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On the number of irregular assignments on a graph

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Abstract

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Let G be a simple graph which has no connected components isomorphic to K_1 or K_2, and let $\mathbb{Z}^+$ be the set of positive integers. A function $\omega : E(G) \to \mathbb{Z}^+$ is called an assignment on G, and for an edge e of G, $\omega(e)$ is called the weight of e. We say that ω is of strength s if $s = \max\{\omega(e): e \in E(G)\}$. The weight of a vertex in G is defined to be the sum of the weights of its incident edges. We call an assignment ω irregular if distinct vertices have distinct weights. Let $\mathrm{Irr}(G, \lambda)$ be the number of irregular assignments on G with strength at most λ. We prove that

$$|\mathrm{Irr}(G, \lambda) - \lambda^q + c_1\lambda^{q-1}| = O(\lambda^{q-2}), \quad \lambda \to \infty$$

where $q = |E(G)|$ and c_1 is a constant depending only on G. An explicit expression for c_1 is given. Analysis of this expression enables us to determine which graph with q edges has the least number of irregular assignments of strength at most λ, for λ sufficiently large.

1. Introduction

Let G be a simple graph with $|E(G)| = q$ and $|V(G)| = v$, and assume G has no connected components isomorphic to K_1 or K_2 (K_n is the complete graph on n vertices). A function $\omega : E(G) \to \mathbb{Z}^+$ is called an *assignment* on G, and for an edge e of G, $\omega(e)$ is called the *weight* of e. We say that ω is of *strength* $s(\omega)$ if $s(\omega) = \max\{\omega(e): e \in E(G)\}$. The *weight of a vertex* $x \in V(G)$ is defined to be the sum of the weights of its incident edges, and is denoted $wt(x)$. We call an assignment ω *irregular* if distinct vertices have distinct weights. The *irregularity strength* $s(G)$ *of* G is defined as $s(G) = \min\{s(\omega): \omega$ is an irregular assignment on $G\}$.

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One motivation for studying $s(G)$ stems from problems related to highly irregular graphs and multigraphs (see [1–3]). Suppose an assignment ω on a graph G is given. Then we can consider a multigraph $G^\star$ obtained in the following way: each edge e of G of weight $\omega(e)$ is replaced by $\omega(e)$ parallel edges. For any $x \in V(G) = V(G^\star)$, the degree of x in $G^\star$ is equal to $wt(x)$. If ω is an irregular assignment on G, then all vertices of $G^\star$ have distinct degrees (which never happens in a simple graph!). Therefore, $s(G)$ is the least of the maximum edge multiplicities of multigraphs which have G as an underlying graph and in which the degrees of all vertices are distinct.

The problem of studying $s(G)$ was proposed by Chartrand et al. in [4]. It proved to be rather hard, even for very simple graphs ([4–5, 8, 10–12, 17]). The recent study of irregular assignments established several connections between the concept of irregularity and hypergraph theory, integer matrix designs, finite projective geometry, etc. ([13–14, 19]). An excellent survey on the subject was written by Lehel [18].

In this paper we continue the study of irregular assignments, but we shift our attention from $s(G)$ to the number of irregular assignments on G with strength at most λ. Let us denote this number by $\mathrm{Irr}(G, \lambda)$. Our initial hope was that $\mathrm{Irr}(G, \lambda)$ would relate to $s(G)$ in a way similar to which the chromatic polynomial of a graph relates to the chromatic number of the graph. To some extent, we found this to be the case. For example, we noticed the following.

(i) $s(G)$ is the least positive value of λ such that $\mathrm{Irr}(G, \lambda) > 0$.

(ii) For some graphs G, $\mathrm{Irr}(G, \lambda)$ is a polynomial function of λ of degree q:
if $K_{1,n}$ denotes a star with n edges, then $\mathrm{Irr}(K_{1,n}, \lambda) = \lambda(\lambda-1)(\lambda-2) \cdots (\lambda - n + 1)$;
if C_n denotes a cycle of length n, then $\mathrm{Irr}(C_3, \lambda) = \lambda(\lambda-1)(\lambda-2)$;
if P_n denotes a path with n edges, then $\mathrm{Irr}(P_3, \lambda) = \lambda(\lambda-1)^2$;
if G is the graph with $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 4\}\}$, then $\mathrm{Irr}(G, \lambda) = \lambda^4 - 3\lambda^3 - 2\lambda$.

