On The Greatest Number of 2 and 3 Colorings of a (V, E)-Graph*

Felix Lazebnik

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE

ABSTRACT

Let $\mathscr{F}$ denote the family of simple undirected graphs on v vertices having e edges ((v, e)-graphs) and $P(\lambda, G)$ be the chromatic polynomial of a graph G. For the given integers v, e, λ, let $f(v, e, \lambda)$ denote the greatest number of proper colorings in λ or less colors that a (v, e)-graph G can have, i.e., $f(v, e, \lambda) = \max\{P(\lambda, G): G \in \mathscr{F}\}$. In this paper we determine $f(v, e, 2)$ and describe all graphs G for which $P(2, G) = f(v, e, 2)$. For $f(v, e, 3)$, a lower bound and an upper bound are found.

1. INTRODUCTION

The definitions in this paper are based on [3]. All graphs we consider are undirected labeled graphs without loops and multiple edges. $V(G)$ and $E(G)$ denote a set of vertices and edges of G, respectively. The number of elements of a finite set A is denoted by $|A|$. We write $v = v(G) = |V(G)|$ and $e = e(G) = |E(G)|$. By $c(G)$ we denote the number of connected components of graph G. For any positive integer λ, a *proper λ-coloring* of a labeled graph G is a mapping of $V(G)$ into the set $\{1, 2, \ldots, \lambda\}$ (the set of colors) such that no two adjacent vertices of G have the same image. The *chromatic number* of a graph G, denoted $\chi(G)$, is the least λ (number of colors) for which there exists a proper coloring of G. Let $P(\lambda) = P(\lambda, G)$ denote the number of proper λ-colorings of

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G. This function was introduced by Birkhoff [2] and it turned out to be a polynomial function of λ.

In this paper we consider the following:

Problem. Let $\mathcal{F} = \mathcal{F}_{v,e}$ be a family of all graphs having v vertices and e edges ((v, e)-graphs). Let $\lambda = 2$ or 3 be the number of colors. Denote by $f(v, e, \lambda)$ the greatest number of proper λ-colorings that a (v, e)-graph can have, i.e., $f(v, e, \lambda) = \max\{P(\lambda, H): H \in \mathcal{F}\}$. Determine explicitly or find bounds for $f(v, e, \lambda)$. Describe, if possible, all the extremal graphs G, i.e., all $G \in \mathcal{F}$ such that $f(v, e, \lambda) = P(\lambda, G)$.

In section 2 we present a complete solution of the problem for $\lambda = 2$. In section 3 we show some results for the case when $\lambda = 3$. The main results are Theorems 2.1, 2.5, 3.3, and 3.6.

The problem was motivated by the analysis of the running time of the backtrack algorithm for the graph coloring problem (see Wilf [6]; Bender and Wilf [1]). Another source of related problems is a paper by Wright [8], where an asymptotic approximation to the number $M_{v,e}$ = the total number of proper λ-colorings of all (v, e)-graphs was found for a fixed λ, large v, and all e. This result immediately allows us to obtain an asymptotic approximation to $av(v, e, \lambda) = av(P(\lambda, G))$ = the average value of the chromatic polynomial of a graph, where average is understood to be taken over all (v, e)-graphs G. In relation to our problem, the role of the function $av(v, e, \lambda)$ is the following: for large v a lower bound for $f(v, e, \lambda)$ can be obtained from it easily. We will use this observation in section 3. Tomescu ([4], [5]) considered problems similar to our problem but for different families of graphs (all graphs on v vertices whose chromatic number is equal to k).

Let xy be an edge of G. Then by $G - xy$ we mean the graph obtained from G by deleting xy. By G/xy we mean a graph obtained from G by identifying the vertices x and y, i.e., (i) by deleting both x and y and all the edges incident to them, and (ii) by introducing a new vertex z and connecting z to both all the neighbors of x different from y and all the neighbors of y different from x in G.

