NEW CONSTRUCTIONS OF BIPARTITE GRAPHS ON m, n VERTICES, WITH MANY EDGES, AND WITHOUT SMALL CYCLES [1]

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Abstract

For arbitrary odd prime power q and $s \in (0,1]$ such that q^s is an integer, we construct a doubly–infinite series of (q^5, q^{3+s})–bipartite graphs which are biregular of degrees q^s and q^2 and of girth 8. These graphs have the greatest number of edges among all known (n,m)–bipartite graphs with the same asymptotics of $\log_n m$, $n \to \infty$. For $s = 1/3$, our graphs provide an explicit counterexample to a conjecture of Erdős which states that an (n,m)–bipartite graph with $m = O(n^{2/3})$ and girth at least 8 has $O(n)$ edges. This conjecture was recently disproved by de Caen and Székely [2], who established the existence of a family of such graphs having $n^{1+1/57+o(1)}$ edges. Our graphs have $n^{1+1/15}$ edges, and so come closer to the best known upper bound of $O(n^{1+1/9})$.

1. Introduction.

All graphs we consider are simple. The *order* (*size*) of a graph G is the number of its vertices (edges). We denote the order of G by $v = v(G)$, and the size of G

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by $e = e(G)$. If G contains a cycle, the *girth* of G, denoted $g(G)$, is the length of its shortest cycle. Let $f(n,m)$ denote the maximum size of a bipartite graph whose partitions have the cardinalities n, m $(n \geq m)$ and whose girth is at least 8. It is well known that $f(n,n) = \Theta(n^{4/3})$, and it is easy to show that $f(n,m) = \Theta(n)$ for $m = O(n^{1/2})$. Recently, de Caen and Székely [2], and independently Faudree and Simonovits [4], proved that $f(n,m) = O(n^{2/3}m^{2/3})$. The remarks above show that this upper bound is asymptotically tight when $n = m$, or $m = O(n^{1/2})$. In [2], using generalized quadrangles, the authors demonstrated that $f(n,m) = \Omega(n^{2/3}m^{2/3})$ also when $m \sim n^{4/5}$ and $m \sim n^{7/8}$.

Another important result in [2] was a disproof of an old conjecture of Erdős (see e.g. [3]) that $f(n,m) = O(n)$ for $m = O(n^{2/3})$. For n, $m \sim n^{\delta}$ $(1/2 < \delta < 7/10)$ they proved the existence of a 4– and 6–cycle free bipartite graph with $n^{1+\epsilon}$ edges $(\epsilon = \epsilon(\delta) > 0)$, which implies that the size of such a graph can be a superlinear function of its order. The partitions of their graphs have cardinalities $t^{1+\sigma}$ and $t^{2+\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}$ and the number of edges is $t^{2+2\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}$ $(0 < \sigma < 1/5)$. Let us denote these graphs by CS_{σ}. In particular, setting $\sigma = 1/37$, they obtained a graph with $m \sim n^{2/3}$ and $n^{1+1/57+o(1)}$ edges. As the authors pointed out, this disproved Erdős' conjecture, but fell well short of their upper bound $O(n^{1+1/9})$. The proof of existence is very clever and not easy; it blends some results and techniques from combinatorial number theory and set systems.

The main results of this paper are the following:

For arbitrary odd prime power q, and arbitrary real number s, $0 < s \leq 1$, such that q^s is an integer, we construct an (n,m)–bipartite graph $B_s = B_s(q)$ such that the partition sets of B_s have $n = q^5,\ m = q^{3+s}$ vertices, $e(B_s) = q^{5+s}$, $g(B_s) = 8$, and B_s is biregular of degrees q^s and q^2. Let $x_0 = \frac{75-\sqrt{5265}}{180} \approx .0135$, $x_1 = \frac{85-\sqrt{6885}}{170} \approx .0119$ and $x_2 = \frac{150-\sqrt{17280}}{290} \approx .0639$. For each $s \in (0,1]$, there exists a unique $\sigma \in (x_1, x_2]$ such that the graphs B_s and CS_σ have the same asymptotics of $\log_n m$. Similarly, for each $\sigma \in (x_1, x_2]$, there exists a unique $s \in (0,1]$ such that the graphs B_s and CS_σ have the same asymptotics of $\log_n m$. Moreover for all such $\sigma \in (x_0, x_2]$ and corresponding s, $e(B_s) > e(CS_\sigma)$. In particular, for $m = n^{2/3}$, the sizes of our graphs are $n^{1+1/15}$.