In contrast, $\mathrm{Irr}(C_4, \lambda)$ is not a polynomial of λ. Indeed, it would otherwise have degree at most four by Theorem A below. However, this is inconsistent with its values for $\lambda = 2, 3, 4, 5, 6$.

Even for some small graphs the computation of $\mathrm{Irr}(G, \lambda)$ may not be very easy (e.g., try to compute $\mathrm{Irr}(P_4, \lambda)$). We do not know any reasonable way of computing $\mathrm{Irr}(G, \lambda)$ for an arbitrary graph G. However, we prove the following in the next section.

Theorem A. $|\mathrm{Irr}(G, \lambda) - \lambda^q + c_1\lambda^{q-1}| = O(\lambda^{q-2})$, $\lambda \to \infty$, *where* c_1 *is a constant depending on* G *only.*

In Section 3, we use an explicit expression for c_1 to establish the following extremal result.

Theorem B. *For all but finitely many* $q \in \mathbb{Z}^+$, $\mathrm{Irr}(K, \lambda) < \mathrm{Irr}(G, \lambda)$, $\lambda \to \infty$, *where*

$$K = \begin{cases} K_{1,n} + K_{1,n}, & q = 2n, \\ K_{1,n} + K^{\star}_{1,n}, & q = 2n+1 \end{cases}$$

and G *is an arbitrary graph with* q *edges,* $G \neq K$. *(Here* $K^{\star}_{1,n}$ *denotes the graph obtained from* $K_{1,n}$ *by replacing a single edge with a path of length* 2.)

Also in Section 3, we specialize to the family $\mathbb{F}_q$ of forests with q edges. Here we extend our results to include not only the extremal graph K, but forests $K = K_1, K_2, \ldots, K_r$, where K_i is extremal in $\mathbb{F}_q \setminus \{K_1, K_2, \ldots, K_{i-1}\}$, with $r = 2$ for q even and $r = 7$ for q odd.

2. Proof of Theorem A: The behavior of Irr(G, λ) for large λ

The idea of the proof is to compute $\mathrm{Irr}(G, \lambda)$ by using inclusion-exclusion and then find asymptotics for some terms in the formula as $\lambda \to \infty$. Let us number all unordered pairs of distinct vertices of G by integers from 1 to $\binom{v}{2}$:

$$\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_{\binom{v}{2}}, j_{\binom{v}{2}}\}.$$

Let Ω be the set of all (both irregular and not irregular) assignments on G of strength at most λ. Obviously, $|\Omega| = \lambda^q$. For each k, $1 \leqslant k \leqslant \binom{v}{2}$, we define A_k as

$$A_k = \{\omega\colon \omega \text{ is an assignment on } G,\ 1 \leqslant s(\omega) \leqslant \lambda,\ wt(i_k) = wt(j_k)\}$$

For each $I \subseteq \{1, 2, \ldots, \binom{v}{2}\}$, let

$$A_I = \bigcap_{i \in I} A_i; \qquad A_\emptyset = \Omega;$$

and let

$$S_m = \sum_{|I|=m} |A_I|, \quad 0 \leqslant m \leqslant \binom{v}{2}.$$

Then by inclusion-exclusion,

$$\mathrm{Irr}(G, \lambda) = \sum_{m=0}^{\binom{v}{2}} (-1)^m S_m. \tag{2.1}$$

In order to compute S_m, we use the following well-known facts. We adopt the convention that whenever a lower index in any binomial coefficient is negative, then the binomial coefficient is equal to 0.

Proposition 2.1. *Let* n, k *be integers,* $n \geqslant 1$, $k \geqslant 0$. *Then the number of (ordered) solutions of the equation* $n = x_1 + x_2 + \cdots + x_k$ *with the* x_i*'s being positive integers is* $\binom{n-1}{k-1}$.

Proposition 2.2. *Let n, k be integers, $n \geq 1$, $k \geq 0$. Then*

$$\sum_{i=1}^{n} i^k \sim \frac{n^{k+1}}{k+1}, \quad n \to \infty.$$

As an immediate corollary to Proposition 2.2 we observe the following.