By K_v, $\overline{K}_v$, T_v, and $K_{m,n}$ we denote correspondingly the complete graph on v vertices (any two vertices are joined by an edge), the completely disconnected graph on v vertices (no edges at all), a tree on v vertices, and a complete bipartite graph on $m + n$ vertices. By $G + H$ we denote the disjoint union of graphs G and H. The following proposition describes some properties of $P(\lambda, G)$ that we are going to use later. Proofs can be found in [3, pp. 145–148].

Preposition 1. Let G be a graph with v vertices, e edges, and $G_1, G_2, \ldots, G_{c(G)}$ be all connected components of G. Then

(i) $P(\lambda, G) = P(\lambda, G_1) \cdot P(\lambda, G_2) \cdots P(\lambda, G_{c(G)})$.

(ii) Reduction Formula: Let a and b be two adjacent vertices of G. Then

$$P(\lambda, G) = P(\lambda, G - ab) - P(\lambda, G/ab). \tag{1.1}$$

(iii) $P(\lambda, \overline{K}_v) = \lambda^v$, $P(\lambda, K_v) = \lambda(\lambda - 1) \cdots (\lambda - v + 1)$, $P(\lambda, T_v) = \lambda(\lambda - 1)^{v-1}$.

(iv) For a connected graph G, $P(\lambda, G) \leq \lambda(\lambda - 1)^{v-1} (\lambda \geq 2)$. The equality sign occurs if and only if G is a tree.

CASE $\lambda = 2$

In this section we will find $f(v, e, 2)$ and describe all extremal graphs. It is well known that a graph is 2-colorable if and only if it has no odd cycles or if and only if each of its connected components is a bipartite graph. Let

$$\mathcal{B} = \{G: G \in \mathcal{F}_{v,e} \text{ and each connected component of } G \text{ is a bipartite graph}\}.$$

Then $f(v, e, 2) = \max\{P(2, G): G \in \mathcal{B}\}$ and our goal is finding $f(v, e, 2)$ and all $G \in \mathcal{B}$ such that $P(2, G) = f(v, e, 2)$.

Theorem 2.1

$$f(v, e, 2) = \begin{cases} 2^v, & \text{if } e = 0, \\ 2^{v - \lceil 2\sqrt{e} \rceil + 1}, & \text{if } 0 < e \leq \lfloor v^2/4 \rfloor, \\ 0, & \text{if } e > \lfloor v^2/4 \rfloor. \end{cases}$$

Proof.   For $e = 0$, $G = \overline{K}_v$ and the result is obvious.

For $e > \lfloor v^2/4 \rfloor$, the result follows immediately from Turán's Theorem (see [3, p. 17]), since G contains a triangle.

To prove the rest of the theorem we observe that if $G \in \mathcal{B}$, then

$$P(2, G) = 2^{c(G)},$$

which follows from Proposition 1(i) and the fact that each connected component of G can be colored in exactly 2 different ways. Let

$$c = c(v, e) = \max\{c(G) \mid G \in \mathcal{B}\}.$$

Then

$$f(v, e, 2) = 2^c,$$

and the problem is reduced to determining c. We do this by means of the following three lemmas:

Lemma 2.2.   The least number of vertices a bipartite graph G with e edges may have is $\lceil 2\sqrt{e} \rceil$, and for any $e \geq 1$ these exists a bipartite graph G with e edges and $\lceil 2\sqrt{e} \rceil$ vertices.

Proof.   Let a bipartite graph G, $G \in K_{n,m}$, have $m + n$ vertices and e edges. Since $m + n$ is an integer and

$$m + n \geq 2\sqrt{m \cdot n} \geq 2\sqrt{e},$$

the least possible value for $m + n$ is $\lceil 2\sqrt{e} \rceil$.

To prove the second part of the lemmas, i.e., the existence of the graph G with e edges and $\lceil 2\sqrt{e} \rceil$ vertices, we deonte $\lfloor \sqrt{e} \rfloor = p$ and consider the following three cases:

Case 1.   If $e = p^2$, then we can only have $G = K_{p,p}$. Here $m = n = p$, $m + n = 2p = \lceil 2\sqrt{e} \rceil$.

