GENERAL PROPERTIES OF SOME FAMILIES OF GRAPHS DEFINED BY SYSTEMS OF EQUATIONS

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ABSTRACT. In this paper we present a simple method for constructing infinite families of graphs defined by a class of systems of equations over commutative rings. We show that the graphs in all such families possess some general properties including regularity and bi-regularity, existence of special vertex colorings, and existence of covering maps — hence, embedded spectra — between every two members of the same family. Another general property, recently discovered, is that nearly every graph constructed in this manner edge-decomposes either the complete, or complete bipartite, graph which it spans.

In many instances, specializations of these constructions have proved useful in various graph theory problems, but especially in many extremal problems. A short survey of the related results is included. We also show that the edge-decomposition property allows one to improve existing lower bounds for some multicolor Ramsey numbers.

1. INTRODUCTION

In the last several years some algebraic constructions of graphs have appeared in the literature. Many of these constructions were motivated by problems from extremal graph theory, and, as a consequence, the graphs obtained were primarily of interest in the context of a particular extremal problem. In the case of the graphs appearing in [?], [?]–[?], [?], the authors recently discovered that they exhibit many interesting properties *beyond* those which motivated their construction. Moreover, these properties tend to remain present even when the constructions are made far more general. This latter observation forms the motivation for our paper.

Before proceeding, we establish some notation. Given a graph Γ (by which we shall always mean 'undirected graph, without loops or multiple edges'), we denote the vertex set of Γ by $V(\Gamma)$ and the edge set by $E(\Gamma)$. Elements of $E(\Gamma)$ will be written as xy, where $x, y \in V(\Gamma)$ are the corresponding adjacent vertices. For a vertex v of Γ, let $N(v) = N_\Gamma(v)$ denote its neighborhood in Γ.

By R we will mean an arbitrary commutative ring. Suppose R has multiplicative identity element 1_R. Recall that $\alpha \in R$ is called a *unit* provided there exists $\beta \in R$ for which $\alpha\beta = 1_R$. (Here, β is a unit as well, often denoted by α^{-1} to emphasize its unique dependence on α.) If there exists a positive integer k for which the k-fold sum $1_R + \cdots + 1_R$ is zero, then the *least* such integer is denoted by $char(R)$ and it is called the *characteristic of* R, see [?]. If no such integer exists we define $char(R)$ to be zero. We mention that $char(R)$ must be prime or zero when R is an integral domain. Finally, we denote the sum $1_R + 1_R$ by 2_R provided $char(R) \neq 2$.

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We write R^n to denote the Cartesian product of n copies of R, and we refer to its elements as *vectors*.

The paper is organized as follows. In Section **??** we provide definitions of the families of graphs which are the main objects of this paper. Trying to keep matters simple, there we concentrate only on certain basic constructions. All rather straightforward generalizations are postponed until Section **??**, and related proofs can be found in our report [**?**]. In Section **??** we establish the general properties of graphs defined in Section **??**. In Section **??** we survey applications of certain specializations of the graphs defined in Section **??**. In Section **??** we also suggest some open problems.

2. Main Constructions

2.1. **Bipartite version.** Let $f_i : R^{2i-2} \to R$ be an arbitrary function, $i \geq 2$. We define the bipartite graph $B\Gamma_n = B\Gamma(R; f_2, \dots, f_n)$ as follows. The set of vertices $V(B\Gamma_n)$ is the disjoint union of two copies of R^n, one denoted by P_n and the other by L_n. Elements of P_n will be called *points* and those of L_n *lines*. In order to distinguish points from lines we introduce the use of parentheses and brackets: if $a \in R^n$, then $(a) \in P_n$ and $[a] \in L_n$. We define edges of $B\Gamma_n$ by declaring point $(p) = (p_1, p_2, \dots, p_n)$ and line $[l] = [l_1, l_2, \dots, l_n]$ to be adjacent if and only if the following $n-1$ relations on their coordinates hold:

$$
\begin{aligned}
p_2 + l_2 &= f_2(p_1, l_1) \\
p_3 + l_3 &= f_3(p_1, l_1, p_2, l_2) \\
\dots &\quad \dots \\
p_n + l_n &= f_n(p_1, l_1, p_2, l_2, \dots, p_{n-1}, l_{n-1})
\end{aligned}
\tag{2.1}
$$

For a function $f_i : R^{2i-2} \to R$, we define $\overline{f_i} : R^{2i-2} \to R$ by the rule

$$\overline{f_i}(x_1, y_1, \dots, x_{i-1}, y_{i-1}) = f_i(y_1, x_1, \dots, y_{i-1}, x_{i-1}).$$

We call f_i *symmetric* if the functions f_i and $\overline{f_i}$ coincide. The following is trivial to prove.

Proposition 1. *Graphs $B\Gamma(R; f_2, \dots, f_n)$ and $B\Gamma(R; \overline{f_2}, \dots, \overline{f_n})$ are isomorphic, an explicit isomorphism being given by $\varphi : (a) \leftrightarrow [a]$.*

We now define our second fundamental family of graphs for which we require that all functions be symmetric.

2.2. **Ordinary version.** Let $f_i : R^{2i-2} \to R$ be symmetric for all $2 \leq i \leq n$. We define $\Gamma_n = \Gamma(R; f_2, \dots, f_n)$ to be the graph with vertex set $V(\Gamma_n) = R^n$, where distinct vertices (vectors) $a = \langle a_1, a_2, \dots, a_n \rangle$ and $b = \langle b_1, b_2, \dots, b_n \rangle$ are adjacent if and only if the following $n-1$ relations on their coordinates hold:

$$
\begin{aligned}
a_2 + b_2 &= f_2(a_1, b_1) \\
a_3 + b_3 &= f_3(a_1, b_1, a_2, b_2) \\
\dots &\quad \dots \\
a_n + b_n &= f_n(a_1, b_1, a_2, b_2, \dots, a_{n-1}, b_{n-1})
\end{aligned}
\tag{2.2}
$$

For R a finite ring of cardinality r, the bipartite graphs $B\Gamma_n$ of **??** are easily seen to be r-regular. (This will be proved in Section **??**.) In contrast, graphs Γ_n

are hardly ever regular, though at most two degrees can occur, viz. r and $r-1$. (For details, see Corollary **??** in Section **??**.)

For the graphs Γ_n of **??**, our requirement that all functions f_i be symmetric is necessary to ensure that adjacency be symmetric. Without this condition one obtains not graphs, but digraphs. We see no real obstruction to an analogous theory of digraphs, which, in fact, could prove quite interesting. Note that in the bipartite constructions of **??** there is no such distinction; thus the f_i may be arbitrary in that case.

2.3. **Polarities.** Let Γ be a bipartite graph with bipartition $P \cup L$. A *polarity* of Γ is an order two automorphism which interchanges P and L. (*Note*: The term 'polarity' comes from classical geometry, where it is synonymous with *order two correlation*, e.g., see [**?**]. As a polarity in this latter sense induces an order two automorphism of the (bipartite) incidence graph of the geometry, we make no distinction between the geometric polarity and its induced automorphism.) Henceforth, we denote a bipartite graph Γ having polarity π by the pair (Γ, π). A vertex $v \in P$ is called an *absolute point* of (Γ, π) if $vv^\pi \in E(\Gamma)$, where $v^\pi \in L$ is the image of v under π. Let $Abs(\Gamma, \pi)$ denote the set of absolute points of (Γ, π).

