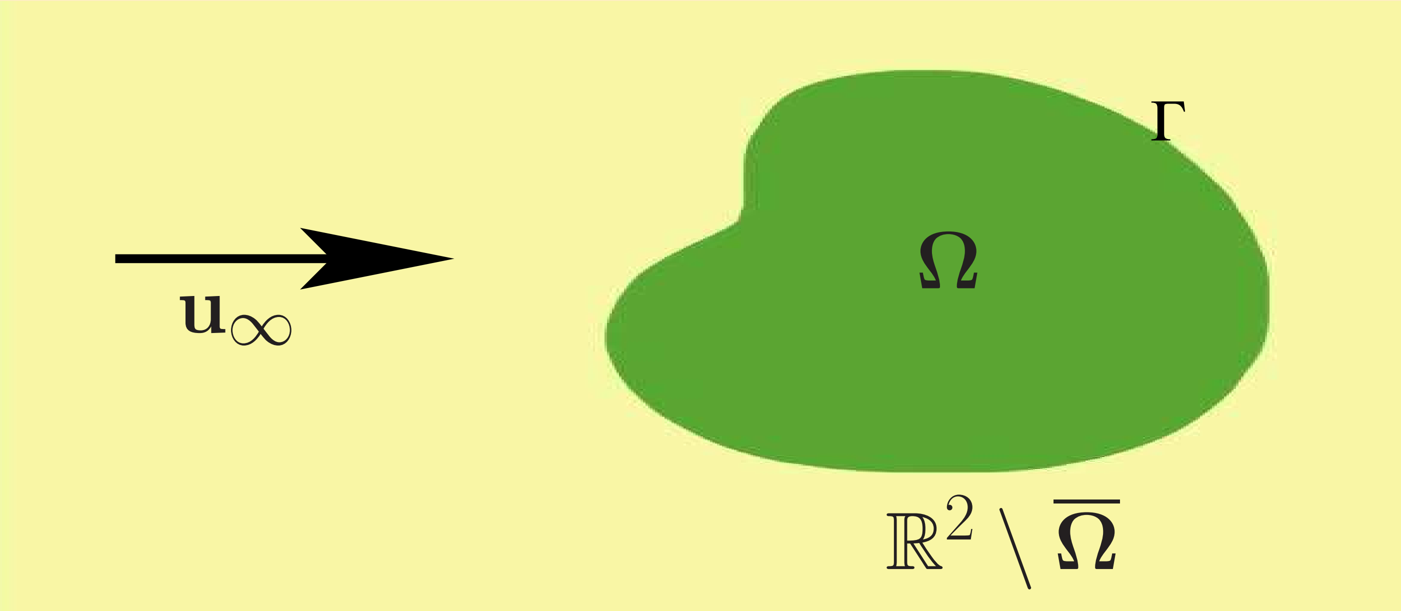
Costabel and Wendland (1986) gave a systematic approach for a general class of boundary integral operators associated with strongly elliptic boundary value problems. For such class of BIOs, Gårding's inequality is a consequence of strong ellipticity of the corresponding BVP for the PDE.

- Costabel, M. and Wendland, W. L.: *Strongly ellipticity of boundary integral operators.* J. Reine Angew. Mathematik **372** (1986) 34-63.
- Hsiao, G. C., and Wendland, W. L., *Boundary Integral Equations*, Applied Mathematical Series, **164**, Springer-Verlag, 2008

Exterior Boundary Value Problems

Viscous flow past an obstacle





$$\mathbf{u} := \frac{\mathbf{Q}}{|\mathbf{Q}_\infty|}, \quad p := \frac{DP}{\mu|\mathbf{Q}_\infty|}$$

$$\underline{\underline{\sigma}} := \frac{D\underline{\underline{\mathbf{\Sigma}}}}{\mu|\mathbf{Q}_\infty|} = -p\underline{\underline{I}} + \left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$$

$$\boxed{Re := \frac{D|\mathbf{Q}_\infty|}{\mu/\rho} << 1}$$

Dimensionless BVP (P_{Re})

$$-\Delta \mathbf{u} + Re(\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p = \mathbf{0}; \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega^c,$$
$$\mathbf{u}|_\Gamma = \mathbf{0} \quad \text{on} \quad \Gamma,$$
$$\mathbf{u} \to \mathbf{u}_\infty, \quad p \to 0 \quad \text{as} \quad |x| \to \infty.$$

The force exerted on the obstacle by the fluid is

$$\mathbf{f} := \frac{\mathbf{F}}{\mu |\mathbf{Q}_\infty|} = \int_\Gamma \underline{\underline{\boldsymbol{\sigma}}}[\hat{n}]\, ds.$$

Stokes Paradox

The reduced problem (P_0) has no solution.

A modified Stokes problem

$$-\Delta \mathbf{u}_0 + \nabla p_0 = \mathbf{0}; \quad \nabla \cdot \mathbf{u}_0 = 0 \quad \text{in} \quad \Omega^c,$$
$$\mathbf{u}_0|_\Gamma = \mathbf{q} \quad \text{on} \quad \Gamma,$$
$$\mathbf{u} - \mathbf{A}\, log|x| = O(1), \quad p_0 = o(1) \quad \text{as} \quad |x| \to \infty,$$

where $\mathbf{A}$ is any given vector in $\mathbb{R}^2$ and $\mathbf{q}$ is a prescribed function satisfying

$$\int_\Gamma \mathbf{q} \cdot \hat{n}\, ds = 0.$$

Solution of the modified Stokes problem

Theorem [Hsiao and MacCamy 1973]

For given $(\mathbf{q}, \mathbf{A}) \in (C^{1+\alpha}(\Gamma))^2 \times \mathbb{R}^2$ with $\int_\Gamma \mathbf{q} \cdot \hat{n}\, ds = 0$, there exists a unique "solution" pair $(\mathbf{u}, p)$ of the modified Stokes problem which can be represented by the simple-layer of the form

$$\mathbf{u}(x) = \int_\Gamma \underline{\underline{\mathbf{E}}}(x,y) \cdot \boldsymbol{\sigma}(y) ds_y + \boldsymbol{\omega}, \quad p(x) = \int_\Gamma \underline{\varepsilon}(x,y) \cdot \boldsymbol{\sigma}(y) ds_y$$

for $x \in \Omega^c$, and $(\boldsymbol{\sigma}, \boldsymbol{\omega}) \in (C^\alpha(\Gamma))^2 \times \mathbb{R}^2$ is the unique solution of the system

$$V\boldsymbol{\sigma} + \boldsymbol{\omega} = \mathbf{q} \quad \text{on} \quad \Gamma \quad \text{and} \quad -\frac{1}{4\pi}\int_\Gamma \boldsymbol{\sigma}\, ds = \mathbf{A}$$

subject to the constrain: $\int_\Gamma \boldsymbol{\sigma} \cdot \hat{n}\, ds = 0$.

- The simply-layer BIO V:

$$V\boldsymbol{\sigma}(x) \quad := \quad \int_\Gamma \underline{\underline{\boldsymbol{E}}}(x,y)\cdot\boldsymbol{\sigma}(y)ds_y, \quad x\in\Gamma.$$

- The fundamental solution $(\underline{\underline{\mathbf{E}}}, \underline{\epsilon})$ of the Stokes equations defined by

$$-\Delta\underline{\underline{\mathbf{E}}}(x,y)+\nabla\underline{\epsilon}(x,y)=\delta(|x-y|)\underline{\underline{\mathbf{I}}}$$
$$div\ \underline{\underline{\mathbf{E}}}(x,y)=\mathbf{0}$$

can be computed explicitly :

$$\underline{\underline{\mathbf{E}}}(x,y) \quad = \quad -\frac{1}{4\pi}\{\log|x-y|+(1+\gamma_o-\log 4)\}\underline{\underline{I}}+\frac{(x-y)(x-y)^t}{|x-y|^2},$$
$$\underline{\varepsilon}(x,y) \quad = \quad \frac{1}{2\pi}\nabla_y\log|x-y| \quad \text{with the Euler's constant } \gamma_o.$$

Main results

Theorem [Hsiao and MacCamy 1982]

- Let $\mathcal{D}$ be any compact subset of $\bar{\Omega}^c$. Then the classical solution $(\mathbf{u}, p)$ of the viscous flow problem admits the asymptotic expansions:

$$\begin{aligned}
\mathbf{u}(x) &= \sum_{k=1}^{2} \frac{\mathbf{u}_k(x)}{(\log\ Re)^k} + O((\log\ Re)^{-3}),\\
p(x) &= \sum_{k=1}^{2} \frac{p_k(x)}{(\log\ Re)^k} + O((\log Re)^{-3})
\end{aligned}$$

as $Re \to 0^+$, uniformly on $\mathcal{D}$.