Case 2.   If $p^2 < e \leq p^2 + p$, then

$$2p < 2\sqrt{e} \leq 2\sqrt{p^2 + p} < 2p + 1.$$

So $\lceil 2\sqrt{e} \rceil = 2p + 1$ and G can be taken as $K_{p,p} \subset G \subseteq K_{p+1,p}$ Here $n = p + 1$, $m = p$.

Case 3.   If $p^2 + p < e < (p + 1)^2$, then

$$2p + 1 < 2\sqrt{p^2 + p} + 1 < 2\sqrt{e} < 2p + 2.$$

So $\lceil 2\sqrt{e} \rceil = 2p + 2$ and G can be taken as $K_{p+1,p} \subset G \subset K_{p+1,p+1}$. Here $m = n = p + 1$.   ∎

We call each connected component of G different from K_1 a *nontrivial* component of G.

Lemma 2.3.   Let $G \in \mathcal{B}$ and two nontrivial components of G have together at least 4 edges. Then there exists $G' \in \mathcal{B}$, such that $c(G') > c(G)$.

Proof.   Let G_1 and G_2 be two nontrivial components of G and $e(G_1 + G_2) = e_0 \geq 4$. Then $v(G_1 + G_2) = v_0 \geq e_0 + 2$. For any integer $n \geq 4$, $n \geq 2\sqrt{n}$. Since n is an integer, then $n \geq \lceil 2\sqrt{n} \rceil$. Therefore

$$v_0 - \lceil 2\sqrt{e_0} \rceil \geq e_0 + 2 - \lceil 2\sqrt{e_0} \rceil = (e_0 - \lceil 2\sqrt{e_0} \rceil) + 2 \geq 2.$$

Consider a bipartite graph H with $\lceil 2\sqrt{e_0} \rceil$ vertices and e_0 edges. It exists by Lemma 2.2. Take $H' = H + \overline{K}_{v-\lceil 2\sqrt{e} \rceil}$. Then H' has v_0 vertices, e_0 edges, and $c(H') \geq 3$. Replacing $G_1 + G_2$ in G by H' we get graph G', $G' \in \mathcal{B}$, which has more connected components that G.   ∎

Let $\mathcal{B}' \subseteq \mathcal{B}$ be defined as

$$\mathcal{B}' = \{G \in \mathcal{B} \mid G \text{ has only } one \text{ nontrivial component.}\}$$

Lemma 2.4. For any $G \in \mathcal{B}$, there exists $G' \in \mathcal{B}'$ such that

$$c(G') \geq c(G).$$

Proof. If $G \in \mathcal{B}'$, then $G' = G$ (this is always the case for $v = 2$ or 3).

Let G have at least two nontrivial components. If the total number of their edges is greater or equal to 4, then the statement of the lemma follows from Lemma 2.3. Therefore we assume that the total number of edges in any two nontrivial components of G is less than 4. Then the set of nontrivial components of G can be one of the following:

(i) $\{K_{2,1}$ and $e - 2$ copies of $K_{1,1}\}$
(ii) $\{e$ copies of $K_{1,1}\}$

In both cases one obtains G' by replacing all nontrivial components of G by $K_{e,1}$. Note that $c(G') = c(G)$. ∎

Now we are able to finish the proof of Theorem 2.1. As it follows from Lemma 2.4, $c = \max\{c(G): G \in \mathcal{B}\} = \max\{c(G'): G' \in \mathcal{B}\}$ and we can restrict our search of c from $\mathcal{B}$ to $\mathcal{B}'$.