The method we use in this paper is quite elementary and the presentation is self–contained. It is motivated by the idea of embedding Chevalley group geometries in Lie algebras [7] and it has proven to be succesful in producing bipartite graphs of high girth with many edges [5,6,8,9]. A new additional simple technique within the method, namely a certain way of constructing induced subgraphs, is introduced here to build graphs $B_s(q)$. We hope that it may become a useful tool in the future.

2. Construction of graphs $B_s(q)$.

Let q be a odd prime power, F_q be the field of q elements, and F_{q^2} be the quadratic extention of F_q. Let $P = \{(p) = (p_1, p_2, p_3) \mid p_1 \in F_q,\ p_2 \in F_{q^2},\ p_3 \in F_q\ \}$ and $L = \{[l] = [l_1, l_2, l_3] \mid l_1 \in F_{q^2},\ l_2 \in F_{q^2},\ l_3 \in F_q\ \}$. We define a bipartite graph with partition sets P and L. The parentheses and brackets will allow us to distinguish vectors from different partitions. We will refer to vectors from P as *points* and vectors from L as *lines*. We define adjacency I between points and lines (and write $(p)I[l]$) as follows: point $(p) = (p_1, p_2, p_3)$ is adjacent to line $[l] = [l_1, l_2, l_3]$

if and only if the following conditions are satisfied:

$$\begin{cases} l_2 - p_2 = l_1 p_1 \\ l_3 - p_3 = \bar{l}_1 p_2 + l_1 \bar{p}_2 \end{cases} \tag{1}$$

where $x \to \bar{x}$ denotes the involutory automorphism of F_{q^2} with fixed field F_q. This defines a bipartite graph $B = B(q)$ whose vertex partition sets are P and L, and whose edges are $\{(p), [l]\}$ if and only if $(p)I[l]$.

Remark. There is an alternate description of graphs $B(q)$ which comes from the geometry of the rank two twisted Chevalley group ${}^2A_3(q)$. The graph $B(q)$ arises in this case as a certain induced subgraph of the incidence graph of this geometry. Using [7] it is possible to obtain the description of $B(q)$ presented in this paper.

Theorem 1. *For every odd prime power q, the bipartite graph $B = B(q)$ satisfies the following properties:*

(a) B is biregular of order $q^4 + q^5$ and size q^6. The degree of every point is q^2, and the degree of every line is q.

(b) $g(B) = 8$.

Proof. (a) Obviously, $v(B) = |P| + |L| = q^4 + q^5$. It is immediate from (1) that for a fixed $(p) \in V(B)$, the components of a line $[l] \in V(B)$ adjacent to (p) are determined uniquely by the value of l_1, which can be any element of F_{q^2}. Therefore, the degree of (p) in B is q^2. In the same way we obtain that the degree of a line $[l]$ in B is q. Therefore B is biregular and $e(B) = q^6$.

Our proof of part (b) will be facilitated by the following observations. First we notice that a graph G contains no $2k$–cycle, $k \geq 2$, if there is at most one simple path of length k between any two of its vertices. We will show that any pair of vertices of B is connected by at most one simple path of length k, $k = 2, 3$. This will imply that $g(B) \geq 8$ since, being a bipartite graph, B contains no odd cycles. Then we show that $g(B) = 8$.