- The force $\mathbf{f}$ exerted on the obstacle by the fluid admits the asymptotic expansion

$$\mathbf{f} = 4\pi \left\{ \frac{\mathbf{A}_1}{(log\ Re)} + \frac{\mathbf{A}_2}{(log\ Re)^2} \right\} + O((log\ Re)^{-3})$$

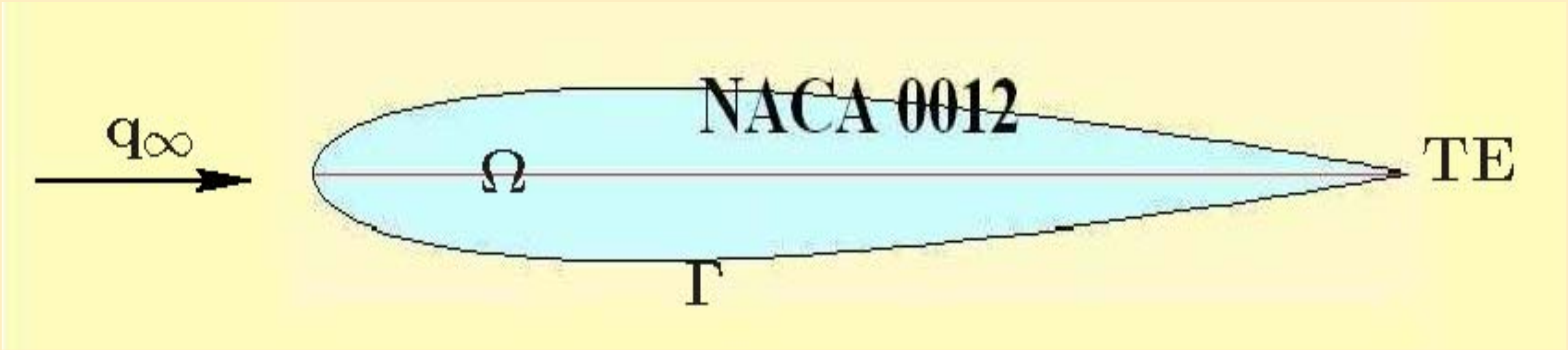
as $Re \to 0^+$.

Here $(\mathbf{u}_k, p_k)$ are solutions of the modified Stokes problem with $\mathbf{q} = \mathbf{0}$ and $\mathbf{A} = \mathbf{A}_k$ for $k = 1, 2$. The constant vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ are coincided with those obtained by the singular perturbation procedure developed by Hsiao and MacCamy in [1973]. Moreover the constant vector $\mathbf{A}_1$ is independent of the shape of the obstacle. In particular,

$$\mathbf{A}_1 = -\hat{\mathbf{e}}_1,$$

if $\mathbf{u}_\infty = \hat{\mathbf{e}}_1$ (the unit vector in the x_1 -direction).

Potential flow past an airfoil



The NACA airfoils are airfoil shapes for aircraft wings developed by the National Advisory Committee for Aeronautics (NACC). The shape of the NACA airfoils is described using a 4 -digit series following the word NACA. The formula for the shape of a NACA 00xx foil, with xx being replaced by the percentage of thickness to chord.

Formulation (velocity $\mathbf{q} = (q_1, q_2)'$)

$$\nabla \times \mathbf{q} \quad := \quad \frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2} = 0 \quad \text{in} \quad \Omega^c$$

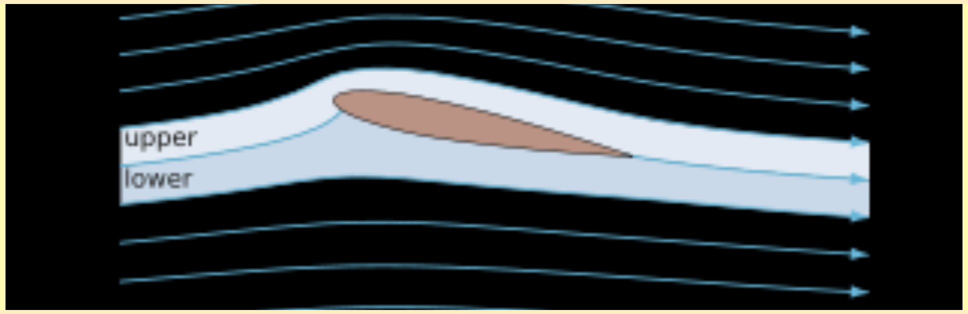
$$\nabla \cdot \mathbf{q} \quad := \quad \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0 \quad \text{in} \quad \Omega^c$$

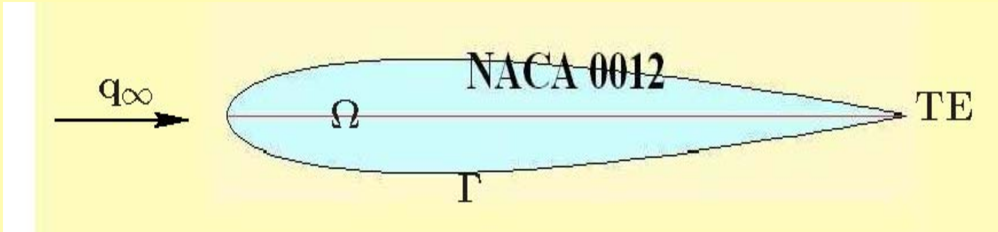
$$\mathbf{q} \cdot \hat{n}|_\Gamma = 0 \quad \text{on} \quad \Gamma$$

$$\mathbf{q} - \mathbf{q}_\infty = o(1) \quad \text{as} \quad |x| \to \infty$$

$$\lim_{\Omega^c \ni x \to \text{TE}} \quad |\mathbf{q}(x)| = |\ \mathbf{q}\ |_{TE} \quad \text{exists at TE}$$

+ (*Kutta-Joukowski cond.*)





- Circulation

$$\kappa := \int_\Gamma \mathbf{q} \cdot dx = -\int_\Gamma \frac{\partial u}{\partial n} ds$$

- The Kutta-Joukowski condition states that *the circulation around the airfoil should be such that the perturbed velocity* $\mathbf{q} - \mathbf{q}_\infty$ *should be finite and continuous at the trailing edge TE.*

Formulation (Stream Function $\mathbf{q} = (\nabla\psi)^{\perp}$)

$$u := \psi - \psi_\infty, \quad \psi_\infty := -\mathbf{q}_\infty{}^{\perp} \cdot x$$

$$-\Delta u = 0 \quad \text{in} \quad \Omega^c,$$
$$u|_\Gamma = -\psi_\infty \quad \text{on} \quad \Gamma,$$
$$u + \frac{\kappa}{2\pi} log|x| = \omega + O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty$$

+ (*Kutta-Joukowski cond.*)

Remark: A subsonic flow past a given profile Γ is uniquely determined by its free stream velocity $\mathbf{q}_\infty$ and its circulation κ (or equivalently the Kutta -Joukowski condition at TE).

Theorem

Given $\psi_\infty := -\mathbf{q}_\infty^\perp \cdot x$, the problem of potential flow past an airfoil has a unique solution pair $(\kappa, u) \in \mathbb{R} \times H^{2+\epsilon}_{loc}(\Omega^c)$.

Green's Representation for u $(\mu := u|_\Gamma,\ \sigma := \partial u/\partial n|_\Gamma)$:

$$u(x) = \int_\Gamma \mu(x) \frac{\partial E}{\partial n_y}(x,y) ds_y - \int_\Gamma E(x,y)\sigma(y) ds_y + \omega, \ \ x \in \Omega^c$$

$$\sim \frac{1}{2\pi}\left(\int_\Gamma \sigma \ ds\right) log|x| + \omega + O(|x|^{-1})$$

$$\Longrightarrow \int_\Gamma \sigma ds = -\kappa.$$

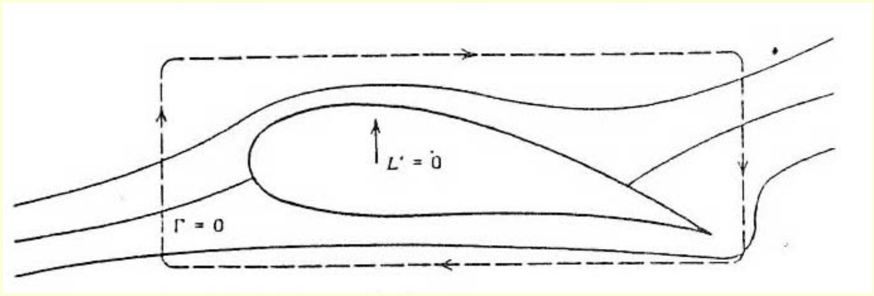
Boundary Integral Equations

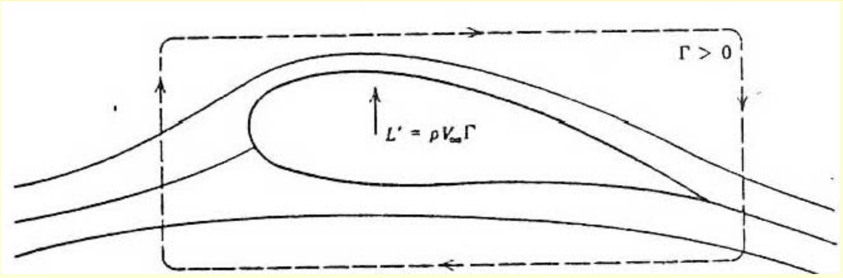
$$V\sigma - \omega = f, \quad \int_\Gamma \sigma ds = -\kappa$$