Let $G' \in \mathcal{B}'$ and let H be the only nontrivial component of G'.
If

$$v(H) > 2\lceil \sqrt{e(H)} \rceil = \lceil 2\sqrt{e} \rceil,$$

then by Lemma 2.2, H can be replaced by H', which has the same number of edges but less vertices. The obtained graph will have more connected components than G. Therefore, $v(H) = \lceil 2\sqrt{e} \rceil$, $G = H + \overline{K}_{v - \lceil 2\sqrt{e} \rceil}$, $c = v - \lceil 2\sqrt{e} \rceil + 1$ and the proof of Theroem 2.1 is complete. ∎

Now we are going to describe all extremal graphs $G \in \mathcal{B}$, i.e., such that

$$c = c(G) = \max\{c(F) | F \in \mathcal{B}\}.$$

If G *has more than one nontrivial component,* then it follows from the proof of Lemma 2.4 that either

(i) $G = eK_{1,1} + \overline{K}_{v-2e}$ or
(ii) $G = K_{2,1} + (e - 2)K_{1,1} + \overline{K}_{v-2e+1}$.

In case (i), $1 \leq e \leq 3$. Otherwise the union of four independent edges of G could be replaced by $K_{2,2} + \overline{K}_4$, which would result the increase of $c(G)$ by one.

In case (ii), $e = 2$ or 3. Otherwise the part of G, $K_{2,1} + 2K_{1,1}$ could be replaced by $K_{2,2} + \overline{K}_3$, which would result the increase of $c(G)$ by one.

Let G *have only one nontrivial component,* i.e., $G \in \mathcal{B}'$, and let H be the nontrivial component of G. Then $e(H) = e \geq 1$, and by Lemma 2.2, $v(H) = \lceil 2\sqrt{e} \rceil$. Let $p = \lceil \sqrt{e} \rceil$.

If $p^2 = e$, then $H = K_{p,p}$ (proved in Lemma 2.2).

If $p^2 < e \leq p^2 + p$ then $\lceil 2\sqrt{e} \rceil = 2p + 1$ (proved in Lemma 2.2). Since H is a bipartite graph, it is a subgraph of some $K_{n,m}$ such that $m + n = 2p + 1$, $mn \geq e$, $n \geq m$. Then for some nonnegative integer a, $n = p + 1 + a$, $m = p - a$, and

$$mn = (p + 1 + a)(p - a) \geq e, \quad \text{or}$$

$$a^2 - a - (p^2 + p - e) \geq 0, \quad \text{or}$$

$$0 \leq a \leq \lfloor \sqrt{.25 + p^2 + p - e} + .5 \rfloor. \tag{2.1}$$

Therefore H can always be obtained from some $K_{p+1+a,p-a}$ by deleting $(p + 1 + a)(p - a) - e$ edges, where a is a nonnegative integer given by (2.1). We denote any graph H described above by $H(p + 1, p, a)$.

For example, for $e = 38$ we have $p = 6$, $v(H) = 13$; possible values for a are 0, 1, 2. Therefore H can be obtained by deleting 4 edges from $K_{7,6}$, or 2 edges from $K_{8,5}$, or 0 edges from $K_{9,6}$.

If $p^2 + p < e < (p + 1)^2$, then $\lceil 2\sqrt{e} \rceil = 2p + 2$ (proved in Lemma 2.1). Since H is a bipartite graph, it is a subgraph of $K_{n,m}$ such that $m + n = 2p + 2$, $mn \geq e$, $n \geq m$. Then for some nonnegative integer a, $m = p + 1 + a$, $n = p + 1 - a$, and

$$mn = (p + 1)^2 - a^2 \geq e, \quad \text{or}$$

$$0 \leq a \leq \lfloor \sqrt{(p + 1)^2 - e} \rfloor. \tag{2.2}$$

Therefore H can always be obtained from some $K_{p+1+a,p+1-a}$ by deleting $(p + 1)^2 - a^2 - e$ edges, where a is a nonnegative integer given by (2.2). We denote any graph H described above by $H(p + 1, p + 1, a)$.