Proposition 2. *Let f_i be symmetric for all $2 \le i \le n$. Then the isomorphism $(a) \leftrightarrow [a]$ of Proposition* **??** *is a polarity π of $B\Gamma(R; f_2, \ldots, f_n)$. If 2_R is a unit then the absolute points of $(B\Gamma_n, \pi)$ are described by*

$$Abs(B\Gamma_n, \pi) = \{(a) = (a_1, \ldots, a_n) \mid a_i = 2_R^{-1} f_i(a_1, a_1, \ldots, a_{i-1}, a_{i-1}), 2 \le i \le n\}.$$

If $char(R) = 2$, then $(a) \in Abs(B\Gamma_n, \pi)$ if and only if $f_i(a_1, a_1, \ldots, a_{i-1}, a_{i-1}) = 0$ for all $2 \le i \le n$.

Proof. It is immediate that the symmetry of f_i implies that graphs $B\Gamma(R; f_2, \ldots, f_n)$ and $B\Gamma(R; \overline{f_2}, \ldots, \overline{f_n})$ coincide, and that the isomorphism of Proposition **??** is a polarity in this case. Clearly $(a) \in P_n$ is an absolute point of $(B\Gamma_n, \pi)$ if and only if $(a)[a] \in E(B\Gamma_n)$, which occurs precisely when $a_i + a_i = f_i(a_1, a_1, \ldots, a_{i-1}, a_{i-1})$ for all i. □

The *polarity graph* Γ^π of (Γ, π) is the graph with vertex set $V(\Gamma^\pi) = P$ and edge set $E(\Gamma^\pi) = \{uv^\pi \mid uv \in E(\Gamma),\ u \in P,\ v \in L,\ u \ne v^\pi\}$. (Note that the requirement $u \ne v^\pi$ is needed to prevent the occurrence of loops in Γ^π; without it there would be a loop at each vertex $u \in Abs(\Gamma, \pi)$.)

When all functions f_i are symmetric, there is a very natural connection between the bipartite and ordinary versions of our graphs defined in **??** and **??**. Namely, as we now prove, the polarity graph of $B\Gamma(R; f_2, \ldots, f_n)$ is isomorphic to $\Gamma(R; f_2, \ldots, f_n)$. (As $\Gamma(R; f_2, \ldots, f_n)$ and $B\Gamma(R; f_2, \ldots, f_n)$ have the same underlying ring and sequence of functions, we denote them more briefly as Γ_n and $B\Gamma_n$, respectively.)

Theorem 1. *Suppose $f_i : R^{2i-2} \to R$ are symmetric for all $2 \le i \le n$, and let π be the polarity of Proposition 2. Then $(B\Gamma_n)^\pi$ is isomorphic to Γ_n.*

Proof. Clearly, the vertex sets $V(\Gamma_n)$ and $P = V((B\Gamma_n)^\pi)$ can be identified via

$$a = \langle a_1, a_2, \ldots, a_n \rangle \leftrightarrow (a) = (a_1, a_2, \ldots, a_n).$$

Then $ab \in E(\Gamma_n) \iff a \ne b \ \&\ \forall i (a_i + b_i = f_i(a_1, b_1, \ldots, a_{i-1}, b_{i-1})) \iff (a) \ne [b]^\pi \ \&\ (a)[b] \in E(B\Gamma_n) \iff (a)(b) = (a)[b]^\pi \in E((B\Gamma_n)^\pi)$. □

2.4. **Induced subgraphs.** One nice feature of the graphs we consider in this paper is the amount of control and flexability one has in defining induced subgraphs. Let $B\Gamma_n$ be the bipartite graph defined in Section **??**, and let A and B be arbitrary subsets of R. We set

$$P_{n,A} = \{(p) = (p_1, p_2, \ldots, p_n) \in P_n \mid p_1 \in A\}$$
$$L_{n,B} = \{[l] = [l_1, l_2, \ldots, l_n] \in L_n \mid l_1 \in B\}$$

and define $B\Gamma_n[A, B]$ to be the subgraph of $B\Gamma_n$ induced on the set of vertices $P_{n,A} \cup L_{n,B}$. Since we restrict the range of only the first coordinates of vertices of $B\Gamma_n$, graph $B\Gamma_n[A, B]$ can alternately be described as the bipartite graph with bipartition $P_{n,A} \cup L_{n,B}$ and adjacency relations as given in (**??**). This is a valuable observation as it enables one to "grow" the graph $B\Gamma_n[A, B]$ directly, without ever having to construct $B\Gamma_n$. In the case where $A = B$, we shall abbreviate $B\Gamma_n[A, A]$ by $B\Gamma_n[A]$.

Similarly, for arbitrary $A \subseteq R$ we define $\Gamma_n[A]$ to be the subgraph of Γ_n induced on the set $V_{n,A}$ of all vertices having respective first coordinate from A. Again, explicit construction of Γ_n is not essential in constructing $\Gamma_n[A]$; the latter graph is obtained by applying the adjacency relations in (**??**) directly to $V_{n,A}$. (Note that when $A = R$ one has $B\Gamma_n[R] = B\Gamma_n$ and $\Gamma_n[R] = \Gamma_n$.)

3. Properties

3.1. **Neighbor-complete and star-complete colorings.** One of the most important properties of graphs $B\Gamma_n$ and Γ_n defined in the previous section is the following: for every vertex v and every element $\alpha \in R$ there exists a unique neighbor of v whose first coordinate is α (see proof of Theorem **??**). Similar statements, with obvious modifications, hold for graphs $B\Gamma_n[A, B]$ and $\Gamma_n[A]$, and we leave such verification to the reader. But before embarking on proofs, we first consider a generalization of this property in terms of special vertex colorings.

Let C be a nonempty set (set of colors). A *neighbor-complete coloring* of a graph Γ is a vertex coloring $\rho : V(\Gamma) \to C$ such that, given any vertex $v \in \Gamma$, the restriction of ρ to the neighbor set $N_\Gamma(v)$ is a bijection. Thus, every color in C is uniquely represented among the neighbors of each vertex of the graph.

Clearly a neighbor-complete coloring is never proper, since every vertex has the same color as exactly one of its neighbors. This also implies that the set of monochromatic edges is a perfect matching of the graph, hence its order must be even.

If we replace the neighbor set $N_\Gamma(v)$ in the definition of a neighbor-complete coloring by $N_\Gamma(v) \cup \{v\}$, we obtain the corresponding definition of a *star-complete coloring*. Star-complete colorings are always proper.

Removing the matching consisting of monochromatic edges from a graph with a neighbor-complete coloring, one obtains a star-complete coloring of the resulting graph.

Non-trivial examples of graphs possessing either neighbor-complete colorings or star-complete colorings are not easy to construct. Remarkably, graphs $B\Gamma_n$ always admit neighbor-complete colorings, while the graphs $B\Delta_n$ and Δ_n introduced in Section **??** always admit star-complete colorings.

Theorem 2. *(i) Graph $B\Gamma_n$ admits a neighbor-complete coloring.*
(ii) Let π be the polarity of $B\Gamma_n$ given by $\pi : (a) \leftrightarrow [a]$. If $Abs(B\Gamma_n, \pi) \neq P_n$, then graph Γ_n admits a neighbor-complete coloring if and only if $Abs(B\Gamma_n, \pi) = \emptyset$.
(iii) Let π be the polarity of $B\Gamma_n$ given by $\pi : (a) \leftrightarrow [a]$. If $Abs(B\Gamma_n, \pi) \neq \emptyset$, then graph Γ_n admits a star-complete coloring if and only if $Abs(B\Gamma_n, \pi) = P_n$.