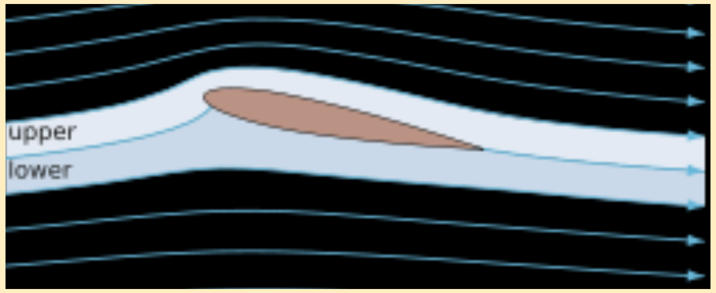
$$+(\textit{Kutta-Joukowski cond.})$$

$$f := \{(-1 + \theta_x/2\pi)I + K\}(-\psi_\infty)$$

Circulation and flow pattern







Solution Procedure: $\sigma = \sigma_0 - \kappa\sigma_1, \quad \omega = \omega_0 - \kappa\omega_1,$

$$V\sigma_0 - \omega_0 = f, \qquad V\sigma_1 - \omega_1 = 0,$$
$$\int_\Gamma \sigma_0 ds = 0, \qquad \int_\Gamma \sigma_1 ds = 1.$$

$$\kappa := \lim_{\rho\to 0} \frac{\rho^\beta \sigma_0}{\rho^\beta \sigma_1}, \quad \beta > 0 \text{ (to be determined)}$$

$$\sigma_i = c_i\sigma_s + \text{regular term}, i = 0, 1$$

$$\sigma_s = O(\rho^{\frac{\pi}{2\pi-\alpha\pi}-1}) = O(\rho^{-\frac{1-\alpha}{2-\alpha}}), \quad \beta := \tfrac{1-\alpha}{2-\alpha}$$

$$\Longrightarrow \sigma_0 - \kappa\sigma_1 = (c_0 - \kappa c_1)\sigma_s + \text{ regular term}$$

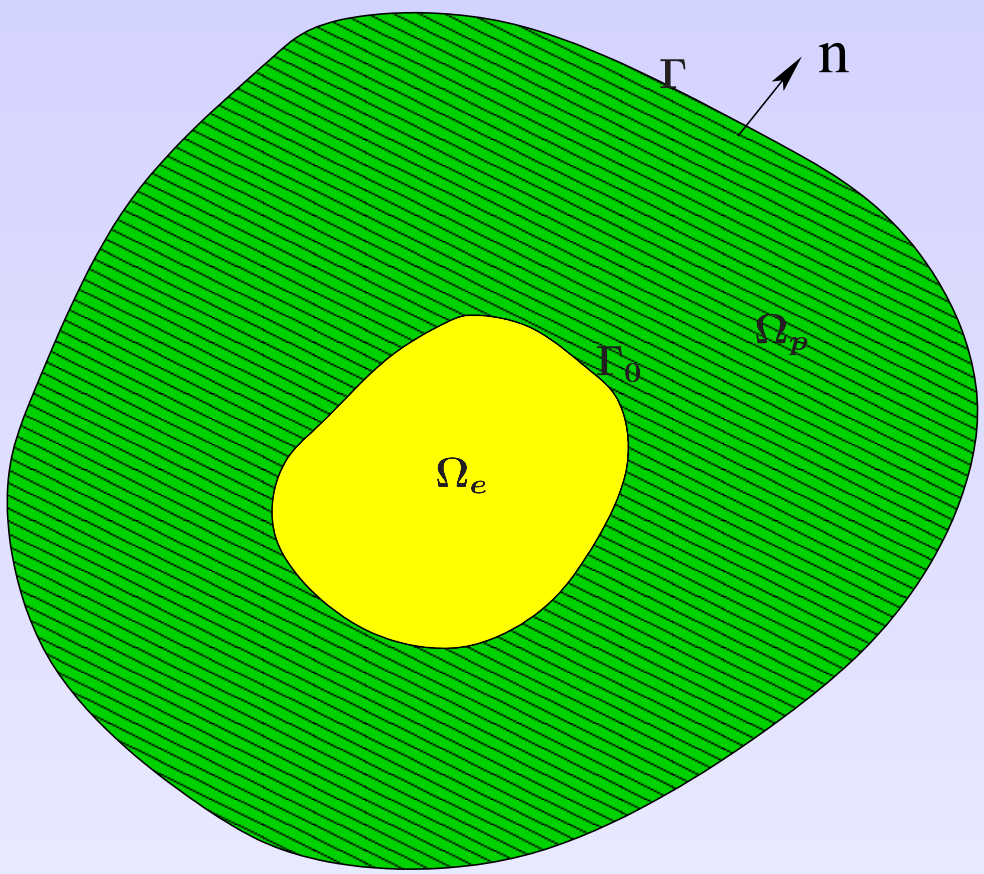
$$\kappa = c_0/c_1$$

(see e.g., Hsiao (1990), Hsiao, Marcozzi & Zhang (1993))

Transmission Problems

A nonlinear interface problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain with boundary Γ. We assume that Ω is decomposed into two subdomains: Ω_e and Ω_p :



- $\Omega_e \subset \Omega$, an inner region with linear elastic material
- $\Omega_p = \overline{\Omega} \setminus \Omega_e$, an annual region with elasto-plastic material .

The interface problem can be formulated as follows:

For given body force $\mathbf{f}$ in Ω_p, find the displacement field $\mathbf{u}$ satisfying

$$-L(\mathbf{u}) := -\operatorname{div} \underline{\underline{\sigma}}(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \Omega_e$$
$$-N(\mathbf{u}) := -\operatorname{div} \underline{\underline{\sigma}}_p(\mathbf{u}) = \mathbf{f} \quad \text{in} \quad \Omega_p$$
$$\mathbf{u}|_\Gamma = \mathbf{0} \quad \text{on } \Gamma,$$
$$\mathbf{u}^- = \mathbf{u}^+, \ \mathbf{T}_p(\mathbf{u})^- = \mathbf{T}(\mathbf{u})^+ \quad \text{on} \quad \Gamma_0,$$

Here the traction operators $\mathbf{T}_p$ and $\mathbf{T}$ are defined by

$$\mathbf{T}_p(\mathbf{u})^- = \underline{\underline{\sigma}}_p(\mathbf{u})\hat{n}|_{\Gamma_0} \quad \text{and} \quad \mathbf{T}(\mathbf{u})^+ = \underline{\underline{\sigma}}(\mathbf{u})\hat{n}|_{\Gamma_0},$$

where $\hat{n}$ denotes the outward unit normal to Γ_0.

Linear elastic material

For the linear elastic material in Ω_e, we assume the standard Hooke's law for the isotropic material,

$$\underline{\underline{\sigma}}(\mathbf{u})(x) = \lambda \operatorname{div} \mathbf{u}(x)\underline{\underline{I}} + 2\mu\underline{\underline{\varepsilon}}(\mathbf{u})(x)$$

for all $x \in \Omega_e$, where λ and μ are the Lamé constants such that $\mu > 0$, and $3\lambda + 2\mu > 0$. Hence, the partial differential equation in Ω_e is linear, namely the Lamé equation,

$$L(\mathbf{u}) := \operatorname{div} \underline{\underline{\sigma}}(\mathbf{u}) = \mathbf{0}$$

The elasto-plastic material

For the elasto-plastic material in Ω_p, we assume the Hencky-Mises stress-strain relation:

$$\begin{aligned} \underline{\underline{\boldsymbol{\sigma}}}_p(\mathbf{u})(x) &= \left[\kappa(x) - \frac{2}{3}\tilde{\mu}(x, G(\mathbf{u})(x))\right] \operatorname{div}\mathbf{u}(x)\underline{\underline{I}} \\ &+ 2\tilde{\mu}(x, G(\mathbf{u})(x))\underline{\underline{\varepsilon}}(\mathbf{u})(x) \end{aligned}$$

where $\kappa : \Omega_p \to \mathbb{R}$ is the bulk modulus, $\tilde{\mu} : \Omega_p \times \mathbb{R}^+ \to \mathbb{R}$ the Lamé function, and

$$G(\mathbf{u}) := \underline{\underline{\varepsilon}}^*(\mathbf{u}) : \underline{\underline{\varepsilon}}^*(\mathbf{u}) \quad \text{with} \quad \underline{\underline{\epsilon}}^*(\mathbf{u}) := \underline{\underline{\varepsilon}}(\mathbf{u}) - \frac{1}{3}(\operatorname{div}\mathbf{u})\underline{\underline{\mathbf{I}}}$$

being the deviator of the small strain tensor.

Here the functions κ and $\tilde{\mu}$ are supposed to be continuous, and $\tilde{\mu}(x, \cdot)$ to be continuously differentiable in $\mathbb{R}^+$ such that the following estimates hold:

$$0 < \kappa_0 < \kappa(x) \leq \kappa_1$$

for all $x \in \bar{\Omega}_p$, while

$$0 < \mu_0 \leq \tilde{\mu}(x, \eta) \leq \frac{3}{2}\kappa(x), \quad \text{and}$$
$$0 < \mu_1 \leq \tilde{\mu}(x, \eta) + 2\left(\frac{\partial}{\partial \eta}\tilde{\mu}(x, \eta)\right)\eta \leq \mu_2$$

for all $(x, \eta) \in \bar{\Omega}_p \times \mathbb{R}^+$.

