IRREGULARITY STRENGTHS FOR CERTAIN GRAPHS

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ABSTRACT

A network $G(\omega)$ of strength s is a graph G in which each edge is assigned a positive integer weight, the largest of which is s. The weight of a vertex in $G(\omega)$ is the sum of the weights of its incident edges. $G(\omega)$ is called irregular if distinct vertices have different weights. The irregularity strength $s(G)$ of G is the minimum strength among all irregular networks having G as the underlying graph. In this paper, the authors determine $s(G)$ for G a wheel, a $2 \times n$ grid, and a k-cube. An asymptotic result is also obtained for certain product graphs.

1. Introduction

Throughout, G will be a simple graph with no K_2 component and at most one isolated vertex. A *network* $G(\omega)$ consists of a graph G together with an assignment $\omega : E(G) \rightarrow Z^+$. The *strength* $s(G(\omega))$ of $G(\omega)$ is defined by $s(G(\omega)) = max\{\omega(e) : e \in E(G)\}$. For each vertex v of G, we define the *weight* $wt(v)$ of v in $G(\omega)$ by

$$wt(v) = \sum_{e \text{ incident to } v} \omega(e),$$

and we call $G(\omega)$ *irregular* if $wt(v) = wt(w)$ implies $v = w$. The *irregularity strength* $s(G)$ of G is now defined to be $min\{s(G(\omega)) : G(\omega) \text{ is irregular}\}$.

In [3], Chartrand et al. proposed the problem of studying $s(G)$. They proved $(3p-2q)/3 \leq s(G) \leq 2p-3$ for G of order p and size q (that is, with p vertices and q edges). In [13], a stronger lower bound for $s(G)$ was obtained:

$$s(G) \geq \lambda(G) = max\left\{\left(\sum_{k=i}^{j} n_k + i - 1\right)/j : i \leq j\right\},$$

where n_k is the number of vertices of degree k in G. (Indeed, let $G(\omega)$ be an irregular network of strength s, and A the set of vertices whose degrees in G are elements of $\{i, i+1, ..., j\}$. Then the smallest weight in $G(\omega)$ of any vertex of A is at least i, so that the largest is at least $|A| + i - 1 = \sum_{k=i}^{j} n_k + i - 1$. Since the largest vertex weight in $G(\omega)$ is clearly no greater than js, we have $s \geq \lambda(G)$.) In [13], the upper bound of $2p-3$ is also improved, to $p-1$, for connected graphs on at least 4 vertices.

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There are not a great many graphs for which the irregularity strength is known, and the problem can be quite difficult even for very simple graphs. In [3], it was determined that $s(K_n) = 3$, $s(K_{2n,2n}) = 3$ and $s(P_n) = n/2$, $(n+1)/2$ or $(n+2)/2$, depending on the congruence of n modulo 4. (Here K_n and $K_{2n,2n}$ denote the complete and complete bipartite graphs, respectively, and P_n denotes the path of order n.) In [9] it was shown that $s(K_{2n+1,2n+1}) = 4$. Work has also been done on dense graphs, binary trees, and disjoint unions of paths, cycles and complete graphs (c.f. [4],[8],[15]). Further results can be found in the references.

In this paper, we determine the irregularity strength of the wheel, the $2 \times n$ grid and the k-dimensional cube. Except for small cases (5 or fewer vertices) and the case $n \equiv 1 \pmod 6$ for the grid, we obtain $s(G) = \lceil \lambda(G) \rceil$, where $\lceil\ \rceil$ denotes the ceiling function. In the exceptional cases, we obtain $s(G) = \lceil \lambda(G) \rceil + 1$.

If G is an r-regular graph on n vertices then $\lambda(G) = (n+r-1)/r$. It has been conjectured (See [7] or [16]) that $s(G) \leq n/r + c$ for all r-regular graphs of order n, where c is some constant (depending only on r and n). In the final section, we give some corroborating evidence in support of this conjecture by looking at graphs which may be obtained by taking the product of an arbitrary regular graph with the d-dimensional cube Q_d.

We close this section with a simple observation concerning $\lambda(T)$ when T is a tree.