Collecting all the results of this section we get a complete description of the extremal graphs:

Theorem 2.5. Let $G = G(v, e)$ be an extremal (v, e)-graph, i.e., $f(v, e, 2) = P(2, G)$, $0 \leq e \leq \lfloor v^2/4 \rfloor$ and $p = \lfloor \sqrt{e} \rfloor$. Then $G(v, 0) = \overline{K}_v$; $G(v, 1) = K_{1,1} + \overline{K}_{v-2}$; $G(v, 2) = 2K_{1,1} + \overline{K}_{v-4}$ or $K_{2,1} + \overline{K}_{v-3}$; $G(v, 3) = 3K_{1,1} + \overline{K}_{v-6}$ or $K_{2,1} + K_{1,1} + \overline{K}_{v-5}$.

For $4 \leq e = p^2$, $G(v, e) = K_{p,p} + \overline{K}_{v-2p}$;

For $4 \leq p^2 < e \leq p^2 + p$, $G(v, e) = H(p + 1, p, a) + \overline{K}_{v-2p-1}$;

For $4 \leq p^2 + p < e < (p + 1)^2$, $G(v, e) = H(p + 1, p + 1, a) + \overline{K}_{v-2p-2}$. ∎

CASE $\lambda = 3$

A Lower Bound for $f(v, e, 3)$

A lower bound for $f(v, e, 3)$, $0 \leq e \leq v^2/4$, will be obtained as a lower bound for the total number of 3-colorings of a particular (v, e)-graph defined below.

Suppose $V(K_{m,n}) = A \cup B$, $|A| = n$, $|B| = m$. Define $K_{m,n,0} = K_{m,n}$. By $K_{m,n,p}$, $(1 \le p \le m)$, we mean a graph obtained from $K_{m,n}$ by adding one more vertex to A and joining it with p vertices of B. (See Figure 1.)

Graph $K_{m,n,p}$ has $m + n + 1$ vertices and $mn + p$ edges. We will call $K_{m,n,p}$ a *semicomplete bipartite* graph. Notice that $K_{m,n,m} = K_{m,n+1,0} = K_{m,n+1}$. Clearly the complete bipartite graph is also semicomplete. The definition is motivated by the fact that we suspect the extremal graphs to be semicomplete bipartite if $e \le v^2/4$.

Now we compute the number of proper colorings of the graph $K_{m,n,p}$ if 3 (or fewer) colors are used.

Lemma 3.1

$$P(3, K_{m,n,p}) = \begin{cases} 3 \cdot (2^n + 2^m - 2), & \text{if } p = 0, \quad (3.1) \\ 3 \cdot (2^{n+1} + 2^m + 2^{m-p+1} - 4), & \text{if } 1 \le p \le m. \quad (3.2) \end{cases}$$

Proof.
   (i) Let $p = 0$ and $m = 1$. Then $P(3, K_{1,n,0}) = P(3, K_{1,n}) = 3 \cdot 2^n$ and the statement is proved.
   (ii) Let $p = 0$ and $m \ge 2$. If we join vertices B_{m-1} and B_m by edge t, then by the reduction formula (1.1)

$$P(3, K_{m,n}) = P(3, K_{m,n} + t) + P(3, K_{m,n}/t).$$

We have $P(3, K_{m,n} + t) = (3 \cdot 2) \cdot 1^n \cdot 2^{m-2} = 3 \cdot 2^{m-1}$, since a choice of 2 different colors for B_{m-1} and B_m determines the color for each of A_i, $1 \le i \le n$, uniquely and each of B_j, $1 \le j \le m - 2$, can be colored independently in any of 2 colors different from one chosen for $A_1, A_2, \ldots, A_n$. Graph $K_{m,n}/t$ is isomorphic to the graph $K_{m-1,n}$. Thus, we obtain a recurrence

$$P(3, K_{m,n}) = 3 \cdot 2^{m-1} + P(3, K_{m-1,n}).$$

By solving it we get

$$P(3, K_{m,n}) = \sum_{i=m-1}^{1} (3 \cdot 2^i) + P(3, K_{1,n}) = 3 \cdot (2^m - 2) + 3 \cdot 2^n$$