The interface problem can be formulated as follows:

For given body force $\mathbf{f}$ in Ω_p, find the displacement field $\mathbf{u}$ satisfying

$$-L(\mathbf{u}) := -\operatorname{div} \underline{\underline{\sigma}}(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \Omega_e$$

$$-N(\mathbf{u}) := -\operatorname{div} \underline{\underline{\sigma}}_p(\mathbf{u}) = \mathbf{f} \quad \text{in} \quad \Omega_p$$

$$\mathbf{u}|_\Gamma = \mathbf{0} \quad \text{on } \Gamma,$$

$$\mathbf{u}^- = \mathbf{u}^+, \; \mathbf{T}_p(\mathbf{u})^- = \mathbf{T}(\mathbf{u})^+ \quad \text{on} \quad \Gamma_0,$$

Here the traction operators $\mathbf{T}_p$ and $\mathbf{T}$ are defined by

$$\mathbf{T}_p(\mathbf{u})^- = \underline{\underline{\sigma}}_p(\mathbf{u})\hat{n}|_{\Gamma_0} \quad \text{and} \quad \mathbf{T}(\mathbf{u})^+ = \underline{\underline{\sigma}}(\mathbf{u})\hat{n}|_{\Gamma_0},$$

where $\hat{n}$ denotes the outward unit normal to Γ_0.

Reduction to non-local problem

From the Betti formula, we represent $\mathbf{u}$ in Ω_e in the form

$$
\begin{aligned}
\mathbf{u}(x) &= \int_{\Gamma_0} \underline{\underline{\mathbf{E}}}(x,y)\mathbf{T}(\mathbf{u})^-(y)ds_y - \int_{\Gamma_0} (\mathbf{T}_y(\underline{\underline{\mathbf{E}}}(x,y)))^t\mathbf{u}^-(y)ds_y \\
&= \mathcal{S}(\mathbf{T}(\mathbf{u})^-) - \mathcal{D}(\mathbf{u}^-), \quad x \in \Omega_e,
\end{aligned}
$$

from which, we now arrive at the system of BIOs on Γ_0:

$$
\begin{aligned}
\mathbf{u}^- &= V(\mathbf{T}(\mathbf{u})^-) + \left(\frac{1}{2}I - K\right)\mathbf{u}^-, \\
\mathbf{T}(\mathbf{u})^- &= \left(\frac{1}{2}I + K'\right)(\mathbf{T}(\mathbf{u})^-) + W\mathbf{u}^-
\end{aligned}
$$

where V, K, K' and W are the four basic boundary integral operators:

Four basic boundary integral operators

$$
\begin{aligned}
V\boldsymbol{\sigma}(x) &:= \int_{\Gamma_0} \underline{\underline{\mathbf{E}}}(x,y)\cdot\boldsymbol{\sigma}(y)ds_y \quad (\textit{simple-layer}),\\
K\boldsymbol{\mu}(x) &:= \int_{\Gamma_0} \Big(\mathbf{T}_y\underline{\underline{\mathbf{E}}}(x,y)\Big)^t\cdot\boldsymbol{\mu}(y)ds_y \quad (\textit{double-layer})\\
K'\boldsymbol{\sigma}(x) &:= \int_{\Gamma_0} \mathbf{T}_x\underline{\underline{\mathbf{E}}}(x,y)\cdot\boldsymbol{\sigma}(y)ds_y \quad (\textit{transpose of } K),\\
W\boldsymbol{\mu}(x) &:= -\mathbf{T}_x\int_{\Gamma_0} \Big(\mathbf{T}_y\underline{\underline{\mathbf{E}}}(x,y)\Big)^t\cdot\boldsymbol{\mu}(y)ds_y \quad (\textit{hypersingular}).
\end{aligned}
$$

$$
\underline{\underline{\mathbf{E}}}(x,y) = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\left\{\frac{1}{|x-y|}\underline{\underline{I}} + \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)\frac{(x-y)(x-y)^t}{|x-y|^3}\right\}
$$

is the fundamental tensor for the Lamé equation .

$$\boxed{\begin{aligned} \mathbf{u}^- &= V\left(\mathbf{T}(\mathbf{u})^-\right) + \left(\frac{1}{2}I - K\right)\mathbf{u}^-, \\ \mathbf{T}(\mathbf{u})^- &= \left(\frac{1}{2}I + K'\right)\mathbf{T}(\mathbf{u})^- + W\mathbf{u}^- \end{aligned}}$$

Now we use the interface conditions:

$$\mathbf{u}^- = \mathbf{u}^+, \quad \mathbf{T}_p(\mathbf{u})^+ = \mathbf{T}(\mathbf{u})^- := \boldsymbol{\sigma}$$

$$V\boldsymbol{\sigma} - \left(\frac{1}{2}I + K\right)\mathbf{u}^+ = \mathbf{0} \quad \text{on} \quad \Gamma_0$$

$$\mathbf{T}_p(\mathbf{u})^+ = (\frac{1}{2}I + K')\boldsymbol{\sigma} + W\mathbf{u}^+ \quad \text{on} \quad \Gamma_0$$

Nonlocal boundary problem

Nonlocal boundary problem for the unknowns **u** and $\boldsymbol{\sigma}$:

$$-N(\mathbf{u}) = \mathbf{f} \quad \text{in} \quad \Omega_p$$

$$\mathbf{u}|_\Gamma = \mathbf{0} \quad \text{on } \Gamma,$$

$$\mathbf{T}_p(\mathbf{u})^+ = (\frac{1}{2}I + K')\boldsymbol{\sigma} + W\mathbf{u}^+ \quad \text{on} \quad \Gamma_0$$

together with the nonlocal boundary condition

$$V\boldsymbol{\sigma} - \left(\frac{1}{2}I + K\right)\mathbf{u}^+ = \mathbf{0} \quad \text{on } \Gamma_0.$$

Variational formulation

$$(H^1_\Gamma(\Omega_p))^3 = \{\mathbf{v} \in (H^1(\Omega_p))^3 : \mathbf{v}|_\Gamma = 0\}.$$

Given $\mathbf{f} \in (L^2(\Omega_p))^3$, find $(\mathbf{u}, \boldsymbol{\sigma}) \in (H^1_\Gamma(\Omega_p))^3 \times (H^{-1/2}(\Gamma_0))^3$ such that

$$A_{\Omega_p}(\mathbf{u}, \mathbf{v}) + B_{\Gamma_0}((\mathbf{u}, \boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\lambda})) = \int_{\Omega_p} \mathbf{f} \cdot \mathbf{v} \, dx.$$

for all $(\mathbf{v}, \boldsymbol{\lambda}) \in (H^1_\Gamma(\Omega_p))^3 \times (H^{-1/2}(\Gamma_0))^3$.

In the formulation, B_{Γ_0} is the bilinear form on Γ_0,

$$B_{\Gamma_0}((\mathbf{u}, \boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\lambda})) := \langle \left(\frac{1}{2} I + K'\right) \boldsymbol{\sigma}, \mathbf{v}^+ \rangle_{\Gamma_0} + \langle W\mathbf{u}^+, \mathbf{v}^+ \rangle_{\Gamma_0}$$

$$+ \langle \boldsymbol{\lambda}, V\boldsymbol{\sigma} \rangle_{\Gamma_0} - \langle \boldsymbol{\lambda}, \left(\frac{1}{2} I + K\right) \mathbf{u}^+ \rangle_{\Gamma_0}$$

An operator equation

Let $\mathcal{H}^*$ denote the dual of $\mathcal{H} := (H^1_\Gamma(\Omega_p))^3 \times (H^{-1/2}(\Gamma_0))^3$.