Proposition 2.3. *Let a, b, c, λ be positive integers and let p(t) be a polynomial with leading term $a_0 t^m$. Then*

$$\sum_{t=b}^{a\lambda + c} p(t) \sim \frac{a_0 a^{m+1}}{m+1} \lambda^{m+1}, \quad \lambda \to \infty.$$

We now introduce a function that will be used throughout the remainder of this paper. For $x \in V(G)$, let $d_x = d_x(G)$ denote the degree of x in G. Let $f(s, t)$ be a function defined on $\{(s, t): 1 \leq s \leq t; s \in Z, t \in Z\}$ by the formula

$$f(s, t) = \frac{s^{s+t-1}}{(s+t-1)(s-1)!\,(t-1)!}.$$

Lemma 2.4.

$$S_1 = \sum_{k=1}^{\binom{v}{2}} |A_k| \sim c_1 \lambda^{q-1}, \quad \lambda \to \infty,$$

where

$$c_1 = c_1(G) = \sum_{\substack{\{x, y\} \notin E(G) \\ d_x \leq d_y}} f(d_x, d_y) + \sum_{\substack{\{x, y\} \in E(G) \\ d_x \leq d_y}} f(d_x - 1, d_y - 1)$$

Proof. First assume that the kth pair of vertices $\{i_k, j_k\} = \{x, y\}$ is not an edge of G and $d_x \leq d_y$. Since G has no isolated vertices, $d_x \geq 1$. Then

$$|A_k| = \left[\sum_{n=d_y}^{d_x \lambda} \binom{n-1}{d_x - 1}\binom{n-1}{d_y - 1}\right] \lambda^{q - d_x - d_y}.$$

Indeed, using Proposition 2.1, we construct the expression in brackets, which counts the number of ways of assigning labels from 1 to λ to the edges incident to x or y in order to get $wt(x) = wt(y)$; the remaining $q - d_x - d_y$ edges can then be assigned any labels from 1 to λ. Then, using Proposition 2.3 with

$$a_0 = \frac{1}{(d_x - 1)!\,(d_y - 1)!} \quad \text{and} \quad m = (d_x - 1) + (d_y - 1),$$

we obtain

$$|A_k| \sim [f(d_x, d_y) \lambda^{d_x + d_y - 1}] \lambda^{q - d_x - d_y} = f(d_x, d_y) \lambda^{q-1}, \quad \lambda \to \infty.$$

Next assume that the kth pair of vertices $\{i_k, j_k\} = \{x, y\}$ is an edge of G and $1 \le d_x \le d_y$. Then similarly to above (first assigning weight ω_1 to $\{x, y\}$)

$$\begin{aligned}|A_k| &= \left[\sum_{\omega_1=1}^{\lambda} \sum_{n=\omega_1+d_y-1}^{\omega_1+(d_x-1)\lambda} \binom{n-1-\omega_1}{d_x-2}\binom{n-1-\omega_1}{d_y-2}\right]\lambda^{q-d_x-d_y+1} \\ &= \left[\sum_{\omega_1=1}^{\lambda} \sum_{n'=d_y-1}^{(d_x-1)\lambda} \binom{n'-1}{d_x-2}\binom{n'-1}{d_y-2}\right]\lambda^{q-d_x-d_y+1} \\ &= \lambda \sum_{n'=d_y-1}^{(d_x-1)\lambda} p(n')\lambda^{q-d_x-d_y+1}\end{aligned}$$

where

$$p(n') = \binom{n'-1}{d_x-2}\binom{n'-1}{d_y-2}.$$

Using Proposition 2.3 with

$$a_0 = \frac{1}{(d_x-2)!\,(d_y-2)!} \quad \text{and} \quad m = (d_x-2)+(d_y-2),$$

we obtain

$$\begin{aligned}|A_k| &\sim [f(d_x-1, d_y-1)\lambda^{d_x+d_y-2}]\lambda^{q-d_x-d_y+1} \\ &= f(d_x-1, d_y-1)\lambda^{q-1}, \quad \lambda \to \infty.\end{aligned}$$

This completes the proof of the lemma. □

In order to finish the proof of Theorem A we will show that all addends in the right hand side of (2.1) corresponding to $m \ge 2$ are $O(\lambda^{q-2})$, $\lambda \to \infty$.