Proposition 1

Let T be a tree. Then

$$\lambda(T) = \begin{cases} n_1, & \text{if } n_1 \geq n_2; \\ (n_1+n_2)/2, & \text{if } n_1 \leq n_2. \end{cases}$$

Proof. Let i and j be chosen so that $\lambda(T) = (n_i + ... + n_j + i - 1)/j$. Clearly $n_j \neq 0$, so $n_1 \geq j > i-1$, whence $i = 1$. (Otherwise $(n_1 + n_2 + ... + n_j)/j > \lambda(T)$.) Let $n = |V(T)|$. Then $n = n_1+n_2+...+n_t$ and $2(n-1) = n_1+2n_2+...+tn_t$, where t is the maximum degree in T. From this we easily obtain $n_1 > n_3+2n_4+...+(j-2)n_j$. Suppose $j \geq 3$. As $n_1 \leq \lambda(T)$, $jn_1 \leq n_1 + n_2 + ...n_j$, whence $jn_3 + 2jn_4 + ... + j(j-2)n_j < n_1 + n_2 + ... + n_j$. Thus $n_1 + n_2 > (j-1)n_3 + (2j-1)n_4 + ... + (j(j-2)-1)n_j \geq 2n_3+2n_4+...+2n_j$. It now follows that $j(n_1+n_2) \geq 3(n_1+n_2) > 2(n_1+n_2+...+n_j)$, and we obtain the contradiction $(n_1+n_2)/2 > (n_1+n_2+...+n_j)/j = \lambda(T)$. Thus $j = 1$ or 2 and the proof is complete. ■

We have some evidence in support of the following

Conjecture

Let T be a tree. Then $s(T) = \lceil \lambda(T) \rceil$ or $\lceil \lambda(T) \rceil + 1$.

This conjecture is known to be true for paths [3] and some other families of trees (See [4], e.g.). Proposition 1 also suggests an answer to a question of Chartrand (see [16]): What is the largest integer $n(p)$ such that whenever a tree T with p pendant vertices has order $n \leq n(p)$, then $s(T) = p$? If the strength $s(T)$ is given by $\lceil \lambda(T) \rceil$, then Proposition 1 implies

$$n(p) = \begin{cases} 4, & \text{if } p = 2; \\ 2p+1, & \text{if } p > 2. \end{cases}$$

Finally, we should mention that Proposition 1 is true also for forests. In this case, however, it is known from [15] that $\lambda(G)$ does not always give a very good bound for $s(G)$.

2. The wheel

Theorem 2

Let G be a wheel on n vertices. Then $\lceil \lambda(G) \rceil = \lceil (n+1)/3 \rceil$ and

$$s(G) = \begin{cases} \lceil \lambda(G) \rceil, & \text{if } n \geq 6; \\ \lceil \lambda(G) \rceil + 1, & \text{if } n = 4 \text{ or } 5. \end{cases}$$

Proof. For $n \geq 7$, irregular assignments are given in Figure 1, in cases dependent on n modulo 3. The reader can readily check the result for $n \leq 6$. ■

3. The $2 \times n$ grid

Theorem 3

Let G be the $2 \times n$ grid, $n \geq 4$. Then

$$s(G) = \begin{cases} \lceil \lambda(G) \rceil, & \text{if } n \not\equiv 1 \pmod 6; \\ \lceil \lambda(G) \rceil + 1, & \text{if } n \equiv 1 \pmod 6. \end{cases}$$

Proof. Note that, for $n \geq 4, \lceil \lambda(G) \rceil = \lceil (2n+1)/3 \rceil$. We first show $s(G) \neq \lceil \lambda(G) \rceil$ when $n \equiv 1 \pmod 6$. Suppose then that $n = 6k+1$ and, by way of