Define a nonlinear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}^*$ on $\mathcal{H}$ by putting

$$[\mathcal{A}(\mathbf{u}, \boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\lambda})] := A_{\Omega_p}(\mathbf{u}, \mathbf{v}) + B_{\Gamma_0}((\mathbf{u}, \boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\lambda})).$$

Then the nonlocal boundary problem can be reformulated in the form of an operator equation for the unknowns $(\mathbf{u}, \boldsymbol{\sigma})$:

$$\mathcal{A}(\mathbf{u}, \boldsymbol{\sigma}) = \mathcal{F},$$

where $\mathcal{F} \in \mathcal{H}^*$ is defined by

$$[\mathcal{F}, (\mathbf{v}, \boldsymbol{\lambda})] := \int_{\Omega_p} \mathbf{f} \cdot \mathbf{v} \, dx.$$

Main results

Theorem

Under the assumptions on the bulk modulus κ and on the Lamé function $\tilde{\mu}$, we have the following:

- *The operator $\mathcal{A}$ is both strongly monotone and Lipschitz continuous on $\mathcal{H}$.*
- *Given $\mathbf{f} \in (L^2(\Omega_p))^3$, there exists a unique solution $(\mathbf{u}, \boldsymbol{\sigma}) \in \mathcal{H}$ of the operator equation:*

$$\mathcal{A}(\mathbf{u}, \boldsymbol{\sigma}) = \mathcal{F}.$$

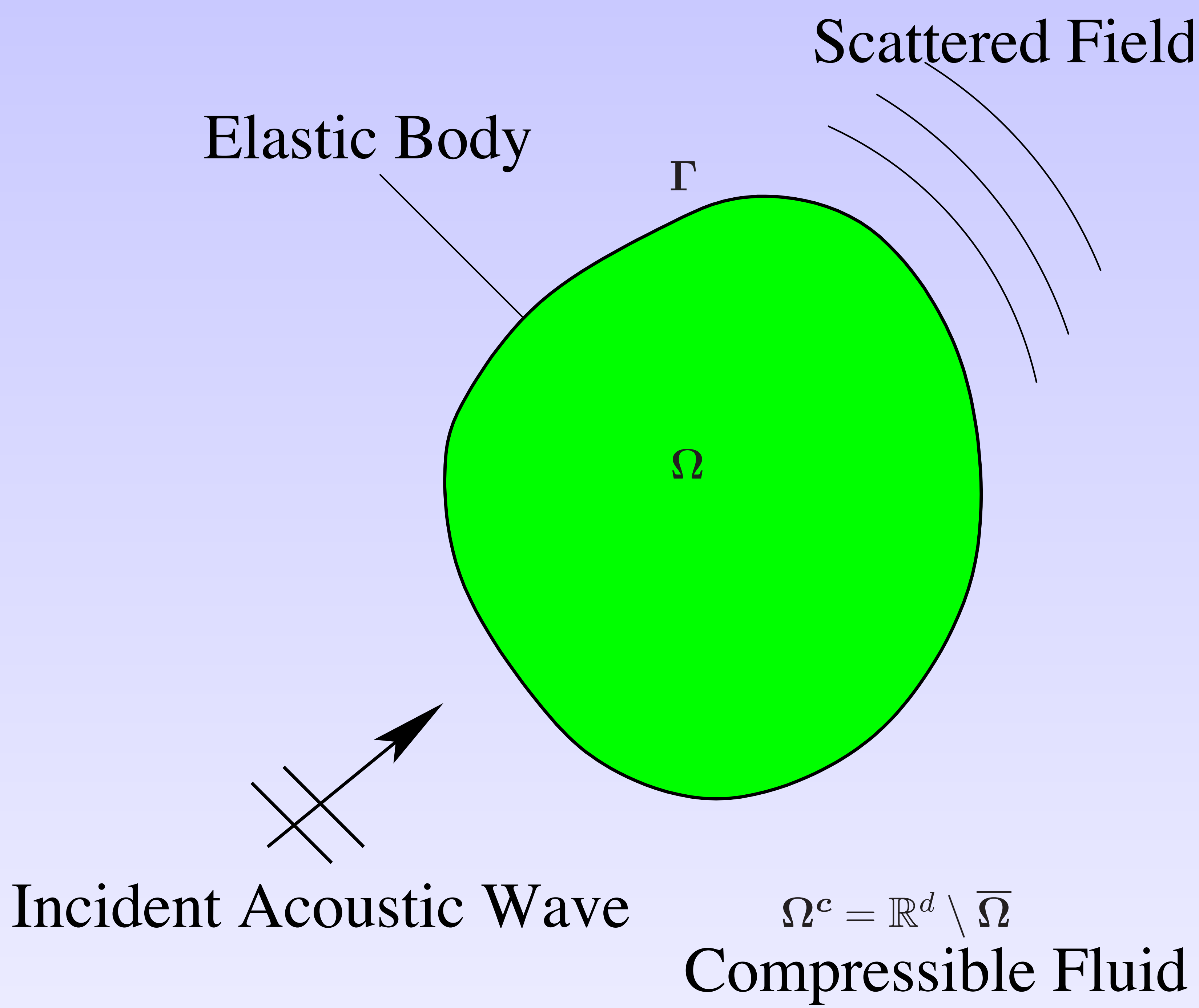
(see, e.g., Gatica and Hsiao (1989, 1990))

Remarks:

Linear-Nonlinear Transmission Problems

- Strongly monotone type:
  Polizzoto (1987), Gatica & Hsiao (1989, 1990, 1992, 1995), Stephan & Costabel (1990), Carstensen (1992), Gatica & Wendland (1994)
- Non-monotone type:
  Gatica & Hsiao (1992), Berger, Warneke, & Wendland (1993, 1994), Feistauer, Hsiao, Kleinman & Tezaur (1994, 1995).

Fluid-solid interaction in $\mathbb{R}^d$, $d = 2, 3$



Find $\mathbf{u} \in (H^1(\Omega))^d$ in Ω and $p^s \in H^1_{loc}(\Omega^c)$ in Ω^c such that

$$(E)\quad \begin{cases} \Delta^*\mathbf{u} + \rho\omega^2\mathbf{u} = \mathbf{0} \quad \text{in} \quad \Omega \quad (\text{elastic body}) \\ \Delta p^s + k^2 p^s = 0 \quad \text{in} \quad \Omega^c \quad (\text{fluid}) \end{cases}$$

$$(B)\quad \left.\begin{cases} \mathbf{T}(\mathbf{u})^- = -(p^{s+} + p^{inc})\hat{n} \\ \mathbf{u}^- \cdot \hat{n} = \dfrac{1}{\rho_f\omega^2}\left(\dfrac{\partial p^{s+}}{\partial n} + \dfrac{\partial p^{inc}}{\partial n}\right) \end{cases}\right\} \quad \text{on} \quad \Gamma$$

$$(C)\quad \frac{\partial p^s}{\partial r} - ikp^s = o(r^{-(d-1)/2}) \quad \text{as} \quad r = |x| \longrightarrow \infty.$$

Basic idea

The basic idea is agin to transform the original transmission problem to an equivalent nonlocal boundary problem for u in the bounded domain Ω together with an appropriate boundary integral equation for p^{s+} on the interface Γ in terms of the transmission conditions on the interface. For instance, if we represent p^s in Ω^c,

$$\begin{aligned} p^s(x) &= \int_\Gamma \frac{\partial E}{\partial n_y}(x,y)p^{s+}(y)ds_y - \int_\Gamma E(x,y)\frac{\partial p^{s+}(y)}{\partial n_y}ds_y \\ &= \mathcal{D}(p^{s+})(x) - \mathcal{S}(\frac{\partial p^{s+}}{\partial n})(x), \end{aligned}$$

then we arrive at a non-local boundary problem of the form

Nonlocal boundary problem

Given $p^{inc} \in H^{1/2}(\Gamma)$, find $(u, p^{s+}) \in (H^1(\Omega))^3 \times H^{1/2}(\Gamma)$ satisfying

$$\Delta^* \mathbf{u} + \rho\omega^2 \mathbf{u} = \mathbf{0} \quad \text{in} \quad \Omega$$

$$\mathbf{T}(\mathbf{u}) = -(p^{s+} + p^{inc})\hat{n} \quad \text{on} \quad \Gamma$$

$$Wp^{s+} + \rho_f\omega^2(\frac{1}{2}I + K')(\mathbf{u}^- \cdot \hat{n}) = (\frac{1}{2} + K')\frac{\partial p^{inc}}{\partial n} \quad \text{on} \quad \Gamma$$

Theorem

Assume that the homogeneous BVP of (E), (B), (C) has no traction free solution and that k is not an exceptional value. Then the corresponding non-local boundary problem has unique solution $(\mathbf{u}, p^{s+})$ *in* $(H^1(\Omega))^3 \times H^{1/2}(\Gamma)$.

(see, e.g., Hsiao (1989), Hsiao, Kleinman & Roach (2000).)

Representations of p^s in Ω^c

Method	Rep. of p^s in Ω^c	Traction $\mathbf{T}(\mathbf{u})$ on Γ
1	$p^s = \mathcal{D}p^{s+} - \mathcal{S}\sigma$ $p^{s+} \in H^{1/2}(\Gamma), \quad \sigma \in H^{-1/2}(\Gamma)$ $\sigma := \rho_f \omega^2(\mathbf{u}^- \cdot \hat{n}) - \frac{\partial p^{inc}}{\partial n}$	$-(p^{s+} + p^{inc})\mathbf{n}$
2(a)		*(same as above)*
2(b)		*(same as above)*
3		$-((\frac{1}{2}I + K)p^{s+} - V\sigma + p^{inc})\hat{n}$
4		$-(p^{s+} + p^{inc})\hat{n}$
5	$p^s = \mathcal{D}\mu, \quad \mu \in H^{1/2}(\Gamma)$	$-((\frac{1}{2}I + K)\mu + p^{inc})\hat{n}$
6(a)	$p^s = -\mathcal{S}\phi$ $\phi \in H^{-1/2}(\Gamma)$	$-(-V\phi + p^{inc})\hat{n}$
6(b)		*(same as above)*
7	$p^s = \mathcal{D}\mu + i\eta\mathcal{S}\mu, \quad \mu \in H^{1/2}(\Gamma)$	$-((\frac{1}{2}I + K)\mu + i\eta V\mu + p^{inc})\hat{n}$