$$= 3 \cdot (2^n + 2^m - 2),$$

which proves the lemma for $p = 0$.
   (iii) Let $p = 1$. Then

$$P(3, K_{m,n,1}) = 2 \cdot P(3, K_{m,n}). \quad (3.3)$$

(iv) Finally, we assume $2 \leq p \leq m$. In this case

$$P(3, K_{m,n,p}) = P(3, K_{m,n,p} + t) + P(3, K_{m,n,p}/t)$$
$$= 3 \cdot 2^{m-1} + P(3, K_{m-1,n,p-1}).$$

Applying (3.1) and (3.3) we solve this recurrence and obtain

$$P(3, K_{m,n,p}) = \sum_{i=m-1}^{m-p+1} (3 \cdot 2^i) + P(3, K_{m-p+1,n,1})$$
$$= 3 \cdot (2^m - 2^{m-p+1}) + 2 \cdot 3 \cdot (2^n + 2^{m-p+1} - 2)$$
$$= 3 \cdot (2^{n+1} + 2^m + 2^{m-p+1} - 4),$$

which proves Lemma 3.1 for $1 \leq p \leq m$. ∎

Notice that for $p = m$ we have $P(3, K_{m,n,m}) = P(3, K_{m,n+1})$ as it should be.

Lemma 3.2. (modified and restricted version of Lemma 3.1). Let $e = mn + p$, where $1 \leq p \leq m \leq n$ (in case $p = m$ we require $n \geq m + 1$). Take $n = \lceil e/m \rceil - 1$ and $p = e - m \cdot n = e - m \cdot (\lceil e/m \rceil - 1)$. Then

$$P(3, K_{m,n,p}) = 3 \cdot (2^{n+1} + 2^m + 2^{m-p+1} - 4). \tag{3.4}$$

Proof. If $\lceil e/m \rceil = e/m$, then $n + 1 = e/m$ and $p = m$. Therefore

$$P(3, K_{m,n,p}) = P(3, K_{m,n,m}) = P(3, K_{m,n+1,0}) = P(3, K_{m,n+1})$$

and (3.4) gives the same result as (3.1). If $\lceil e/m \rceil > e/m$, then (3.4) gives the same result as (3.2). The lemma is proved.

Suppose m, n, p satisfy the conditions of Lemma 3.2. Then the graph $K_{m,n,p}$ has $n + m + 1$ vertices. If $v > n + m + 1$, then

$$P(3, K_{m,n,p} + \overline{K}_{v-(n+m+1)}) = P(3, K_{m,n,p}) \cdot 3^{v-(n+m+1)}$$
$$= \frac{2^{n+1} + 2^m + 2^{m-p+1} - 4}{3^{n+m}} \cdot 3^v. \tag{3.5}$$

Denoting the coefficient at 3^v in (3.5) by $a_m = a_{m(e)}$ we find that

$$P(3, K_{m,n,p} + \overline{K}_{v-(n+m+1)}) = a_m \cdot 3^v. \tag{3.6}$$

A Lower Bound for $f(v, e, 3)$

Our goal in this section is to get a lower bound for $f(v, e, 3)$ for $e \leq v^2/4$. We consider the following subset $\mathscr{SB}$ of $\mathscr{F}$:

Graph $G \in \mathscr{F}$ belongs to $\mathscr{SB}$ if and only if

(i) $G = K_{m,n} + \overline{K}_{v-(m+n)}$, $m \cdot n = e$;
(ii) $G = K_{m,n,p} + \overline{K}_{v-(m+n+1)}$, $1 \leq p \leq m$, $mn + p = e$.