Lemma 2.5. *For all $m \ge 2$, $S_m = O(\lambda^{q-2})$, $\lambda \to \infty$.*

Proof. Let $I \subseteq \{1, 2, \ldots, \binom{v}{2}\}$ with $|I| = m$. By definition $A_I = \bigcap_{i \in I} A_i$. Let us renumber pairs of vertices in such a way that $I = \{1, 2, \ldots, m\}$ and let, for each $i \in I$, the ith pair be $\{x_i, y_i\}$. Let $C = \{x_i, y_i : i \in I\}$. Consider a binary relation ϕ on C defined as follows. For any $x, y \in C$, $x\phi y$ if and only if one of the following three conditions is satisfied:

(i) $x = y$,

(ii) $\{x, y\}$ is an ith pair, $i \in I$,

(iii) among the first m pairs there exists a sequence of pairs with the following property: The first pair of the sequence contains x, the last pair of the sequence contains y, and each two consecutive pairs of the sequence have exactly one element in common.

It is clear that ϕ is an equivalence relation on C. Let $C_1, C_2, \ldots, C_t$ be the equivalence classes with respective representatives $z_1, z_2, \ldots, z_t$. Although the vertex weights can vary from assignment to assignment, it is clear that, for fixed $\omega \in A_I$, all vertices of C_j have common weight $wt(z_j)$. Of course, it is possible that $wt(z_j) = wt(z_k)$ for a given $\omega \in A_I$, even when $j \neq k$.

Let $G' = G[C]$ be the subgraph of G induced by C and set $q' = |E(G')|$. For each $v \in C(= V(G'))$, let d'_v be the degree of v in G', and $d''_v = d_v - d'_v$. Denote the restriction of ω to $E(G')$ by ω'. For every $v \in C$, by $wt'(v)$ we shall mean the weight of v with respect to ω'. For each j, $1 \leqslant j \leqslant t$, and fixed ω, let

$$b_j(\omega) = b_j = \max_{v \in C_j} \{wt'(v) + d''_v\}, \qquad a_j(\omega) = a_j = \min_{v \in C_j} \{wt'(v) + \lambda d''_v\},$$

and w_j be the common weight with respect to ω of the vertices of C_j. We claim

$$|A_I| = \left[\sum_{s_1=1}^{\lambda} \sum_{s_2=1}^{\lambda} \cdots \sum_{s_{q'}=1}^{\lambda} \left\{\prod_{j=1}^{t} \left[\sum_{w_j=b_j}^{a_j} \left(\prod_{v \in C_j} \binom{w_j - wt'(v) - 1}{d''_v - 1}\right)\right]\right\}\right] \lambda^{q+q'-\sum_{v \in C} d_v} \tag{2.2}$$

Indeed, this follows by considering the following three-step process for constructing an arbitrary assignment $\omega \in A_I$.

(1) Order the q' edges of G' in some fixed way and assign the weight s_i to the ith edge, $1 \leqslant s_i \leqslant \lambda$, $1 \leqslant i \leqslant q'$.

(2) For each $v \in C$, label each of the d''_v edges which join v to a point of $V(G) \backslash C$ in such a way that $wt(v) = w_j$ for all $v \in C_j$. (Observe that we may assume $b_j \leqslant w_j \leqslant a_j$ for all j.)

(3) Assign the weights from the set $\{1, 2, \ldots, \lambda\}$ arbitrarily to all edges of G which are incident to no vertex of C. There are $q + q' - \sum_{v \in C} d_v$ such edges.

We now complete the proof by analyzing the asymptotics of the right-hand side of (2.2). By repeated application of Proposition 2.3 (noting that a_j is of the form $a\lambda + c$) we obtain

$$\begin{aligned} |A_I| &= \mathrm{O}(\lambda^{q' + \sum_{j=1}^{t} \{1 + \sum_{v \in C_j} (d''_v - 1)\} + q + q' - \sum_{v \in C} d_v}) \\ &= \mathrm{O}(\lambda^{2q' + t - |C| + q + \sum_{v \in C} d''_v - \sum_{v \in C} d_v}) \\ &= \mathrm{O}(\lambda^{2q' + t - |C| + q - \sum_{v \in C} d'_v}) = \mathrm{O}(\lambda^{q + t - |C|}). \end{aligned}$$

The last equality follows from the fact that $2q' = \sum_{v \in C} d'_v$.