Proof. (i) Color the vertices of $B\Gamma_n$ (whether they be points or lines) by their first coordinate; thus (a) and $[b]$ are colored by a_1 and b_1, respectively. This is clearly a coloring of the vertices of $B\Gamma_n$ in $|R|$ colors.

Fix a vertex $v \in V(B\Gamma_n)$, which we may assume is a point $v = (a) \in P_n$. Then for any α in the color set R, there is a unique line $[b] \in L_n$ which is adjacent to (a) and for which $b_1 = \alpha$. Indeed, with respect to the unknowns b_i the system **??** is triangular, and each b_i is uniquely determined from the values $a_1, \ldots, a_i, b_1, \ldots b_{i-1}$, $2 \leq i \leq n$. Thus for each color $\alpha \in R$, there is a unique neighbor of (a) of that color. This gives a neighbor-complete coloring of $B\Gamma_n$.

(ii) Suppose $Abs(B\Gamma_n, \pi) \neq P_n$. Then Γ_n is regular if and only if $Abs(B\Gamma_n, \pi) = \emptyset$, which implies that Γ_n can admit a neighbor-complete coloring only if $Abs(B\Gamma_n, \pi) = \emptyset$. So assume $Abs(B\Gamma_n, \pi) = \emptyset$ and color the vertices of Γ_n by their respective initial coordinates. Given any vertex $a \in V(\Gamma_n)$ and color $\alpha \in R$, we see from above that there is a unique α-colored line $[b]$ adjacent to point (a) in the graph $B\Gamma_n$. Since (a) is nonabsolute, we clearly have $[b] \neq [a] = a^\pi$ in which case $b \neq a$. Thus b is the unique α-colored neighbor of a in graph Γ_n and the specified coloring of Γ_n is neighbor-complete.

(iii) Suppose $Abs(B\Gamma_n, \pi) \neq \emptyset$. Then Γ_n is regular if and only if $Abs(B\Gamma_n, \pi) = P_n$, which implies that Γ_n can admit a star-complete coloring only if $Abs(B\Gamma_n, \pi) = P_n$. So assume $Abs(B\Gamma_n, \pi) = P_n$ and again color the vertices of Γ_n by their first coordinates. As seen from above, no two neighbors of $a \in V(\Gamma_n)$ have the same color, so it suffices to show that the color a_1 of a is distinct from the color of any of its neighbors. By way of contradiction, let $b \in V(\Gamma_n)$ be an a_1-colored neighbor of a. Then the line $[b]$ is clearly the unique a_1-colored neighbor of point (a) in the graph $B\Gamma_n$. But as (a) is absolute we must therefore have $[b] = [a]$ in which case $b = a$, the desired contradiction. Hence the specified coloring of Γ_n is indeed star-complete. □

In most instances, $Abs(B\Gamma_n, \pi)$ is a nonempty proper subset of P_n, but the two extremes alluded to in Theorem **??** actually can occur. For example, when R has characteristic 2, we can get $Abs(B\Gamma_n, \pi) = P_n$ by taking all functions f_i to be identically zero while $Abs(B\Gamma_n, \pi) = \emptyset$ can be obtained by choosing at least one f_i to be a *nonzero* constant function.

Corollary 1. *Let $r = |R|$. Then all graphs $B\Gamma_n$ are r-regular. A graph Γ_n has precisely $|Abs(B\Gamma_n, \pi)|$ vertices of degree $r-1$ and $r^n - |Abs(B\Gamma_n, \pi)|$ vertices of degree r. Moreover, if 2_R is a unit in R, then $|Abs(B\Gamma_n, \pi)| = r$ and the vertices $a \in V(\Gamma_n)$ of degree $r-1$ are precisely those of the form $\langle a_1, a_2, \ldots, a_n \rangle$, where*

$$a_i = 2_R^{-1} f_i(a_1, a_1, \ldots, a_{i-1}, a_{i-1}), \ \ 2 \leq i \leq n.$$

Graphs Γ_n are r-regular if and only if $Abs(B\Gamma_n, \pi) = \emptyset$, and they are $(r-1)$-regular if and only if $Abs(B\Gamma_n, \pi) = P_n$.

Proof. All claims about regularity follow from Theorem **??** (or, in the case of Γ_n, from the proof of Theorem **??**). The description of vertices $a \in V(\Gamma_n)$ of degree $r-1$ follows from Proposition **??** and Theorem **??**. □

The notion of neighbor-complete colorings was introduced by Ustimenko in [**?**] under the name of "parallelotopic" and further explored by Woldar in [**?**] under the name of "rainbow." In the first paper some group theoretic constructions of graphs possessing neighbor-complete colorings are given; in the second paper purely combinatorial aspects of such colorings are considered.

3.2. **Sequential covers.** The notion of a covering for graphs is analogous to the one in topology. We call Γ a *cover* of graph $\overline{\Gamma}$ (and we write $\Gamma \to \overline{\Gamma}$) if there exists a surjective mapping $\theta : V(\Gamma) \to V(\overline{\Gamma})$ $(v \mapsto \overline{v})$ which satisfies the two conditions:

(i) θ preserves adjacencies, i.e., $\overline{uv} \in E(\overline{\Gamma})$ whenever $uv \in E(\Gamma)$;
(ii) For any vertex $v \in V(\Gamma)$, the restriction of θ to $N(v)$ is a bijection between $N(v)$ and $\overline{N}(\overline{v})$.

We alert the reader that our definition of cover is a bit stronger than that appearing elsewhere in the literature, e.g. in [**?**], where it is required only that θ satisfy (i) and be *injective* in its restriction to $N(v)$ for each $v \in V(\Gamma)$. Note that our condition (ii) ensures that θ be degree-preserving; in particular, any cover of an r-regular graph is again r-regular.

For $k < n$, denote by $\eta = \eta(n,k)$ the mapping $R^n \to R^k$ $(v \mapsto \overline{v})$ which projects $v \in R^n$ onto its k initial coordinates, VIZ.

$$v = \langle v_1, v_2, \ldots, v_k, \ldots v_n \rangle \mapsto \overline{v} = \langle v_1, v_2, \ldots, v_k \rangle.$$

Clearly, η provides a mapping $V(\Gamma_n) \to V(\Gamma_k)$, and its restriction to $V_{n,A} = A \times R^{n-1}$ gives mappings $V(\Gamma_n[A]) \to V(\Gamma_k[A])$. In the bipartite case, we further impose that η preserve vertex type, i.e. that

$$(p) = (p_1, p_2, \ldots, p_k, \ldots p_n) \mapsto \overline{(p)} = (p_1, p_2, \ldots, p_k),$$
$$[l] = [l_1, l_2, \ldots, l_k, \ldots l_n] \mapsto \overline{[l]} = [l_1, l_2, \ldots, l_k].$$

Here, η induces, in obvious fashion, the mappings $V(B\Gamma_n[A]) \to V(B\Gamma_k[A])$.

In what follows, the functions f_i $(2 \le i \le n)$ for the graphs $B\Gamma_n[A]$ are assumed to be arbitrary, while those for $\Gamma_n[A]$, continue, out of necessity, to be assumed symmetric.