Another observation is the existence of certain automorphisms of B. In order to describe the automorphisms, we make a convention that x,y will denote an arbitrary element of F_q, F_{q^2}, respectively. Let $t_i(z)$, $i = 0, 1, 2, 3$, be the mappings $V(B) \to V(B)$ defined as:

$$
\begin{aligned}
(p)^{t_0(x)} &= (p_1 + x, p_2, p_3)\\
[l]^{t_0(x)} &= [l_1, l_2 + l_1 x, l_3]\\
(p)^{t_1(y)} &= (p_1, p_2 - p_1 y, p_3 + 2p_1 y\bar{y} - 2(p_2\bar{y} + \bar{p}_2 y))\\
[l]^{t_1(y)} &= [l_1 + y, l_2, l_3 - (l_2\bar{y} + \bar{l}_2 y)]\\
(p)^{t_2(y)} &= (p_1, p_2 + y, p_3)\\
[l]^{t_2(y)} &= [l_1, l_2 + y, l_3 + (\bar{l}_1 y + l_1\bar{y})]\\
(p)^{t_3(x)} &= (p_1, p_2, p_3 + x)\\
[l]^{t_3(x)} &= [l_1, l_2, l_3 + x]
\end{aligned}
$$

Lemma 2. *(i) For every $x \in F_q$, $y \in F_{q^2}$, the mappings $t_0(x)$, $t_1(y)$, $t_2(y)$, and $t_3(x)$ are automorphisms of graph B and $t_i^{-1}(z) = t_i(-z)$, $i = 0, 1, 2, 3$.*
(ii) For every edge $\{[l], (p)\}$ of B there exists an automorphism α of B such that $[l]^\alpha = [0, 0, 0]$ and $(p)^\alpha = (0, 0, 0)$. Thus the automorphism group $Aut(B)$ acts transitively on edges, on points, and on lines.

Proof. We shall prove part (i) of the lemma for $t_1(y)$ only. For other $t_i(z)$ it can be done similarly (and actually faster). Let $\{(p), [l]\}$ be an arbitrary edge of B, where $(p) = (p_1, p_2, p_3), [l] = [l_1, l_2, l_3]$. In terms of components their adjacency is represented by the system (1). The condition that $\{(p)^{t_1(y)}, [l]^{t_1(y)}\}$ is an edge of B is given by the following system:

$$\begin{cases} l_2 - (p_2 - p_1 y) = (l_1 + y)p_1 \\ l_3 - (l_2\bar{y} + \bar{l}_2 y) - (p_3 + 2p_1 y\bar{y} - 2(p_2\bar{y} + \bar{p}_2 y)) = (\bar{l}_1 + \bar{y})(p_2 - p_1 y) + (l_1 + y)(\bar{p}_2 - \bar{p}_1 y) \end{cases} \tag{2}$$

It is a straightforward verification that systems (1) and (2) are equivalent for all y. It is obvious that $t_1(y)$ is a well–defined injection on P and L. Therefore it is a bijection. Also $t_1(y)t_1(-y) = t_1(-y)t_1(y)$ is the identity mapping on $V(B)$. Therefore, part (i) of the lemma is proven.

Let $[l] = [l_1, l_2, l_3]$ and $(p) = (p_1, p_2, p_3)$ be any two adjacent vertices of B. Let $\alpha_0 = t_1(-l_1)t_2(-l_2)t_3(-l_3 - \bar{l}_1 l_2 - l_1\bar{l}_2)$. It is easily checked that $[l]^{\alpha_0} = [0, 0, 0]$. Since automorphisms preserve adjacency, $(p)^{\alpha_0} = (p_1', p_2', p_3')$ is adjacent to $[0, 0, 0]$ so that $p_2' = p_3' = 0$ by (1). Also $p_1' = p_1$, since α_0 fixes the first component of every point. We now observe that the desired automorphism is given by $\alpha = \alpha_0 t_0(-p_1)$.■

Now we show that any pair of vertices of B is connected by at most one simple path of length 3. Without loss of generality, we assume that the first vertex on the path is a point, and the last is a line. Let $(p^0)I[l^1]I(p^1)I[l^2]$ be our path. By transitivity on points we may assume $(p^0) = (0, 0, 0)$. We denote the first components of $[l^1]$, (p^1) and $[l^2]$ by x, y and z respectively. The conditions of adjacency (1) on subsequent vertices of the path written in terms of their components allow us to express all components of vertices of the path in terms of x, y, z: $(p^0)I[l^1]$ gives $[l^1] = [x, 0, 0]$, $[l^1]I(p^1)$ gives $(p^1) = (y, -xy, x\bar{x}(y + \bar{y}))$ and $(p^1)I[l^2]$ gives $[l^2] = [z, y(z - x), 2yx\bar{x} - y(x\bar{z} + \bar{x}z)]$. We first show there are no 4–cycles in B. For this, it suffices to check that $[l^2]$ is not adjacent to $(p^0) = (0, 0, 0)$. But adjacency here would imply that $y(z - x) = 0$. Since $(p^1) \neq (p^0)$, then $y \neq 0$, and therefore we get $x = z$. But this implies that $[l^1] = [l^2]$, which is not the case.