But $|C_j| \geqslant 2$ for all j, so $q + t - |C| \leqslant q + t - 2t = q - t$ and hence $|A_I| = \mathrm{O}(\lambda^{q-2})$ if $t \geqslant 2$. Now assume $t = 1$. Then, as $m \geqslant 2$, we have $|C| \geqslant 3$, whence $q + t - |C| \leqslant q - 2$ as well. This proves $|A_I| = \mathrm{O}(\lambda^{q-2})$ for all m-element subsets I of $\{1, 2, \ldots, \binom{v}{2}\}$. As the number of such subsets is clearly independent of λ, the proof of Lemma 2.5 is complete. □

Proof of Theorem A. Follows immediately from Lemma 2.4 and Lemma 2.5. □

3. Proof of Theorem B: an extremal result

Recall the function $f(s, t)$ defined in Section 2. The following facts will be used later in this section. Proofs are straightforward and are omitted. In parts (iv) and (v), Stirling's formula is used.

Lemma 3.1. (i) *For fixed* s, $g(t)=f(s,t)$ *is a decreasing function of* t.

(ii) *For fixed* t, $h(s)=f(s,t)$ *is an increasing function of* s.

(iii) $k(s)=f(s,s)$ *is an increasing function of* s.

(iv) *For fixed* α, $0<\alpha<1$, *let* $s=\lfloor \alpha t \rfloor$. *Then*

$$\frac{f(s,s)}{f(t,t)}=\mathrm{O}(\beta^t), \quad t\to\infty,$$

for some β, $0<\beta<1$.

(v) *For fixed* $s>0$,

$$\frac{f(t-s,t-s)}{f(t,t)}\to \mathrm{e}^{-2s}, \quad t\to\infty.$$

Let

$$f^\star(d_x,d_y)=\begin{cases} f(d_x,d_y), & \{x,y\}\notin E(G),\\ f(d_x-1,d_y-1), & \{x,y\}\in E(G).\end{cases}$$

Observe that $c_1(G)$, defined in Lemma 2.4, can now be expressed as

$$c_1(G)=\sum_{\{x,y\}\subseteq V(G)} f^\star(d_x,d_y).$$

We further define

$$f^\star(G)=\max_{\{x,y\}\subseteq V(G)}\{f^\star(d_x,d_y)\}.$$

Let K be the graph which appears in the statement of Theorem B in the Introduction. For the remainder of this section, let $n=\lfloor q/2\rfloor$ where $q=|E(G)|$.

Lemma 3.2. (i) $f^\star(G)\leqslant f(n,n)=f^\star(K)$.

(ii) If $f^\star(G)>f(n-1,n-1)$, *then* $f^\star(G)$ *is either* $f(n,n)$ *or* $f(n,n+1)$. *Moreover*, $f^\star(G)=f(n,n+1)$ *can occur only when* q *is odd.*

Proof. (i) This follows immediately from Lemma 3.1 and the definition of K.

(ii) Let x, y be vertices of G such that $f^\star(G)=f^\star(d_x,d_y)$.

Case 1: $\{x,y\}\notin E(G)$.

Here $f^\star(G)=f^\star(d_x,d_y)=f(d_x,d_y)$. By Lemma 3.1 (i), $f^\star(G)=f(d_x,d_y)\leqslant f(d_x,d_x)$. If $d_x\leqslant n-1$, we have $f^\star(G)\leqslant f(n-1,n-1)$ from Lemma 3.1(iii). So assume $d_x\geqslant n$. If $q=2n$, then $2n\geqslant d_x+d_y$, whence $d_x=d_y=n$ and $f^\star(G)=f(n,n)$. If $q=2n+1$, then $2n+1\geqslant d_x+d_y$, whence $d_x=n$, $d_y=n$ or $n+1$. Here $f^\star(G)$ is either $f(n,n)$ or $f(n,n+1)$. This proves Lemma 3.2 when $\{x,y\}\notin E(G)$.