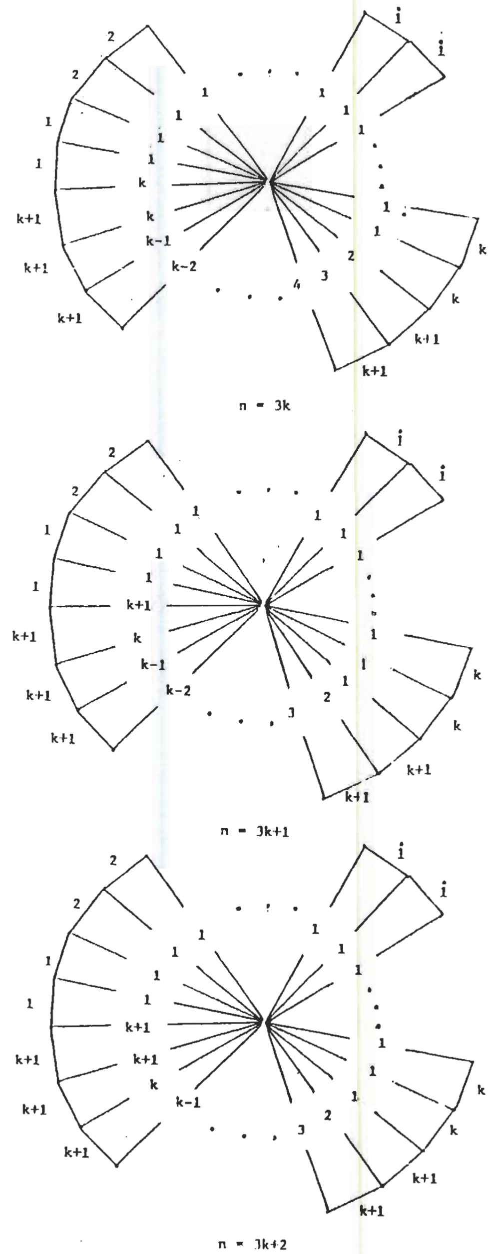

FIGURE 1. Irregular assignments for the wheel

Conjecture

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Proof. Note that, for $n \geq 4, \lceil \lambda(G) \rceil = \lceil (2n+1)/3 \rceil$. We first show $s(G) \neq \lceil \lambda(G) \rceil$ when $n \equiv 1 \pmod 6$. Suppose then that $n = 6k+1$ and, by way of contradiction, $s(G) = \lceil \lambda(G) \rceil = 4k+1$. Then $\{wt(v) : v \in V(G)\} = \{2, 3, ..., 2n+1\}$, so that

$$((2n+1)(2n+2)/2) - 1 = \sum_{v \in V(G)} wt(v) = 2 \sum_{e \in E(G)} \omega(e).$$

But $((2n+1)(2n+2)/2)-1 = (12k+3)(6k+2)-1$, which is odd. Thus $s(G) > \lceil \lambda(G) \rceil$ when $n \equiv 1 \pmod 6$.

To complete the proof we will construct an irregular assignment of the appropriate strength. Let $n = 6k + i$ with $i \in \{0, 1, ..., 5\}$. We will use induction on k to prove a slightly stronger result: G has an irregular assignment ω of the desired strength in which (i) a pair of opposite edges (on the sides with n vertices) have common weight $4k + a_i$, where a_i is given in Table 1, and (ii) $\{wt(v) : v \in V(G)\}$ is $\{2, 3, ..., 2n + 1\}$ if n is even, $\{2, 4, 5, ..., 2n + 2\}$ if n is odd. The start of the induction, for $4 \le n \le 9$, is given in Figure 2.

For general $n \ge 10$, we will construct an assignment on G starting from an assignment on the $2 \times (n-6)$ grid G', as follows. First endow G' with an assignment, using the inductive assumption. Then split G' along the opposite edges having common weight $4(k-1) + a_i$. Insert a 2×6 grid into the gap, creating 4 new edges from the original 2 opposite edges. Assign these four edges the same weight as the original two. Leave the weights of all other edges of G' unchanged. Assign weights to the 2×6 grid as in Figure 3, where a_i and b_i are given in Table 1. It is easy to check that the resulting assignment is irregular, has the desired strength, and satisfies (i) and (ii). ■

i	a_i	b_i
0	0	0
1	1	1
2	2	0
3	3	1
4	3	2
5	4	3

Table 1. The definitions of a_i and b_i.

4. The k-cube

The k-cube Q_k is defined as follows. The set of vertices $V(Q_k)$ is $\{\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) : \text{each } \alpha_i \text{ is 0 or 1}\}$, and vertices α and β are said to be adjacent if they differ in exactly one coordinate. For the remainder of this section, let us fix