We now show B contains no 6–cycles. Let us look at another path $(p^0)I[\tilde{l}^1]I(\tilde{p}^1)I[l^2]$ between $(p^0) = (0,0,0)$ and $[l^2]$. Then $[\tilde{l}^1] = [x', 0, 0]$ and $(\tilde{p}^1) = (y', -x'y', x'\bar{x}'(y' + \bar{y}'))$. Since $(\tilde{p}^1)I[l^2]$, then

$$\begin{cases} y'(z - x') = y(z - x) \\ 2y'x'\bar{x}' - y'(x'\bar{z} + \bar{x}'z) = 2yx\bar{x} - y(x\bar{z} + \bar{x}z) \end{cases} \tag{3}$$

This system can be rewritten as

$$\begin{cases} z(y - y') = xy - x'y' \\ 2yx\bar{x} - 2y'x'\bar{x}' = z(\bar{x}y - \bar{x}'y') + \bar{z}(xy - x'y') \end{cases} \tag{4}$$

Now multiply both sides of the second equation of (4) by $y - y'$. Replacing $z(y - y')$ by $xy - x'y'$ (from (3)), and $\bar{z}(y - y')$ by $\bar{x}y - \bar{x}'y'$ (since $y = \bar{y}, y' = \bar{y}'$), we obtain

$$(2yx\bar{x} - 2y'x'\bar{x}')(y - y') = (xy - x'y')(\bar{x}y - \bar{x}'y') + (\bar{x}y - \bar{x}'y')(xy - x'y').$$

Since q is odd by assumption, this simplifies to $yy'(x - x')(\bar{x} - \bar{x}') = 0$. Hence $y = 0$, $y' = 0$, or $x = x'$, which implies $(p^1) = (p^0)$, $(\tilde{p}^1) = (p^0)$, or $[l^1] = [\tilde{l}^1]$, respectively. Since B contains no 4–cycles and our paths are assumed distinct, none of these cases is possible. Therefore there are no two interior vertex disjoint paths between (p^0) and $[l^2]$, and B is 6–cycle free. Since B is bipartite, $g(B) \geq 8$. Suppose $g(B) > 8$. Counting the vertices at distance 0, 2 or 4 from a fixed line gives the inequality

$$1 + q(q^2 - 1) + q(q^2 - 1)^2(q - 1) \leq |L| = q^5,$$

a contradiction since $q \geq 3$. Thus $g(B) = 8$. ■

We are now ready to describe the graphs $B_s(q)$. Let $S \subseteq F_q$, $s = \log_q |S|$, and $T \subseteq F_{q^2}$, $t = \log_q |T|$.

We define $P_S = S \times F_{q^2} \times F_q$ and $L_T = T \times F_{q^2} \times F_q$. Let $B_{S,T} = B_{S,T}(q)$ be the subgraph of B induced by the set of vertices $P_S \cup L_T$. The proof of the following theorem is an immediate corollary of Theorem 1 and the definition of $B_{S,T}$.

Theorem 3. *For every odd prime power q, the bipartite graph $B_{S,T}(q)$ is biregular of order $q^{3+s} + q^{3+t}$, size q^{3+s+t} and girth at least 8. The degree of every point of P_S is q^t, and the degree of every line from L_T is q^s.*

If $T = F_{q^2}$, we will write $B_s = B_s(q)$ instead of $B_{S,F_{q^2}}$. This abridged notation is justified by the fact that the properties of graphs $B_{S,T}(q)$ with which we are here concerned will not depend on the subset S but only on its cardinality s. The partitions of B_s have cardinalities $n = q^5$ and $m = q^{s+3} = n^{\frac{s+3}{5}}$, and the size of B_s is $e(B_s) = n^{1+s/5}$, i.e. the size is a superlinear function of the order (since $v \sim n$).