Case 2: $\{x,y\}\in E(G)$.

Here $f^\star(G)=f(d_x-1, d_y-1)$. Arguing as in Case 1, if $d_x \leqslant n$, we obtain $f^\star(G) \leqslant f(n-1, n-1)$. So assume $d_x \geqslant n+1$. If $q=2n$, then $2n \geqslant d_x+d_y-1 \geqslant 2n+1$, a contradiction. If $q=2n+1$, then $2n+1 \geqslant d_x+d_y-1$, whence $d_x = d_y = n+1$. Thus $f^\star(G)=f(n, n)$, and the proof is complete. □

Lemma 3.3. *Let G be a graph with q edges satisfying $f^\star(G) \leqslant f(n-1, n-1)$. Then $c_1(G) < (1+\epsilon)e^{-2}c_1(K)$ for any $\epsilon > 0$ and sufficiently large q.*

Proof. We may assume $q \geqslant 16$. Let $d = \lfloor 2q/5 \rfloor$. Then G has at most two vertices with degree at least $d+1$. Therefore

$$c_1(G) = \sum_{\{x,\, y\} \subseteq V(G)} f^\star(d_x, d_y) \leqslant \sum_{d_x \leqslant d} f(d_x, d_x) + f(n-1, n-1),$$

by Lemma 3.1 and the hypothesis of the theorem. Clearly $|V(G)| \leqslant 2q$, whence the number of addends in the summation does not exceed $\binom{2q}{2}$. Thus, by Lemma 3.1(iii), we have

$$c_1(G) \leqslant \binom{2q}{2} f(d, d) + f(n-1, n-1).$$

Clearly, $c_1(K) \geqslant f(n, n)$. Therefore

$$\frac{c_1(G)}{c_1(K)} \leqslant \frac{\binom{2q}{2} f(d, d) + f(n-1, n-1)}{f(n, n)} \leqslant \binom{2q}{2} \frac{f(d, d)}{f(n, n)} + \frac{f(n-1, n-1)}{f(n, n)}.$$

By Lemma 3.1 (iv) and (v), $c_1(G)/c_1(K) < (1+\epsilon)e^{-2}$ for any $\epsilon > 0$ and q sufficiently large. The result follows. □

The significance of Lemma 3.3 is that it allows us to reduce the proof of Theorem B to those graphs G for which $f^\star(G) > f(n-1, n-1)$. By Lemma 3.2 the only possibilities for $f^\star(G)$ are $f(n, n)$ and $f(n, n+1)$. These graphs are easy to describe. They fall into nine families and are depicted in Fig. 1 ($q=2n$) and Fig. 2 ($q=2n+1$). In Table 1, we give the corresponding values of $f^\star(G)$ for these graphs. For each graph in Table 1, we can calculate $c_1(G)$ explicitly from the formula given in Lemma 2.4. The results appear in Table 2.

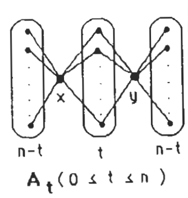

Fig. 1. The graphs G with $f^\star(G) > f(n-1, n-1)$, $q=2n$.

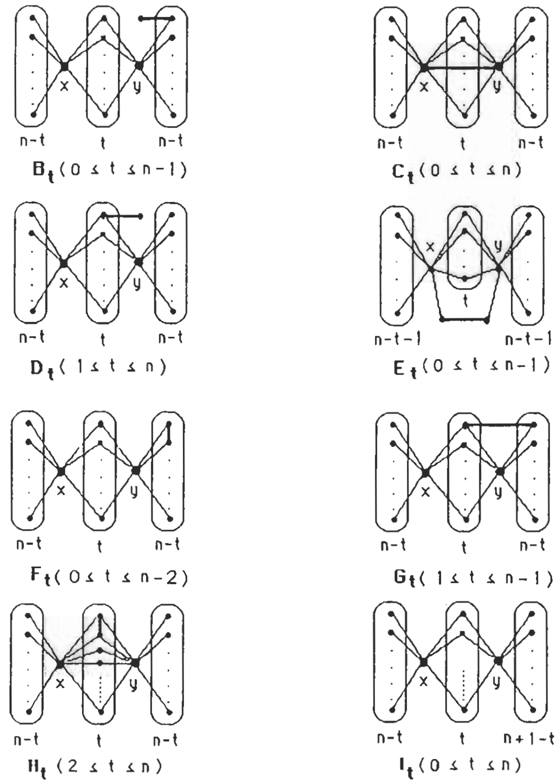

Fig. 2. The graphs G with $f^\star(G) > f(n-1, n-1)$, $q = 2n+1$.