Theorem 3. *For every $A \subseteq R$, and every k, n, $2 \le k < n$, $B\Gamma_n[A] \to B\Gamma_k[A]$. Graph $\Gamma_n[A]$ covers $\Gamma_k[A]$ if and only if no edge of $\Gamma_n[A]$ projects to a loop of $\Gamma_k[A]$ for any $n > k \ge 2$.*

Proof. Obviously the mapping of $\eta : V(B\Gamma_n[A]) \to V(B\Gamma_k[A])$ induced by the canonical projection $\eta(n,k)$ is surjective.

If $ab \in E(B\Gamma_n[A])$, then the respective coordinates of a and b satisfy $a_i + b_i = f_i(a_1, b_1, \ldots, a_{i-1}, b_{i-1})$ for every $i \le n$, so *a fortiori* for every $i \le k$.

This implies $\overline{a}\overline{b} \in E(B\Gamma_k[A])$ unless $\overline{a}\overline{b}$ is a loop in $B\Gamma_k[A]$, in which case $\overline{a} = \overline{b}$. But this is impossible, since a and b have different vertex type. Therefore η satisfies (i).

Let b and c be two distinct neighbors of a in $B\Gamma_n[A]$ and $\overline{b} = \overline{c}$. Then $b_1 = \overline{b_1} = \overline{c_1} = c_1$. Since $ab, ac \in E(B\Gamma_n[A])$ and $b_1 = c_1$, then $b = c$, a contradiction. This proves that the restriction of η to the neighbor set of $a \in V(B\Gamma_n[A])$ is injective, whence (ii) follows from above.

In the case of the graphs $\Gamma_n[A]$, it is entirely possible that an edge fold to a loop in $\Gamma_k[A]$ (see Remark **??**, below). When this occurs it is easy to see that conditions (i) and (ii) are both violated.

On the other hand, if this never occurs then one can fashion a proof for $\Gamma_n[A]$ similar to the one given above for $B\Gamma_n[A]$. □

Remark 1. We here provide an example in which Γ_n is not a cover of Γ_k. Let $n = 3$, $k = 2$, assume R has characteristic 2, and define f_2 to be identically zero and f_3 to be identically 1. Then $ab \in E(\Gamma_3)$ where $a = \langle 0,0,0\rangle$ and $b = \langle 0,0,1\rangle$, but $\overline{a} = \langle 0,0\rangle = \overline{b}$.

Remark 2. One important consequence of Theorem **??**, particularly amenable to girth related Turán type problems in extremal graph theory, is that the girth of $B\Gamma_n[A]$ or $\Gamma_n[A]$ is a non-decreasing function of n. Thus, one hopes to be able to identify certain conditions on the functions f_i which might provide families with unbounded girth. (See Section **??** for such an example.)

3.3. **Embedded spectra.** The spectrum $spec(\Gamma)$ of a graph Γ is defined to be the multiset of eigenvalues of its adjacency matrix. One important property of covers discussed in Section **??** is that the spectrum of any graph embeds (as a multiset, i.e., taking into account also the multiplicities of the eigenvalues) in the spectrum of its cover. This result can be proven in many ways, for example as a consequence of either Theorem 0.12 or Theorem 4.7, both of [**?**], or using the notion of equitable partitions introduced by Schwenk [**?**]. (See [**?**] for a proof based on the latter approach.) As an immediate consequence of this fact and Theorem **??**, we obtain

Theorem 4. *Assume R is finite and let $A \subseteq R$. Then for each k, n, $2 \le k < n$,*

$$spec(B\Gamma_k[A]) \subseteq spec(B\Gamma_n[A]).$$

For graphs $\Gamma_n[A]$, one has $spec(\Gamma_k[A]) \subseteq spec(\Gamma_n[A])$ provided no edge of $\Gamma_n[A]$ projects to a loop of $\Gamma_k[A]$.

3.4. **Edge-decomposing K_n and $K_{m,m}$.** Let Γ and Γ' be graphs. An *edge-decomposition of* Γ *by* Γ' is a collection $\mathcal{C}$ of subgraphs of Γ, each isomorphic to Γ', such that $\{E(\Lambda) \mid \Lambda \in \mathcal{C}\}$ is a partition of $E(\Gamma)$.

We also say in this case that Γ' *decomposes* Γ. It is customary to refer to the subgraphs Λ in $\mathcal{C}$ as *copies* of Γ', in which case one may envision an edge-decomposition of Γ by Γ' as a decomposition of Γ into edge-disjoint copies of Γ'.

Throughout this section we assume R is a *finite* commutative ring of cardinality r. Recall that when R has multiplicative identity element 1_R and $char(R) \neq 2$, we denote by 2_R the element $1_R + 1_R \in R$. Further recall that while there is no restriction on the functions $f_i : R^{2i-2} \to R$ for the bipartite graph $B\Gamma_n$, we must assume all f_i are symmetric for the graph Γ_n.

The purpose of this section is to prove the following two theorems. As usual, K_n will denote the complete graph on n vertices, and $K_{m,m}$ the complete bipartite graph on $m + m$ vertices.

Theorem 5. *$B\Gamma_n$ decomposes K_{r^n,r^n}.*

Proof. Clearly $B\Gamma_n$ spans K_{r^n,r^n}, whence $V(K_{r^n,r^n}) = V(B\Gamma_n) = P_n \cup L_n$. For each $\alpha = \langle \alpha_2, \dots, \alpha_n \rangle \in R^{n-1}$, define the mapping $\phi_\alpha : V(K_{r^n,r^n}) \to V(K_{r^n,r^n})$ by

$$(p) = (p_1, p_2, \dots, p_n) \mapsto (p)^{\phi_\alpha} = (p_1, p_2 + \alpha_2, \dots, p_n + \alpha_n),$$
$$[l] \mapsto [l]^{\phi_\alpha} = [l].$$

Now define $(B\Gamma_n)^{\phi_\alpha}$ to be the subgraph of K_{r^n,r^n} having vertex set $V((B\Gamma_n)^{\phi_\alpha}) = V(K_{r^n,r^n})$ and edge set $E((B\Gamma_n)^{\phi_\alpha}) = \{(p)^{\phi_\alpha}[l]^{\phi_\alpha} \mid (p)[l] \in E(B\Gamma_n)\}$. It is immediate from the description of its edges that subgraph $(B\Gamma_n)^{\phi_\alpha}$ is isomorphic to $B\Gamma_n$, for each $\alpha \in R^{n-1}$; indeed, ϕ_α is an explicit isomorphism.

Thus it remains to verify that $\{E((B\Gamma_n)^{\phi_\alpha}) \mid \alpha \in R^{n-1}\}$ is a partition of $E(K_{r^n,r^n})$, which is tantamount to showing (since R is finite) that sets $E^{\phi_\alpha} := E((B\Gamma_n)^{\phi_\alpha})$ and $E^{\phi_\beta} := E((B\Gamma_n)^{\phi_\beta})$ are disjoint for each pair of distinct vectors $\alpha, \beta \in R^{n-1}$. We do this directly.

Let $(p)[l] \in E^{\phi_\alpha} \cap E^{\phi_\beta}$, in which case $(p)[l] = (a)^{\phi_\alpha}[b]^{\phi_\alpha} = (c)^{\phi_\beta}[d]^{\phi_\beta}$ for certain edges $(a)[b], (c)[d] \in E(B\Gamma_n)$.