If $e \leq v^2/4$, then $\mathscr{SB}$ is not an empty set. For any family $\mathscr{T}$ of graphs, let

$$f(\mathscr{T}) = \max\{P(3, G) \,|\, G \in \mathscr{T}\}.$$

Since $\mathscr{SB} \subset \mathscr{F}$, then

$$f(\mathscr{SB}) \leq f(\mathscr{F}),$$

and any lower bound for $f(\mathscr{SB})$ is also a lower bound for $f(\mathscr{F})$. Using new notations and (3.6) we get

$$f(\mathscr{SB}) = \left[\max_{1 \leq m \leq \lfloor \sqrt{e} \rfloor} \{a_m\} \right] \cdot 3^v.$$

Therefore in order to find a lower bound for $f(\mathscr{SB})$ we can find one for

$$a(e) = a_{q(e)} = \max_{1 \leq m \leq \lfloor \sqrt{e} \rfloor} \{a_m\}$$

(it is easy to show that $q(e)$ is defined uniquely by this equality). Let m, n satisfy the conditions of Lemma 3.2. We define

$$b_m = 2^{n+1}/3^{n+m} \text{ and } b(e) = b_{r(e)} = \max_{1 \leq m \leq \lfloor \sqrt{e} \rfloor} \{b_m\}$$

(it is easy to show that $r(e)$ is defined uniquely by this equality).
Then for all m, $0 \leq m < \lfloor \sqrt{e} \rfloor$,

$$\frac{1}{3} \cdot \frac{2^{e/m}}{3^{(e/m)+m}} \leq b_m/2 \leq a_m \leq 2 \cdot b_m \leq 12 \cdot \frac{2^{e/m}}{3^{(e/m)+m}}.$$

It is easy to see that the function $f(x) = 2^{e/x}/3^{(e/x)+x}$ defined on $[1; \lfloor \sqrt{e} \rfloor]$ takes its maximum at point

$$x_0 = \sqrt{e}/c, \quad \text{where } c = (\log 3)/(\log 1.5).$$

Therefore

$$a(e) = kf(x_0) = k(.2632)^{\sqrt{e}},$$

where k is some constant from the interval $[1/3, 12]$, and a lower bound for $a(e)$ can be obtained if we take $k = 1/3$. This implies the following:

Theorem 3.3. Let $0 \le e \le v^2/4$. Then there is a graph $G \in \mathscr{SB}$ having at least $(.2632)^{\sqrt{e}} \cdot 3^{v-1}$ proper colorings in 3 or fewer colors. Therefore

$$(.2632)^{\sqrt{e}} \cdot 3^{v-1} \le P(3, G) \le f(\mathscr{SB}) \le f(\mathscr{F}) = f(v, e, 3). \qquad (3.7)$$

In order to improve the constant in the lower bound in (3.7), one can try to determine the index $q(e)$ more precisely. There is convincing numerical evidence that for $e \ge 26$, $q(e) = r(e)$, but we could not prove it. Even if this is true, it turned out to be surprisingly difficult to find bounds for $r(e)$ better than ones given below:

Proposition 3.4. If m and n satisfy the conditions of Lemma 3.2, then

$$\sqrt{e/3 + .25} - .5 \le r(e) \le \sqrt{e/2 + .25} - .5 \qquad (3.8)$$

Proof. First let us find when $b_{m+1} \ge b_m$:

$$\frac{2^{\lceil e/(m+1)\rceil}}{3^{\lceil e/(m+1)\rceil + (m+1)}} \ge \frac{2^{\lceil e/m\rceil}}{3^{\lceil e/m\rceil + m}}$$

$$\Longleftrightarrow \lceil e/m\rceil - \lceil e/(m+1)\rceil \ge (\ln 3)/\ln 1.5 \approx 2.709$$

The left side of the inequality represents an integer. So

$$b_{m+1} > b_m \Longleftrightarrow \lceil e/m\rceil - \lceil e/(m+1)\rceil \ge 3 \qquad (3.9)$$

if

$$e/m - e/(m+1) > 3, \qquad (3.10)$$

then

$$\lceil e/m\rceil - \lceil e/(m+1)\rceil > e/m - e(m+1) - 1 > 2.$$

Therefore (3.10) implies (3.9). Solving (3.10) for $m \ge 1$, we obtain the left-hand side of the inequality (3.8).