Lemma 3.4. *Let $\{X_t\}$ be one of the nine families appearing in Table 2, and let $t_1 = t_1(X_t)$ and $t_2 = t_2(X_t)$ be the smallest and largest allowable values of t for $\{X_t\}$ respectively. Then, for n sufficiently large,*

$$\max_{t_1 \leqslant t \leqslant t_2} \{c_1(X_t)\} = c_1(X_{t_1}).$$

Proof. Let $g(t) = c_1(X_t) - f^\star(X_t)$, $t_1 \leqslant t \leqslant t_2$. It is easy to verify that $g(t) = (7/3)t^2 + (-3n + \mathrm{O}(1))t + \mathrm{O}(n^2)$. Clearly, the least value of the quadratic $g(t)$ is attained at $t_0 = 9n/14 + \mathrm{O}(1)$. Since $t_0 > (t_1 + t_2)/2$ for large n, the statement follows from simple geometric reasoning. □

Proof of Theorem B. We can now complete the proof. By Theorem A, $\mathrm{Irr}(G, \lambda) = \lambda^q - c_1\lambda^{q-1} + \mathrm{O}(\lambda^{q-2})$ for large λ. Therefore, minimizing $\mathrm{Irr}(G, \lambda)$ is tantamount to maximizing $c_1(G)$ (over all graphs with q edges).

First observe that $K = A_0$ for $q = 2n$ and $K = B_0$ for $q = 2n+1$. By Lemmas 3.2, 3.3 and 3.4, it suffices to prove (for large n) that $c_1(G) < c_1(K)$ where G is any graph listed in Table 3 with the same number of edges as K, $G \neq K$. For

Table 1
$f^\star(G)$ for graphs of Figs 1 and 2

G	q	$f^\star(G)$
A_t	$2n$	$f(d_x, d_y) = f(n, n)$
B_t	$2n+1$	$f(d_x, d_y) = f(n, n)$
C_t	$2n+1$	$f(d_x - 1, d_y - 1) = f(n, n)$
D_t	$2n+1$	$f(d_x, d_y) = f(n, n)$
E_t	$2n+1$	$f(d_x, d_y) = f(n, n)$
F_t	$2n+1$	$f(d_x, d_y) = f(n, n)$
G_t	$2n+1$	$f(d_x, d_y) = f(n, n)$
H_t	$2n+1$	$f(d_x, d_y) = f(n, n)$
I_t	$2n+1$	$f(d_x, d_y) = f(n, n+1)$