Nonlocal BC's

Method	NLBC on Γ	Weak Form
1	$Wp^{s+} + (\frac{1}{2}I + K')\sigma = 0$	$< \cdot, \overline{q} >_0 = 0, \qquad q \in H^{1/2}(\Gamma)$
2(a)	$(\frac{1}{2}I - K)p^{s+} + V\sigma = 0$	$< \cdot, q >_{1/2} = 0, \qquad q \in H^{1/2}(\Gamma)$
2(b)	*(same as above)*	$< \cdot, \overline{Wq} >_0 = 0, \qquad q \in H^{1/2}(\Gamma)$
3	$Wp^{s+} + (\frac{1}{2}I + K')\sigma = 0$	$< \cdot, \overline{q} >_0 = 0, \qquad q \in H^{1/2}(\Gamma)$
4	$Wp^{s+} + (\frac{1}{2}I + K')\sigma$ $+ i\eta((\frac{1}{2}I - K)p^{+} + V\sigma) = 0$	$< \cdot, \overline{q} >_0 = 0, \qquad q \in H^{1/2}(\Gamma)$
5	$W\mu + \sigma = 0$	$< \cdot, \overline{\nu} >_0 = 0\ , \qquad \nu \in H^{1/2}(\Gamma)$
6(a)	$(\frac{1}{2}I - K')\phi - \sigma = 0$	$< \cdot, \psi >_{-1/2} = 0, \qquad \psi \in H^{-1/2}(\Gamma)$
6(b)	*(same as above)*	$< \cdot, \overline{V\psi} >_0 = 0, \qquad \psi \in H^{1/2}(\Gamma)$
7	$W\mu + i\eta(\frac{1}{2}I - K')\mu + \sigma = 0$	$< \cdot, \overline{\nu} >_0 = 0, \qquad \nu \in H^{1/2}(\Gamma)$

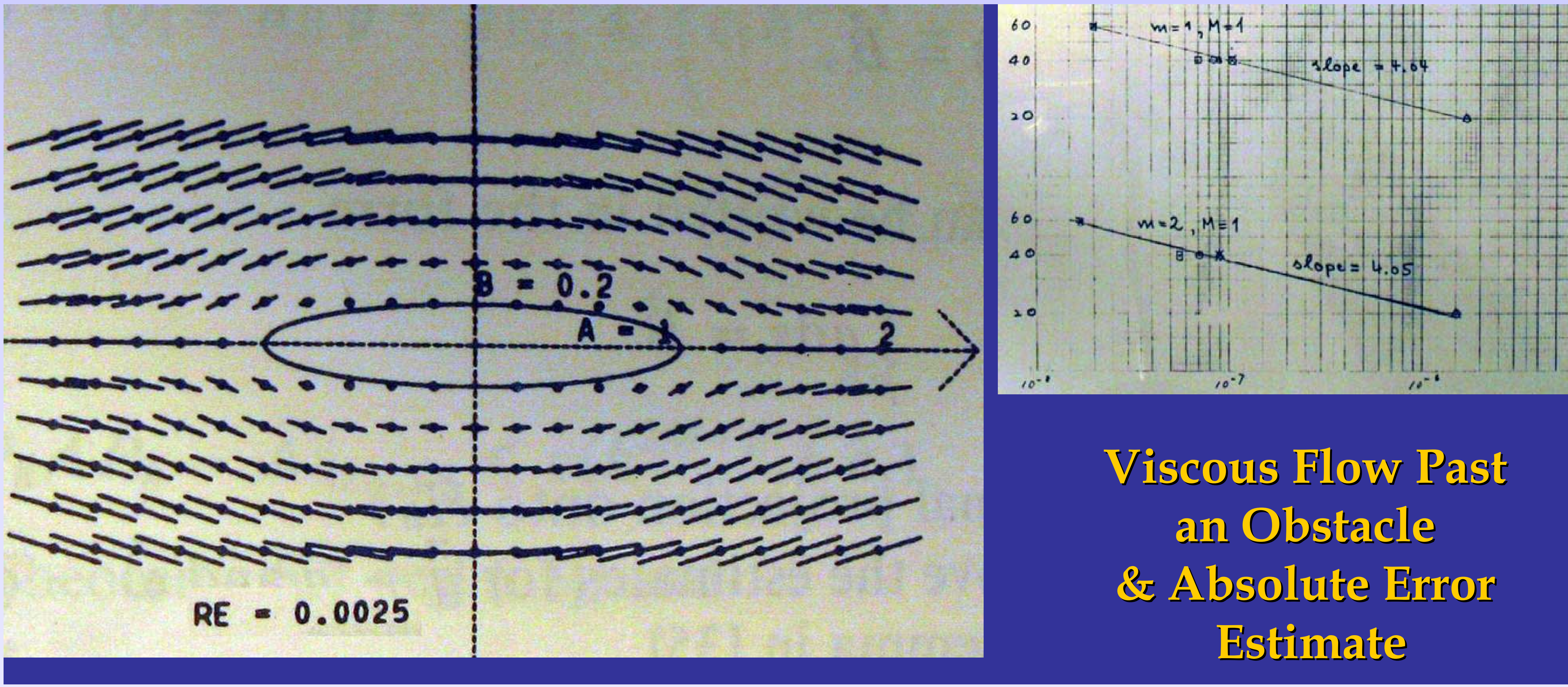
Concluding Remarks

Contact, crack and screen problems

- 2011: Gachechiladze, A., Gachechiladze, Gwinner, J., Natroshvili, D., Contact problems with friction for hemitropic solids: boundary variational inequality approach, *Appl. Anal.* **90** 279-303
- 2004: Natroshvili, D. and Wendland, W. L., Boundary variational inequalities in the theory of interface cracks: In *Operator Theory: Advances and Appl.* **147** 387-402
- 1991: Hsiao, G.C., Stephan, E. P. and Wendland, W. L., On the Dirichlet problem in elasticity for a domain exterior to an arc, ıJ. Comp. Appl. Math. **34** 1-19
- 1990: Han, H., A direct boundary element method for Signorini problems, *Mathematics of Computation*, **55** 115-128.
- 1988: Han, H. and Hsiao, G.C., The boundary element method for a contact problem: In *Theory and Applications of Boundary Elements,* Q. Du and M. Tanaka eds., Tsinghua University Press, Beijing, 33-38
- 1986: Stephan, E. P., Boundary integral equations for magnetic screens in $\mathbb{R}^3$, *Proc. Royal Soc. Edinburgh* **102A** 189-210