It is also clear that $\lceil e/m\rceil - \lceil e/(m+1)\rceil < e/m + 1 - e/(m+1)$, so the inequality

$$e/m - e/(m+1) < 2 \qquad (3.11)$$

implies $\lceil e/m\rceil - \lceil e/(m+1)\rceil \le 2$ or $b_{m+1} < b_m$. Solving (3.11) for positive m we obtain the right-hand side of (3.8). ∎

If $e > v^2/4$, then a lower bound can be obtained from the results of Wright [8, Theorem 6], which were discussed in Section 1. A proof of the following theorem can be easily obtained from them and we omit it:

Theorem 3.5.   There exists a positive constant k such that

$$f(v, e, 3) > k \cdot \frac{\binom{v^2/3}{e}}{\binom{\binom{v}{2}}{e}} \cdot v \cdot 3^v$$

for sufficiently large v and all $e > v^2/4$.

An Upper Bound for $f(v, e, 3)$

The upper for $f(v, e, 3)$ will be a particular case of an upper bound for $f(v, e, \lambda)$. Using Proposition 1 (iv) we have that for any *connected* graph G on v vertices,

$$P(\lambda, G) \le \lambda(\lambda - 1)^{v-1}.$$

So

$$f(v, e, \lambda) \le \lambda(\lambda - 1)^{v-1} \tag{3.12}$$

Let G have $c = c(G)$ connected components $G_1, G_2, \ldots, G_c$ on $v_1, v_2, \ldots, v_c$ vertices correspondingly and let $c(v, e)$ denote the maximal number of connected components that a graph with v vertices and e edges may have. Then

$$P(\lambda, G) = \prod_{i=1}^{c} P(\lambda, G_i) \le \prod_{i=1}^{c} \lambda(\lambda - 1)^{v_i - 1}$$

$$= \lambda^c (\lambda - 1)^{(\Sigma_{i=1}^{c} v_i) - c} = \lambda^c (\lambda - 1)^{v-c},$$

and

$$f(v, e, \lambda) \le \left(1 - \frac{1}{\lambda}\right)^{v - c(v,e)} \lambda^v.$$

The number $c(v, e)$ can be expressed in terms of v and e in the following way. It is known, see for example [7, p. 27], that for any simple graph G with v vertices, e edges, and c connected components

$$v - c \le e \le (v - c)(v - c + 1)/2.$$

Solving this for c we obtain that the greatest value of c that satisfies this system of inequalities is $c(v, e) = v - \lceil 1/2(\sqrt{1 + 8e} - 1) \rceil$. Therefore we have

$$f(v, e, \lambda) \leq (1 - 1/\lambda)^{\lceil 1/2(\sqrt{1+8e}-1) \rceil} \lambda^v. \tag{3.13}$$

For $\lambda = 3$, we get

$$f(v, e, 3) \leq (2/3)^{\lceil 1/2(\sqrt{1+8e}-1) \rceil} 3^v \leq (2/3)^{\sqrt{2e}-1} 3^v$$

$$\leq (.5636)^{\sqrt{e}-1} 3^v.$$

Combining the obtained upper bound with the lower bound (3.7) and (3.13) we obtain

Theorem 3.6. For $0 \leq e \leq v^2/4$,

$$(.2632)^{\sqrt{e}} \cdot 3^{v-1} \leq f(v, e, 3) \leq (.5636)^{\sqrt{e}-1} 3^v. \tag{3.14}$$

For sufficiently large v and all $e > v^2/4$, a lower bound is given in Theorem 3.5. The upper bound in (3.14) is correct for all e, $0 \leq e \leq v(v - 1)/2$.

Conjecture. For $0 \leq e \leq v^2/4$, $f(v, e, 3) = P(3, G)$ if and only if G is a semicomplete bipartite graph.

ACKNOWLEDGMENT

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