Table 2
Expressions for $c_1(X_t)$

X_t	$c_1(X_t)$
A_t	$\binom{2n-2t}{2}f(1,1) + t(2n-2t)f(1,2) + \binom{t}{2}f(2,2) + 2tf(1,n-1) + (2n-2t)f(1,n) + f(n,n)$
B_t	$\binom{2n-2t}{2}f(1,1) + \{(t+1)(2n-2t)-1\}f(1,2) + \binom{t+1}{2}f(2,2) + (2t+1)f(1,n-1) + (2n-2t+1)f(1,n) + f(2,n) + f(n,n)$
C_t	$\binom{2n-2t}{2}f(1,1) + 2t(n-t)f(1,2) + \binom{t}{2}f(2,2) + 2tf(1,n) + (2n-2t)f(1,n+1) + f(n,n)$
D_t	$\binom{2n-2t+1}{2}f(1,1) + (t-1)(2n-2t+1)f(1,2) + (2n-2t)f(1,3) + \binom{t-1}{2}f(2,2) + (t-1)f(2,3) + (2t-2)f(1,n-1) + (2n-2t+2)f(1,n) + 2f(2,n-1) + f(n,n)$
E_t	$\left\{\binom{2n-2t-2}{2}+1\right\}f(1,1) + (t+2)(2n-2t-2)f(1,2) + \left\{\binom{t+2}{2}-1\right\}f(2,2) + (2t+2)f(1,n-1) + (2n-2t-2)f(1,n) + 2f(2,n) + f(n,n)$
F_t	Same as E_t
G_t	$\binom{2n-2t-1}{2}f(1,1) + \{t(2n-2t-1)+1\}f(1,2) + (2n-2t-1)f(1,3) + \binom{t}{2}f(2,2) + (t-1)f(2,3) + (2t-1)f(1,n-1) + (2n-2t-1)f(1,n) + 2f(2,n-1) + f(2,n) + f(n,n)$
H_t	$\binom{2n-2t}{2}f(1,1) + (t-2)(2n-2t)f(1,2) + (4n-4t)f(1,3) + \left\{\binom{t-2}{2}+1\right\}f(2,2) + (2t-4)f(2,3) + (2t-4)f(1,n-1) + (2n-2t)f(1,n) + 4f(2,n-1) + f(n,n)$
I_t	$\binom{2n-2t+1}{2}f(1,1) + t(2n-2t+1)f(1,2) + \binom{t}{2}f(2,2) + tf(1,n-1) + (n+1)f(1,n) + (n-t)f(1,n+1) + f(n,n+1)$

Table 3
Expressions for $c_1(X_t)$, $n \to \infty$

X_t	$c_1(X_t)$
A_0	$f(n, n) + 2n^2 - n + O(1)$
B_0	$f(n, n) + 2n^2 + O(1)$
C_0	$f(n, n) + 2n^2 - n + O(1)$
D_1	$f(n, n) + 2n^2 - 8n/3 + O(1)$
E_0	$f(n, n) + 2n^2 - 3n + O(1)$
F_0	$f(n, n) + 2n^2 - 3n + O(1)$
G_1	$f(n, n) + 2n^2 - 17n/3 + O(1)$
H_2	$f(n, n) + 2n^2 - 25n/3 + O(1)$
I_0	$f(n, n+1) + 2n^2 + n + O(1)$

$G \neq I_0$, this is evident from the table. For $G = I_0$, we have

$$c_1(K) - c_1(G) = c_1(B_0) - c_1(I_0) = f(n, n) - f(n, n+1) + O(n^2).$$

But

$$f(n, n) - f(n, n+1) \sim ke^{2n}/n^2, \quad n \to \infty,$$

by Stirling's formula. Therefore $c_1(I_0) < c_1(B_0)$ for large n, and Theorem B is proved. □

We now shift our attention to forests with q edges. For this family of graphs, we are able to extend our extremal result of the previous section in the following manner.

Corollary 3.5. *Let $\mathbb{F}_q$ denote the family of forests with q edges.*
(i) *Let $q = 2n$ and $\mathscr{E} = \{A_0, A_1\}$. Then, for all but finitely many q,*

$$\text{Irr}(A_0, \lambda) < \text{Irr}(A_1, \lambda) < \text{Irr}(G, \lambda)$$

for $G \in \mathbb{F}_q \setminus \mathscr{E}$ and λ sufficiently large.
(ii) *Let $q = 2n + 1$ and $\mathscr{O} = \{B_0, C_0, D_1, E_0, B_1, I_0, I_1\}$. Then, for all but finitely many q,*

$$\text{Irr}(B_0, \lambda) < \text{Irr}(C_0, \lambda) < \text{Irr}(D_1, \lambda) < \text{Irr}(E_0, \lambda) < \text{Irr}(B_1, \lambda) < \text{Irr}(I_0, \lambda) < \text{Irr}(I_1, \lambda) < \text{Irr}(G, \lambda)$$

for $G \in \mathbb{F}_q \setminus \mathscr{O}$ and λ sufficiently large.

Proof. First observe that $\mathscr{E}$ (resp., $\mathscr{O}$) consists precisely of all forests among the nine families $\{X_t\}$ for q even (resp., odd). The theorem now follows from Lemma 3.3, Table 3 and the values for $c_1(A_1)$, $c_1(B_1)$, $c_1(I_1)$, which can be computed from Table 2. □

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