As $[l]^{\phi_\alpha} = [l]$ for all $[l] \in L_n$ and $\alpha \in R^{n-1}$, we immediately obtain

$$(a_1, a_2 + \alpha_2, \dots, a_n + \alpha_n) = (a)^{\phi_\alpha} = (c)^{\phi_\beta} = (c_1, c_2 + \beta_2, \dots, c_n + \beta_n)$$

and $[b] = [d]$. In particular $a_1 = c_1$, whence the points (a) and (c), both being neighbors of the line $[b] = [d]$, must be identical (cf. Theorem **??** of Section **??**). Thus it follows that $\alpha = \beta$, which proves the sets E^{ϕ_α}, $\alpha \in R^{n-1}$, are indeed pairwise disjoint.

Since $|E^{\phi_\alpha}| = r^{n+1}$ for every $\alpha \in R^{n-1}$, one now has

$$|\bigcup_{\alpha \in R^{n-1}} E^{\phi_\alpha}| = \sum_{\alpha \in R^{n-1}} |E^{\phi_\alpha}| = \sum_{\alpha \in R^{n-1}} r^{n+1} = r^{n+1} r^{n-1} = r^{2n} = |E(K_{r^n,r^n})|,$$

which proves $\{E((B\Gamma_n)^{\phi_\alpha}) \mid \alpha \in R^{n-1}\}$ is a partition of $E(K_{r^n,r^n})$, as claimed. □

Theorem 6. *Assume R has identity and 2_R is a unit in R. Then Γ_n decomposes K_{r^n}.*

Proof. The idea of the proof follows closely that of Theorem **??**. Namely, for each $\alpha = \langle \alpha_2, \dots, \alpha_n \rangle \in R^{n-1}$ we define the mapping $\phi_\alpha : V(K_{r^n}) \to V(K_{r^n})$ by

$$v = \langle v_1, v_2, \dots, v_n \rangle \mapsto v^{\phi_\alpha} = \langle v_1, v_2 + \alpha_2, \dots, v_n + \alpha_n \rangle,$$

as well as the graph Γ^{ϕ_α} having vertex set $V(\Gamma^{\phi_\alpha}) = V(K_{r^n})$ and edge set $E(\Gamma_n^{\phi_\alpha}) = \{u^{\phi_\alpha} v^{\phi_\alpha} \mid uv \in E(B\Gamma_n)\}$. As before, the isomorphism $\Gamma^{\phi_\alpha} \cong \Gamma_n$ follows directly from the definition of edges in graph Γ^{ϕ_α}; thus it remains only to verify that $\{E(\Gamma_n^{\phi_\alpha}) \mid \alpha \in R^{n-1}\}$ is a partition of $E(K_{r^n})$.

Let $uv \in E(\Gamma_n^{\phi_\alpha}) \cap E(\Gamma_n^{\phi_\beta})$, whence $uv = a^{\phi_\alpha} b^{\phi_\alpha} = c^{\phi_\beta} d^{\phi_\beta}$ for certain edges $ab, cd \in E(\Gamma_n)$.

This gives, without loss of generality,

$$\langle a_1, a_2 + \alpha_2, \dots, a_n + \alpha_n \rangle = \langle c_1, c_2 + \beta_2, \dots, c_n + \beta_n \rangle,$$
$$\langle b_1, b_2 + \alpha_2, \dots, b_n + \alpha_n \rangle = \langle d_1, d_2 + \beta_2, \dots, d_n + \beta_n \rangle.$$

We prove, by induction on coordinates, that $a = c$, $b = d$ and $\alpha = \beta$. The base step is simply $a_1 = c_1$, $b_1 = d_1$ which is apparent. Now as $ab, cd \in E(\Gamma_n)$, one has

$a_i + b_i = f_i(a_1, b_1, \ldots, a_{i-1}, b_{i-1})$ and $c_i + d_i = f_i(c_1, d_1, \ldots, c_{i-1}, d_{i-1})$ for each $2 \le i \le n$, whence $a_i + b_i = c_i + d_i$ follows from the inductive step.

From above, we have the relations $a_i + \alpha_i = c_i + \beta_i$ and $b_i + \alpha_i = d_i + \beta_i$ so, upon adding, we obtain $2_R\alpha_i + a_i + b_i = 2_R\beta_i + c_i + d_i$, whence $\alpha_i = \beta_i$. But now $a_i = c_i$ and $b_i = d_i$ follow at once. This proves $a = c$, $b = d$ and $\alpha = \beta$, from which we conclude that the sets $\{E(\Gamma_n^{\phi_\alpha}) \mid \alpha \in R^{n-1}\}$ are pairwise disjoint.

To complete the proof of the theorem, we merely deduce from Corollary **??** that $|E(\Gamma_n)| = \frac{1}{2}(r^{n+1} - r)$, whence we obtain $|\cup_{\alpha \in R^{n-1}} E(\Gamma^{\phi_\alpha})| = \sum_{\alpha \in R^{n-1}} |E(\Gamma^{\phi_\alpha})| = \sum_{\alpha \in R^{n-1}} \frac{1}{2}(r^{n+1} - r) = \frac{1}{2}(r^{n+1} - r)(r^{n-1}) = |E(K_{r^n})|$.

Thus $\{E(\Gamma_n^{\phi_\alpha}) \mid \alpha \in R^{n-1}\}$ is a partition of $E(K_{r^n})$, as claimed. □

Remark 3. We readily deduce from Theorem **??** that each vertex of K_{r^n} has degree $r-1$ in a unique copy of Γ_n, and degree r in each of the $r^{n-1} - 1$ remaining copies.

Remark 4. Recently, in [**?**], Lazebnik and Mubayi generalized Theorems 5 and 6 to edge-decompositions of complete uniform r-partite hypergraphs and complete uniform hypergraphs, respectively.

4. Examples and Applications

In this section we provide a survey of examples of graphs defined by systems of equations which have had application to extremal type problems. In most instances, the graphs considered are specializations of $B\Gamma_n$, with R taken to be the finite field $\mathbb{F}_q$ of q elements and the functions f_i chosen in such a way as to ensure the resulting graphs have a high degree of symmetry and large girth. (The *girth* of a graph Γ, denoted by $g(\Gamma)$, is the length of a shortest cycle in Γ.)

4.1. **Dense graphs of large girth.** Let $\mathcal{F}$ be a family of graphs. By $ex(\nu, \mathcal{F})$ we denote the greatest number of edges in a graph on ν vertices which contains no subgraph isomorphic to a graph from $\mathcal{F}$. Let C_n denote the cycle of length $n \ge 3$. The best bounds on $ex(\nu, \{C_3, C_4, \cdots, C_{2k}\})$ for fixed k, $2 \le k \ne 5$, are the following:

$$c_k \nu^{1+\frac{2}{3k-3+\epsilon}} \le ex(\nu, \{C_3, C_4, \cdots, C_{2k}\}) \le 90k\nu^{1+\frac{1}{k}} \tag{4.1}$$

The upper bound actually holds for all $k \ge 2$ and ν, and was established by Bondy and Simonovits [**?**] (see also [**?**], [**?**]). The lower bound holds for an infinite sequence of values of ν; c_k is a positive function of k only and $\epsilon = 0$ if k is odd, and $\epsilon = 1$ if k is even. It was established by Lazebnik, Ustimenko and Woldar in [**?**]. For $k = 5$ a better lower bound $c(\nu^{1+1/5})$ is given by the regular generalized hexagon of Lie type B_2 (see [**?**]).