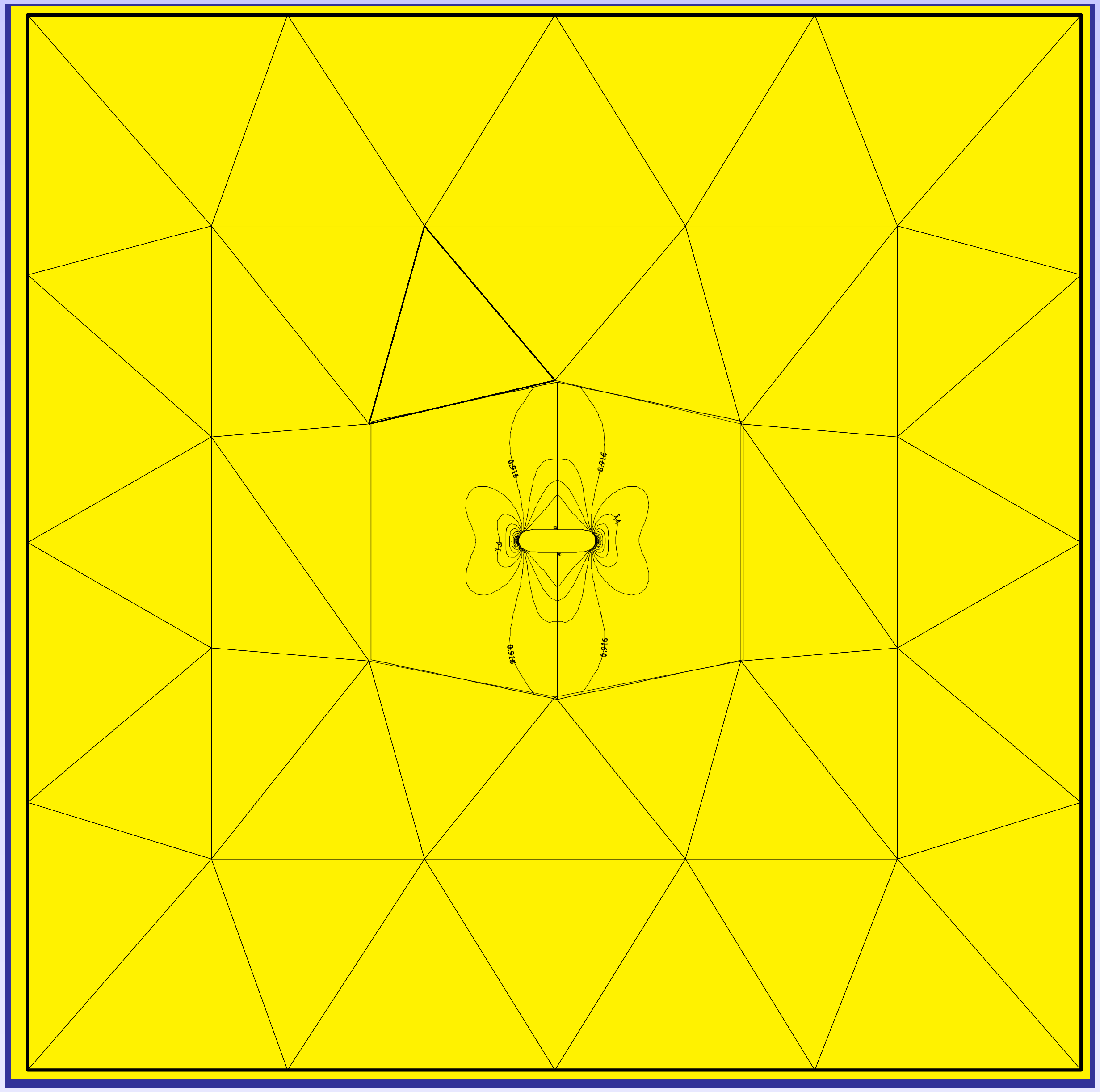
Numerical experiments

Viscous Flow Pattern



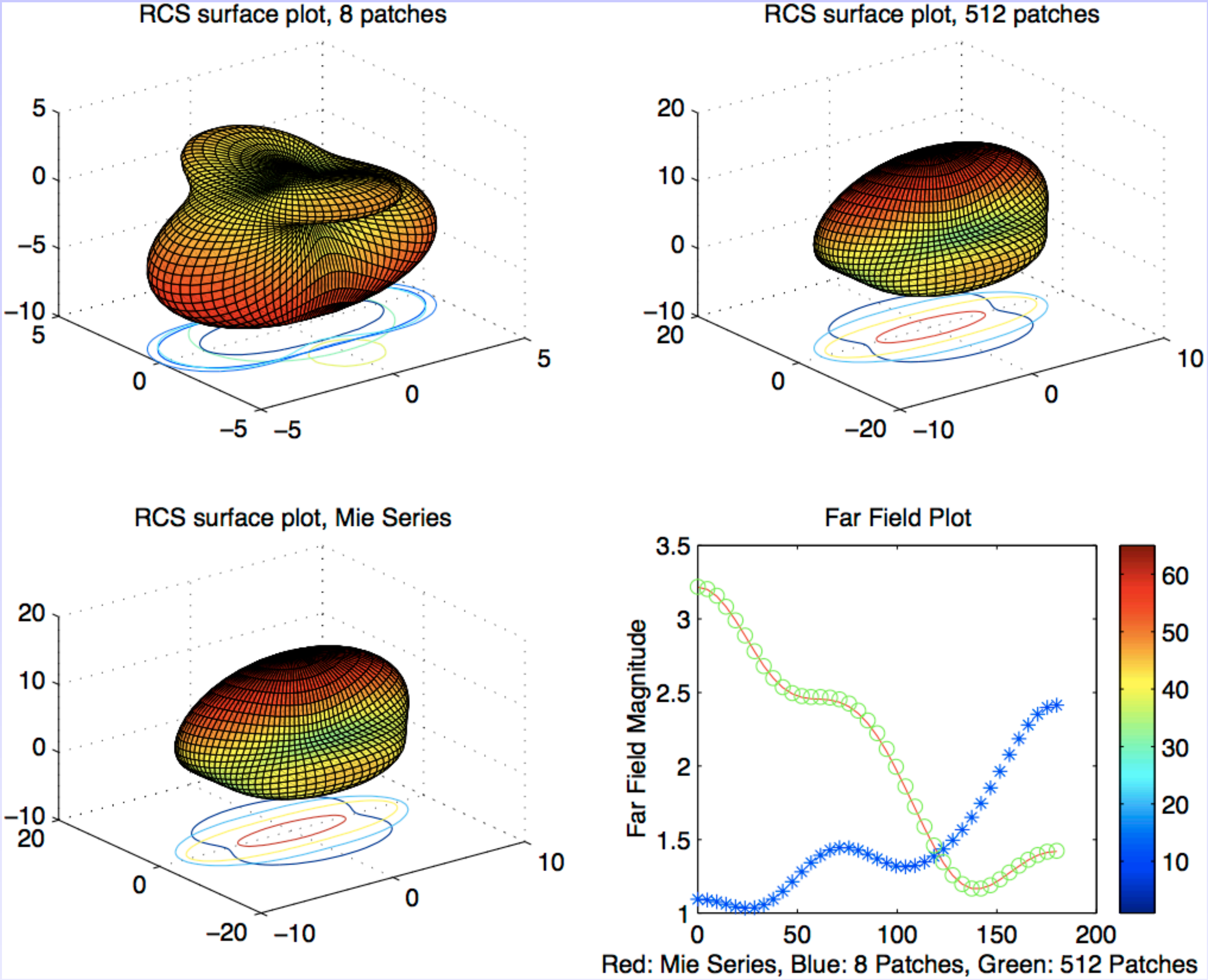
Hsiao, G.C., Kopp, P., and Wendland, W.L., Some applications of a Galerkin-collocation method for boundary integral equations of the first kind, *Math. Meth. Appl. Sci.* **6** (1984) 280-325

High Stress Gradients



Hsiao, G. C., Schnack, and Wendland, W. L., Hybrid coupled finite-boundary element methods for elliptic system of second order, *Comput. Meth. Mech. Engrg.* **190** (2000) 431-485

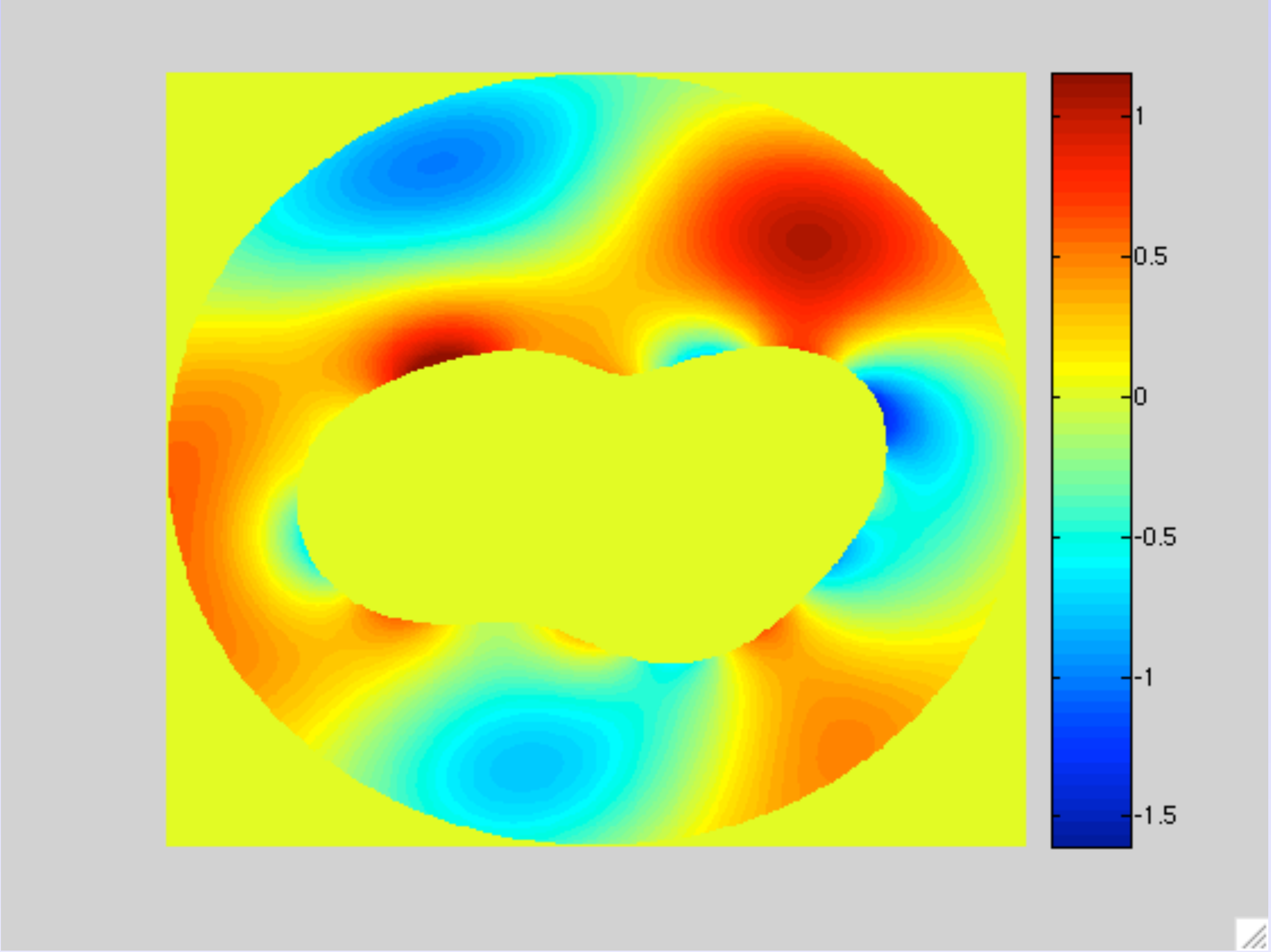
Radar Cross Section



Hsiao, G. C., Kleinman, R. E., and Wang, D. Q., Applications of boundary integral methods in 3-D electromagnetic scattering, *J. Comp. Appl. Math.*, **104** (1999) 89-110