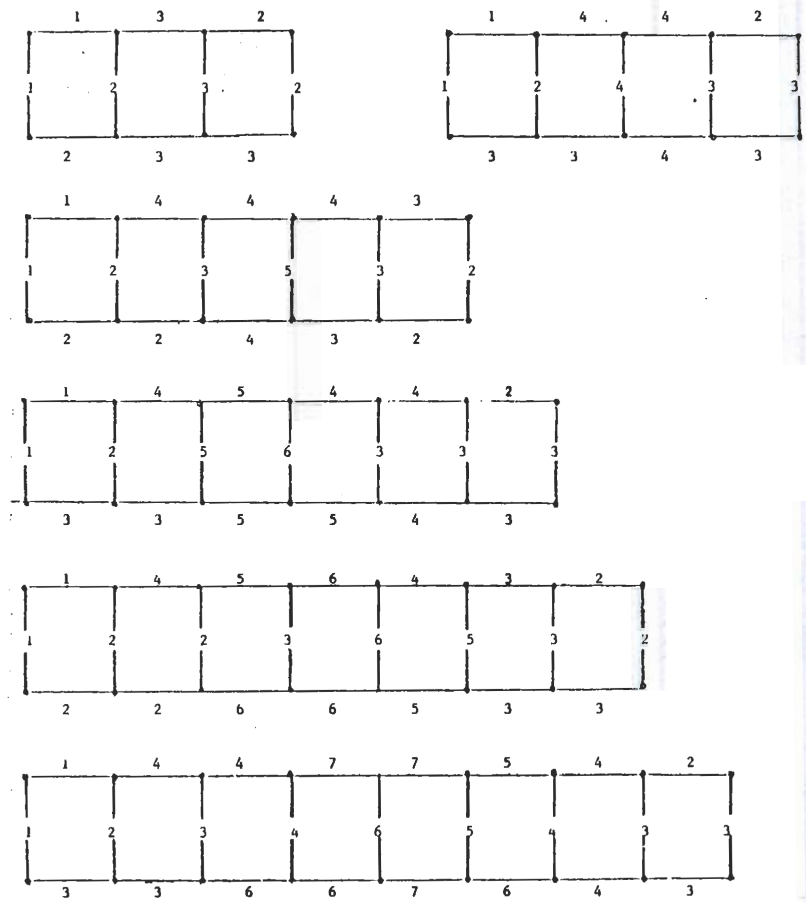

FIGURE 2. Irregular assignments for the 2xn grid, $4 \leq n \leq 9$

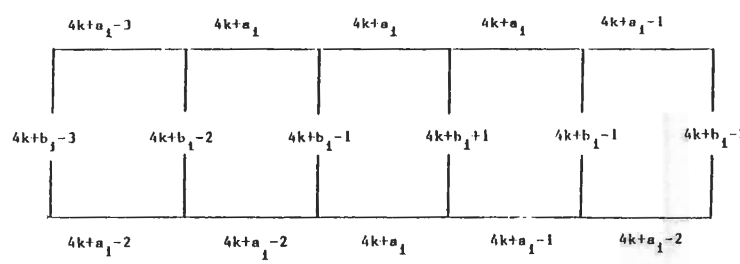

FIGURE 3. The 2x6 grid for the inductive step

$k \geq 2$ and set

$$s = \lceil \lambda(G) \rceil = \left\lceil \frac{2^k + k - 1}{k} \right\rceil, \quad t = \lfloor log_2 s \rfloor.$$

Note that $2^t \leq s < 2^{t+1}$.

In Theorem 4.3 we will construct an assignment of strength s. The basic idea is as follows. Q_k can be thought of as consisting of two copies of Q_{k-1} joined by a matching. Each Q_{k-1} can in turn be thought of as $Q_t \times Q_{k-t-1}$. (For graphs G and H, the product $G \times H$ has vertex set $V(G) \times V(H)$; vertices (v_1, w_1) and (v_2, w_2) are adjacent if $v_1 = v_2$ and w_1 is adjacent to w_2 or if $w_1 = w_2$ and v_1 is adjacent to v_2.) We can thus identify 2^{k-t-1} copies of Q_t in each Q_{k-1}, one for each vertex of Q_{k-t-1}. We will consider three classes of edges in Q_k: (i) edges between the two Q_{k-1}'s; (ii) edges that lie within one of the copies of Q_t; and (iii) all other edges. We will label the edges of Q_k using three algorithms, based on the three classes of edges. The following two technical lemmas enable us to properly label the edges in the t-cubes.

Lemma 4.1

Let $d \in \{t, t+1, t+2, ..., st\}$. Then there exists a network $Q_t(\omega)$ of strength at most s such that $wt(v) = d$ for every $v \in V(Q_t)$.