The lower bound comes from the following construction. Consider the family of graphs $D(n,q) = B\Gamma_n(\mathbb{F}_q; f_2, \ldots, f_n)$, where $f_2 = p_1 l_1$, $f_3 = p_1 l_2$, and for $4 \le i \le n$,

$$f_i = \begin{cases} -p_{i-2} l_1 & i \equiv 0 \text{ or } 1 \mod 4 \\ p_1 l_{i-2} & i \equiv 2 \text{ or } 3 \mod 4 \end{cases}$$

This family was introduced by Lazebnik and Ustimenko in [**?**], where it was shown that the graphs are edge-transitive and, most importantly, $g(D(n,q)) \ge n + 5$ for odd n. Together with Woldar, it was shown in [**?**] that for $n \ge 6$ and q odd, graphs $D(n,q)$ are disconnected, and the order of each component (any two being isomorphic) is at least $2q^{n-\lfloor\frac{n+2}{4}\rfloor+1}$. Let $CD(n,q)$ denote one of these components. It is the family of graphs $CD(n,q)$ which provides the lower bound in the above inequality, being a slight improvement of the previous best lower bound $\Omega(\nu^{1+\frac{2}{3k+3}})$ given by the family of Ramanujan graphs constructed by Margulis [?], and independently by Lubotzky, Phillips and Sarnak [?]. In [?], Lazebnik, Ustimenko and Woldar proved that for all $n \geq 6$ and q odd, the order of $CD(n,q)$ is equal to $2q^{n-\lfloor\frac{n+2}{4}\rfloor+1}$, hence another family is needed if the lower bound in **??** is to be improved. For $n = 2, 3, 5$, the magnitude $\nu^{1+\frac{1}{n}}$ in the upper bound of **??** is attained by $D(n,q)$ ($n = 2, 3$ and q odd) and by the regular generalized hexagon ($n = 5$). Also, it is not hard to show that the graphs $D(n,q)$ and $CD(n,q)$ are vertex-transitive for $n \geq 2$ and $n \neq 3$.

Remark 5. The construction of the graphs $D(n,q)$ was motivated by attempts to generalize the notion of the "affine part" of a generalized polygon, and it was facilitated by results of Ustimenko on the embedding of Chevalley group geometries into their corresponding Lie algebras. In fact, $D(2,q)$ and $D(3,q)$ (q odd) are exactly the affine parts of a regular generalized 3-gon and 4-gon, respectively. (See [?] and [?] for more details.)

For further results on dense graphs with forbidden cycles, see [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

4.2. **Dense (m,n)-bipartite graphs of girth 8.** Let $f(n,m)$ denote the greatest number of edges in a bipartite graph whose bipartition sets have cardinalities n, m ($n \geq m$) and whose girth is at least 8. It is well known that $f(n,n) = \Theta(n^{4/3})$, and it is easy to show that $f(n,m) = \Theta(n)$ for $m = O(n^{1/2})$. In [?] de Caen and Székely, and independently Faudree and Simonovits [?], proved that $f(n,m) = O(n^{2/3}m^{2/3})$. The remarks above show that this upper bound is asymptotically tight when $n = m$, or when $m = O(n^{1/2})$. Using generalized quadrangles, de Caen and Székely demonstrated in [?] that $f(n,m) = \Omega(n^{2/3}m^{2/3})$ also when $m \sim n^{4/5}$ and $m \sim n^{7/8}$.

Another important result in [?] was a disproof of an old conjecture of Erdős (see e.g. [?]) that $f(n,m) = O(n)$ for $m = O(n^{2/3})$. Using some results from combinatorial number theory and set systems, the authors proved the existence of an infinite family of (m,n)-bipartite graphs with $m \sim n^{2/3}$, girth at least 8, and having $n^{1+1/57+o(1)}$ edges. As the authors pointed out, this disproved Erdős' conjecture, but fell well short of their upper bound $O(n^{1+1/9})$.

Using certain induced subgraphs of algebraically defined graphs, Lazebnik, Ustimenko and Woldar [?] constructed explicitly an infinite family of $(n^{2/3}, n)$-bipartite graphs of girth 8 with $n^{1+1/15}$ edges. We give now this construction.

Let q be an odd prime power, and set $P = \mathbb{F}_q \times \mathbb{F}_{q^2} \times \mathbb{F}_q$, $L = \mathbb{F}_{q^2} \times \mathbb{F}_{q^2} \times \mathbb{F}_q$. We define the bipartite graph $\Gamma(q)$ with bipartition $P \cup L$ in which (p) is adjacent to $[l]$ provided

$$\begin{aligned} l_2 + p_2 &= p_1 l_1 \\ l_3 + p_3 &= -(p_2\overline{l_1} + \overline{p_2} l_1). \end{aligned}$$

(Here, $\overline{x}$ denotes the image of x under the involutory automorphism of $\mathbb{F}_{q^2}$ with fixed field $\mathbb{F}_q$.)

In the context of the current paper, $\Gamma(q)$ is closely related to the induced subgraph $B\Gamma_3[\mathbb{F}_q, \mathbb{F}_{q^2}]$ of $B\Gamma_3 = B\Gamma(\mathbb{F}_{q^2}; f_2, f_3)$ with

$$f_2(p_1, l_1) = p_1 l_1$$
$$f_3(p_1, l_1, p_2, l_2) = -(p_2\overline{l_1} + \overline{p_2} l_1)$$

(see Section **??**). Indeed, the only difference is that the third coordinates of vertices of $\Gamma(q)$ are required to come from $\mathbb{F}_q$.

Assuming now that $q^{1/3}$ is an integer, we may further choose $A \subset \mathbb{F}_q$ with $|A| = q^{1/3}$. Set $P_A = A \times \mathbb{F}_{q^2} \times \mathbb{F}_q$, and denote by $\Gamma'(q)$ the subgraph of $\Gamma(q)$ induced on the set $P_A \cup L$. Then the family $\{\Gamma'(q)\}$ gives the desired $(n^{2/3}, n)$-bipartite graphs of girth 8 and $n^{1+1/15}$ edges, where $n = q^2$. (See [**?**] for details.)

4.3. **Bipartite graphs of given bi-degree and girth.** A bipartite graph Γ with bipartition $V_1 \cup V_2$ is said to be *biregular* if there exist integers r, s such that $\deg(x){=}r$ for all $x \in V_1$ and $\deg(y){=}s$ for all $y \in V_2$. In this case, the pair r, s is called the *bi-degree* of Γ. By an *(r, s, t)-graph* we shall mean any biregular graph with bi-degree r, s and girth exactly $2t$.

For which $r, s, t \geq 2$ do (r, s, t)-graphs exist? Trivially, $(r, s, 2)$-graphs exist for all $r, s \geq 2$; indeed, these are the complete bipartite graphs. For all $r, t \geq 2$, Sachs [**?**], and Erdős and Sachs [**?**], constructed r-regular graphs with girth $2t$. From such graphs, $(r, 2, t)$-graphs can be trivially obtained by subdividing (i.e. inserting a new vertex on) each edge of the original graph. The methods of [**?**] and [**?**] differ in spirit. In the paper of Sachs, the graphs are constructed explicitly but are rather sparse in their number of edges. In the joint paper [**?**] Erdős and Sachs established, though without explicit construction, the existence of families of much denser graphs.