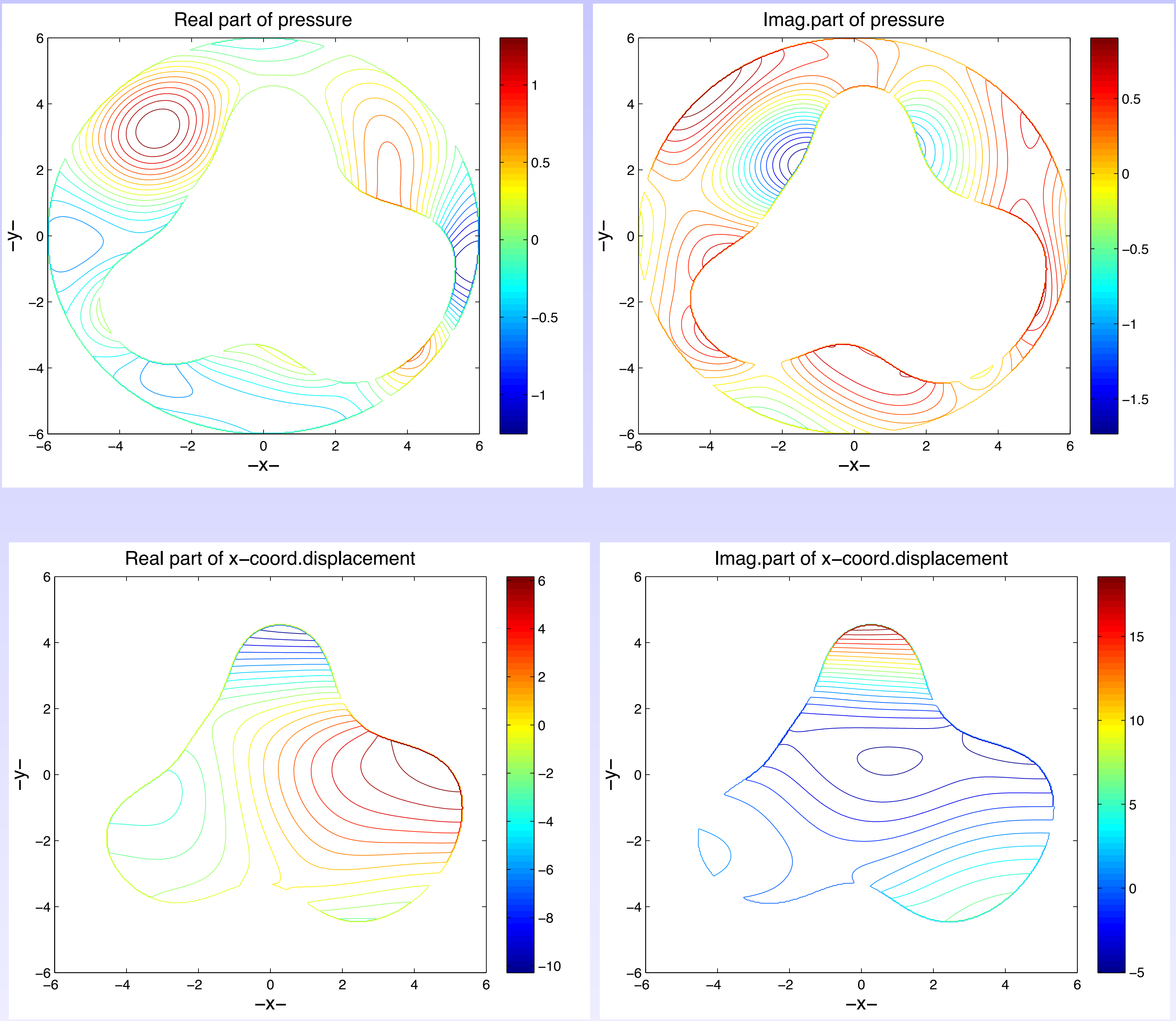
Fluid-Solid interaction

Time-harmonic behaviour of the pressure

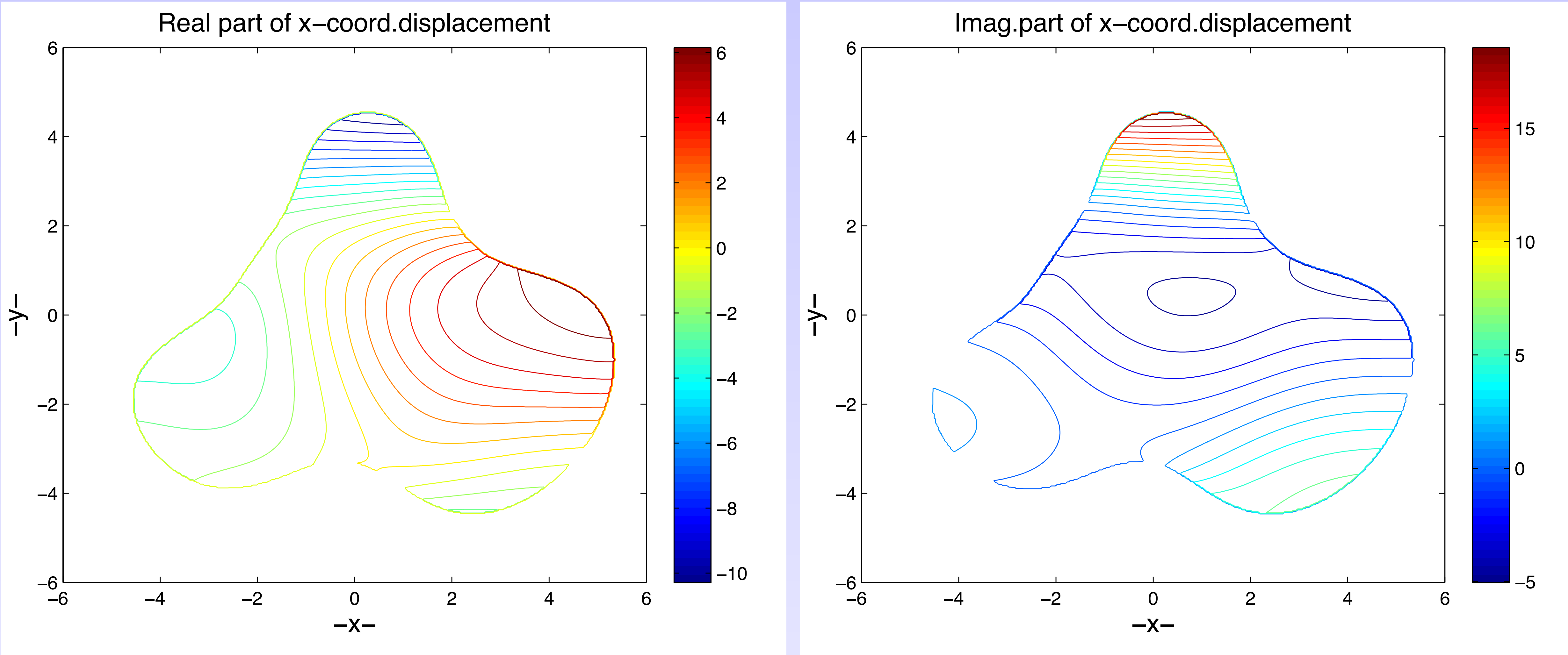


Acoustic source: Plane incident wave from left

$ka = 1.0$, $R = 0.95\lambda$


Real part of pressure
Imag.part of pressure
Real part of x−coord.displacement
Imag.part of x−coord.displacement
−x−
−y−

Displacement Fields



Elschner, J., Hsiao, G.C., and Rathsfeld, A., Reconstruction of elastic obstacles from the far-field data of scattered acoustic waves, *Memoirs on Differential Equations and Mathematical Physics*, **53** (2011) .

Further references

- 1985: Gatica, N. G., and Hsiao, G. C., *Boundary-field equation Methods for a Class of Nonlinear Problems*, Pitman Research Notes in Mathematics Series **331**, Longman
- 2004: Stephan, E. P., Coupling of Boundary Element Methods and Finite Element methods Chapter 13 in *Encyclopedia of Computational Mechanics*, Edited by Erwin Stein, René de Borst and Thomas J.R. Hughes. Volume 1: *Fundamentals*, pp.375-412. © 2004 John Wiley & Sons, Ltd. ISBN: 0–470–84699–2
- 2004: Hsiao, G. C., and Wendland, W. L., *Boundary Element Methods: Foundations and Error Analysis* Chapter 12 in *Encyclopedia of Computational Mechanics*, Edited by Erwin Stein, René de Borst and Thomas J.R. Hughes. Volume 1: *Fundamentals*, pp.339–373. © 2004 John Wiley & Sons, Ltd. ISBN: 0–470–84699–2
- 2008: Hsiao, G. C., and Wendland, W. L., *Boundary Integral Equations*, Applied Mathematical Series, **164**, Springer-Verlag.

“We’ ve-come-a-long-way”-1946 !