It is also easy to see that $g(B_s) = 8$ for all $s \in (0, 1]$ and all $q \geq 3$. Indeed, if $g(B_s) > 8$, then counting the vertices at distance 0, 2 or 4 from a fixed point we get $1 + q^2(q^s - 1) + q^2(q^s - 1)^2(q^2 - 1) \leq |P| = q^{3+s}$, which is equivalent to

$$1 + q^2[q^2(q^s - 1)^2 - (qq^s + (q^s - 1)(q^s - 2))] \leq 0 \tag{5}$$

Obviously (5) does not hold for $q^s = 2, q \geq 3$. Neither does it for $q^s \geq 3$, since in this case

$$1 + q^2[q^2(q^s - 1)^2 - (qq^s + (q^s - 1)(q^s - 2))] \geq 1 + q^2[4q^2 - (2q^2 + 3q - 2)] =$$

$$1 + q^2(2q^2 + 3q - 2) > 0.$$

Therefore $g(B_s) = 8$.

Now we compare graphs B_s with the graphs CS_σ of de Caen and Székely which have the same asymptotics of $\log_n m$. Let n, m ($n \geq m$) denote the cardinalities of partitions in B_s and let a, b ($a \geq b$) denote the cardinalities of the partitions in the de Caen–Székely graphs.

Theorem 4.

(i) *Let* $x_1 = \frac{85-\sqrt{6885}}{170} \approx .0119$ *and* $x_2 = \frac{150-\sqrt{17280}}{290} \approx .0639$. *Then for every* $s \in (0,1]$, *there exists a unique* $\sigma \in (x_1, x_2]$ *such that for graphs* B_s *and* CS_σ, $\log_n m \sim \log_a b$. *Similarly, for every* $\sigma \in (x_1, x_2]$, *there exists a unique* $s \in (0,1]$ *such that for graphs* B_s *and* CS_σ, $\log_n m \sim \log_a b$;

(ii) *Let* $x_0 = \frac{75-\sqrt{5265}}{180} \approx .0135$. *Then for all* $\sigma \in (x_0, x_2]$, $e(B_s) > e(CS_\sigma)$ *whenever* $\log_n m \sim \log_n b$;

(iii) *For* $m = n^{2/3}$, $e(B_s) = n^{1+1/15}$.

Proof. As was mentioned in the Introduction, the cardinalities a, b ($a \geq b$) of the partitions in the de Caen–Székely graphs are $t^{1+\sigma}$ and $t^{2+\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}$, and the size $e(CS_\sigma)$ is $t^{2+2\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}$ $(0 < \sigma < 1/5)$. Comparing the numbers of points in the partitions, it is easy to check that for $0 < \sigma < \frac{3-\sqrt{5}}{6}$, $\log_a b = \frac{1+\sigma}{2+\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}$, $\log_a e(CS_\sigma) = \frac{2+2\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}{2+\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}$, and for $\frac{3-\sqrt{5}}{6} < \sigma < 1/5$, $\log_a b = \frac{2+\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}{1+\sigma}$, and $\log_a e(CS_\sigma) = \frac{2+2\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}{1+\sigma}$. For graphs B_s, $\log_n m = \frac{s+3}{5} \in (\frac{3}{5}, \frac{4}{5}]$. It is easy to check that for $0 < \sigma < 1/5$, $\log_a b \in (\frac{3}{5}, \frac{4}{5}]$ if and only if $\sigma \in (x_1, x_2]$. The condition $\log_n m \sim \log_a b$ is equivalent to $\frac{3+s}{5} \sim \frac{1+\sigma}{2+\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}$. This proves part (i).

It is another trivial verification that $\frac{3+s}{5} \sim \frac{1+\sigma}{2+\sigma-3\sqrt{\sigma(1-\sigma)}+o(1)}$ implies $\log_n e(B_s) >$

$\log_a e(CS_\sigma)$ for all $\sigma \in (x_0, x_2]$. Setting $n = a$, we prove part (ii).

Since $m = n^{2/3}$ if and only if $s = 1/3$, we get $e(B_{1/3}) = n^{1+1/15}$, which proves part (iii). ∎

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