Biregular graphs with girth at least 6 have been studied extensively in the last 150 years in the context of geometric configurations. Calling the vertices of the two bipartition sets 'points' and 'lines,' respectively, we obtain an incidence structure, or geometry, in which each line contains s points and each point is contained in r lines. (The girth condition ensures that no pair of points lie on two distinct lines.) Steiner systems are a special case. From the known constructions, it can be deduced that $(r, s, 3)$ graphs exist for all $r, s \geq 3$. However, apart from certain isolated examples such as even cycles, generalized polygons, and cages, very little is known for $t > 3$.

In [**?**] Füredi, Lazebnik, Seress, Ustimenko and Woldar showed, by explicit construction, that (r, s, t)-graphs exist for all $r, s, t \geq 2$. Their results can be viewed as biregular versions of the results from [**?**] and [**?**]. The paper [**?**] contains two constructions: a *recursive* one and an *algebraic* one. The recursive construction establishes existence for all $r, s, t \geq 2$, but the algebraic method works only for $r, s \geq t$. However, the graphs obtained by the algebraic method are much denser and exhibit the following nice property: one can construct an (r, s, t)-graph Γ such that for all $r \geq r' \geq t \geq 3$ and $s \geq s' \geq t \geq 3$, Γ contains an (r', s', t)-graph Γ' as an induced subgraph.

4.4. **Cages.** Let $k \geq 2$ and $g \geq 3$ be integers. A (k, g)-graph is a k-regular graph with girth g. A (k, g)-*cage* is a (k, g)-graph of minimum order. The problem of determining the order $\nu(k, g)$ of a (k, g)-cage is unsolved for most pairs (k, g) and is extremely hard in the general case. By counting the number of vertices in the breadth-first-search tree of a (k,g)-graph, one easily establishes the following lower bounds for $\nu(k,g)$:

$$\nu(k,g) \geq \begin{cases} \frac{k(k-1)^{(g-1)/2}-2}{k-2}, & \text{for } g \text{ odd;} \\ \frac{2(k-1)^{g/2}-2}{k-2}, & \text{for } g \text{ even.} \end{cases}$$

Graphs whose orders achieve these lower bounds are very special and possess many remarkable properties. Though there is no complete agreement on terminology, they are often referred to as "Moore graphs" when g is odd, and "regular generalized polygons" when g is even. For information on cages, see [?] and the many references therein.

Finding upper bounds for $\nu(k,g)$ is a far more difficult affair; indeed, even the fact that $\nu(k,g)$ is finite is nontrivial to prove. This was first accomplished by Sachs, who in [?] showed by explicit construction that (k,g)-graphs of finite order exist. In the same year, Erdős and Sachs [?] gave, without explicit construction, a much smaller general upper bound on $\nu(k,g)$. Their result was later improved, though only slightly, by Walther [?], [?], and later by Sauer [?]. The following upper bounds are due to Sauer [?]:

$$\nu(k,g) \leq \begin{cases} 2(k-1)^{g-2}, & \text{for } g \text{ odd and } k \geq 4; \\ 4(k-1)^{g-3}, & \text{for } g \text{ even and } k \geq 4. \end{cases}$$

Note that these upper bounds are roughly the squares of the previously indicated lower bounds.

In [?], Lazebnik, Ustimenko and Woldar established general upper bounds on $\nu(k,g)$ which are roughly the $3/2$ power of the lower bounds, and provided explicit constructions for such (k,g)-graphs. The main ingredients of their construction were the algebraically defined graphs $CD(n,q)$ described in Section **??** and certain induced subgraphs of these, manufactured by the method described in Section **??**. The precise result follows.

Theorem 7. [?] *Let $k \geq 2$ and $g \geq 5$ be integers, and let q denote the smallest odd prime power for which $k \leq q$. Then*

$$\nu(k,g) \leq 2kq^{\frac{3}{4}g-a},$$

where $a = 4$, $11/4$, $7/2$, $13/4$ for $g \equiv 0,1,2,3 \pmod 4$, respectively.

4.5. **Polarity graphs.** In [?], Lazebnik, Ustimenko and Woldar showed that bipartite graphs of girth at least $2k+1$ (in particular, generalized polygons) cannot be extremal C_{2k}-free graphs. Generalizing an idea of Brown in [?] and of Erdős, Renyi and Sós in [?], they devised a method enabling one to sometimes improve numerical constants in the lower bounds for $ex(\nu, C_{2k})$, see [?]. Their method utilized polarities in certain rank two geometries of Lie type, and a simple analysis of the resulting polarity graphs (see Section **??**). The obtained graphs were used to refute some conjectures stated in [?] about the values of $ex(\nu, C_{2k})$, and they afforded new examples of graphs which exhibit certain restrictive behavior on the lengths of their cycles. In particular, the authors constructed an infinite family $\{G_i\}$ of C_6-free graphs with $|E(G_i)| \sim \frac{1}{2}|V(G_i)|^{4/3}$ $(i \to \infty)$ which improved the constant in the previously best known lower bound on $ex(\nu, C_6)$ from $2/3^{4/3} \approx 0.462$ (see [?]) to $1/2$.

4.6. **Structure of extremal graphs of large girth.** Let $n \geq 3$, and let Γ be a graph of order ν and girth at least $n+1$ which has the greatest number of edges possible subject to these requirements (i.e. an extremal graph). Must Γ contain an $(n+1)$-cycle? In [**?**] Lazebnik and Wang present several results where this question is answered affirmatively, see also [**?**]. In particular, this is always the case when ν is large compared to n: $\nu \geq 2^{a^2+a+1}n^a$, where $a = n - 3 - \lfloor \frac{n-2}{4} \rfloor$, $n \geq 12$. To obtain this result they used certain generic properties of extremal graphs, as well as of the graphs $CD(n,q)$ described in Section **??**. On the other hand, they proved $(n+1)$-cycles need not occur in extremal graphs of order ν and girth $\geq n+1$ when $\nu = 2n+2 \geq 26$. (Most likely, the lower bound for ν is far too large in the affirmative case.)

4.7. **Multicolor Ramsey Numbers.** Let $k > 1$ be an integer, and let $G_1, \ldots, G_k$ be graphs. The multicolor Ramsey number $r(G_1, \ldots, G_k)$ is defined to be the smallest integer $n = n(k)$ with the property that any k-coloring of the edges of the complete graph K_n must result in a monochromatic subgraph of K_n isomorphic to G_i for some i. (Here, by "monochromatic subgraph" we mean a subgraph all of whose edges have the same color.) When all graphs G_i are identical, one usually abbreviates $r(G, \ldots, G)$ by $r_k(G)$. Clearly, the notion of multicolor Ramsey number is a natural generalization of that of the classical Ramsey number $r(s,t)$. For a survey on multicolor Ramsey numbers, see [**?**]

The edge-decomposition theorem of the previous section (Theorem **??**) immediately implies a lower bound on the multicolor Ramsey number $r(G_1, \ldots, G_{r^{n-1}})$, specifically

$$r(G_1, \ldots, G_{r^{n-1}}) \geq r^n + 1,$$

where $G_1, \ldots, G_{r^{n-1}}$ are any graphs not contained in Γ_n. Indeed, assigning a distinct color to each of the r^{n-1} copies of Γ_n in this decomposition results in an r^{n-1}-coloring of K_{r^n} having no monochromatic G_i, $1 \leq i \leq r^{n-1}$. In most cases, known lower and upper bounds on the numbers $r(G_1, \ldots, G_k)$ or even $r_k(G)$ are far apart. One of the best results is related to $r_k(C_4)$, where C_4 denotes a 4–cycle. Here we have the following result due to Chung and Graham [**?**]:

$$k^2 - k + 2 \leq r_k(C_4) \leq k^2 + k + 1,$$

where the upper bound holds for all $k \geq 1$, and the lower bound holds for $k-1$ being a prime power.

It is easy to see that the graph $\Gamma_2(\mathbb{F}_q; f_2)$, where $f_2 = a_1 b_1$, is C_4–free. Therefore, when q is odd, Theorem **??** implies that $r_q(C_4) \geq q^2 + 1$, which is an improvement to the lower bound $k^2 - k + 2$ mentioned above. With additional effort, the authors are able to show (see [**?**]) that for all odd prime powers q,

$$r_q(C_4) \geq q^2 + 2.$$

5. Concluding remarks.

5.1. **Generalizations.** All constructions in this paper can be carried out in a more general setting in which the ring R is replaced by an arbitrary abelian group G. In such case, the condition that 2_R be a unit in R corresponds to G being 2-divisible (in other words, given any $g \in G$ the equation $x + x = g$ can always be solved for x in G). The condition that $char(R) = 2$ corresponds to every non-identity element of G being an involution. Our motivation for presenting our results at the level of rings stems from the fact that in this case our conclusions are much more uniform.

Most of the results mentioned in previous sections are concerned with the graphs $B\Gamma_n$, Γ_n, and their immediate generalizations $B\Gamma_n[A]$ and $\Gamma_n[A]$. There are several other useful families of graphs closely related to the ones above, and sharing many of their nice properties. Here we briefly mention them. All proofs and additional details can be found in [**?**].

As in [**?**], the *bipartite double* of a graph Γ is the graph $2 * \Gamma$ with vertex set $V(2*\Gamma) = V^1 \cup V^2$, where $V^i = \{v^i \mid v \in V(\Gamma)\}$, $i = 1, 2$, and edge set $E(2*\Gamma) = \{u^i v^j \mid uv \in E(\Gamma), 1 \le i \ne j \le 2\}$.

Our next result gives a stronger connection between the graphs $B\Gamma_n$ and Γ_n. Roughly speaking, it states that when all f_i are symmetric functions, then "$B\Gamma_n$ minus a specified perfect matching" is the bipartite double of "Γ_n minus a matching."

The descriptions of the two matchings are virtually identical, each consisting of the edges adjoining vertices with the same first coordinate. Specifically, we set $E' = \{(a)[b] \in E(B\Gamma_n) \mid a_1 = b_1\}$ and $E'' = \{ab \in E(\Gamma_n) \mid a_1 = b_1\}$, and we define $B\Delta_n$ and Δ_n to be the spanning subgraphs of $B\Gamma_n$ and Γ_n, respectively, with corresponding edge sets $E(B\Delta_n) = E(B\Gamma_n) \setminus E'$ and $E(\Delta_n) = E(\Gamma_n) \setminus E''$. The fact that E' is a perfect matching follows from the proof of Theorem **??**. The matching E'' can be characterized by the property that vertex $a \in V(\Gamma_n)$ is covered by this matching if and only if $(a) \notin Abs(B\Gamma_n, \pi)$. As a consequence, E'' is a *maximal* matching precisely when $\{a \in V(\Gamma_n) \mid (a) \in Abs(B\Gamma_n, \pi)\}$ is an independent set in Γ_n.

Theorem 8. *Assuming notation as above, we have the following:*

(1) *The polarity π of Proposition **??** restricts to a polarity of $B\Delta_n$, also denoted by π. Unlike $(B\Gamma_n, \pi)$ however, graph $(B\Delta_n, \pi)$ has no absolute points.*
(2) *The isomorphism of Theorem **??** restricts to an isomorphism $(B\Delta_n)^\pi \cong \Delta_n$.*
(3) *Graph $B\Delta_n$ is the bipartite double $2 * \Delta_n$ of graph Δ_n.*

It is easy to show that graphs $B\Delta_n$ and Δ_n admit star-complete colorings, and that graph $B\Delta_n$ is $(r-1)$-regular when R is finite of cardinality r. This, together with (3) of Theorem **??**, establishes that Δ_n is $(r-1)$-regular as well; cf. Theorem **??** and Corollary **??** in Section **??**.

In the same manner as in **??**, one obtains the induced subgraphs $B\Delta_n[A, B]$ and $\Delta_n[A]$ of $B\Delta_n$ and Δ_n, respectively, though these may also be envisioned as the subgraphs of $B\Gamma_n[A, B]$ and $\Gamma_n[A]$ obtained by deleting edges which adjoin vertices with the same first coordinates. One can easily show that these graphs satisfy Theorems **??**, **??**.

One can further generalize $B\Delta_n$ and Δ_n as follows. Given any subset A of R for which $A = -A := \{-\alpha \mid \alpha \in A\}$, we define $B\Gamma_n(A)$ (resp., $\Gamma_n(A)$) to be the graph with vertex set $V(B\Gamma_n(A)) = V(B\Gamma_n)$ (resp., $V(\Gamma_n(A)) = V(\Gamma_n)$) and edge set $E(B\Gamma_n(A)) = \{(a)[b] \in E(B\Gamma_n) \mid a_1 - b_1 \in A\}$ (resp., $E(\Gamma_n(A)) = \{ab \in E(\Gamma_n) \mid a_1 - b_1 \in A\}$). In this case, graphs $B\Gamma_n$ introduced in **??** correspond to the graphs $B\Gamma_n(R)$ while those in **??** correspond to $\Gamma_n(R)$. This definition can be extended to $n = 1$ if we assume that the *only* relation defining the edges is $a_1 - b_1 \in A$. Moreover, letting R^* denote the subset of all nonzero elements of R, we now obtain the graphs $B\Delta_n$ and Δ_n as $B\Gamma_n(R^*)$ and $\Gamma_n(R^*)$, respectively.

Again, the condition $A = -A$ is necessary only in the case of graphs $\Gamma_n(A)$ to ensure that adjacency be symmetric. Otherwise, one may certainly investigate the digraphs so obtained.

Graphs $\Gamma_n(A)$ generalize those introduced by Jacobson, Truszczyński and Tuza in [?]; indeed the latter graphs are realized via the specialization: $R = \mathbb{Z}/m\mathbb{Z}$, $n = 1$.

5.2. **Some open questions.** It would be of interest to relate other properties of the graphs introduced in this paper to properties of the ring R and functions f_i used in their definitions. Even in the simplest of cases, say $n = 2$ or 3 and $R = \mathbb{F}_q$ or $R = \mathbb{Z}/m\mathbb{Z}$, one already anticipates great diversity among the graphs, and it would be interesting to characterize which rings and functions produce graphs which have no cycle(s) of given length; or are Cayley graphs; or are vertex- and/or edge-transitive; or are hamiltonian; or have chromatic number at least 4. The questions one can ask are endless.

It would also be of interest to find additional examples of graphs (neither isomorphic to the ones in [?] or in this paper) which admit neighbor-complete or star-complete colorings.

Some preliminary results we have obtained indicate that some of these questions are as interesting over infinite rings R as they are over finite ones.

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