Proof. Let $d = qt + r$ with $1 \leq q \leq s$ and $0 \leq r < t$. Partition the edge set $E(Q_t)$ so that E_1 consists of the edges whose incident vertices differ in one of the coordinates in $\{1, 2, ..., r\}$, provided $r > 0$, $E_1 = \emptyset$ if $r = 0$, and $E_2 = E(Q_t) \setminus E_1$. Now define ω as follows: Give the weight $q + 1$ to every edge in E_1 and the weight q to every edge in E_2. Then for each vertex v, $wt(v) = r(q+1) + (t-r)q = d$. ■

Lemma 4.2

Let

$$d_i = t + (i-1)2^t \quad \text{for} \quad 1 \leq i \leq 2^{k-t-1}$$

$$e_i = 2^{k-1} + st - (s-1)(k-1) + (i-1)2^t \quad \text{for} \quad 1 \leq i \leq 2^{k-t-1} - 1$$

$$e_{2^{k-t-1}} = min\{st, 2^k + st - (s-1)(k-1) - 2^t\}$$

Then $d_i, e_i \in \{t, t+1, t+2, ..., st\}$ for every i.

Proof. The result will clearly follow once we establish the following three inequalities:

(i) $d_{2^{k-t-1}} \leq st$;

(ii) $e_{2^{k-t-1}-1} \leq st$;

(iii) $e_1 \geq t$.

In the ensuing argument, all logarithms have base 2.

(i) Here we are to show $t+(2^{k-t-1}-1)2^t \leq st$, that is, $(s-1)t+2^t \geq 2^{k-1}$. We begin by observing that $2^{k-1} \geq (k-1)^2$ and thus $2^{k-1}+2k-1 \geq k^2$ for $k \geq 5$. Hence $(2^k+2k-1)^2 = 2^{2k}+2^{k+1}(2k-1)+(2k-1)^2 \geq 2^{2k}+2^{k+1}(2k-1) = 2^{k+1}(2^{k-1}+2k-1) \geq 2^{k+1}k^2$ for $k \geq 5$, using the above inequality. In fact, direct substitution shows this also holds for $k = 2, 3, 4$. Rewriting as $[(2^k+2k-1)/k]^2 \geq 2^{k+1}$ and taking logarithms, we obtain $log[(2^k+2k-1)/k] \geq (k+1)/2$. Since $s+1 \geq (2^k+2k-1)/k$ by the definition of s and $t \geq log(s+1)-1$ by the definition of t, we thus have $t \geq (k-1)/2$ or $k \leq 2t+1$.

Again appealing to the definition of s, we have $s-1 \geq (2^k-1)/k$ and hence $(2t+1)(s-1) \geq 2^k-1$ by using the inequality obtained in the previous paragraph. Since $s \leq 2^{t+1}$, this implies $2st+2^{t+1}-2t \geq 2st+s-2t \geq 2^k$, and thus $(s-1)t+2^t \geq 2^{k-1}$ as we wished to show.

(ii) Here we are to prove $st-(s-1)(k-1)+2^k-2^{t+1} \leq st$, or equivalently, $(s-1)(k-1)+2^{t+1} \geq 2^k$. As $2^{t+1} \geq s+1$, we have $(s-1)(k-1)+2^{t+1} \geq (s-1)(k-1)+s+1 = (s-1)k+2$. But $(s-1)k \geq 2^k-1$ by the definition of s. Thus $(s-1)k+2 \geq 2^k$.

(iii) We must prove $st-(s-1)(k-1)+2^{k-1} \geq t$, which is equivalent to $(s-1)(k-t-1) \leq 2^{k-1}$. One can easily verify this inequality directly for $2 \leq k \leq 7$, hence we assume $k \geq 8$. Since

$$\frac{2^k+k-1}{2^{k-1}} = 2 + \frac{k-1}{2^{k-1}} \leq \frac{5}{2} \leq \frac{k}{log(k)}$$

for $k \geq 8$, we have

$$k - log(k) \geq k - \frac{k2^{k-1}}{2^k+k-1} = \frac{k(2^{k-1}+k-1)}{2^k+k-1}$$

and thus certainly

$$log\left(\frac{2^k+k-1}{k}\right) \geq \frac{k(2^{k-1}+k-1)}{2^k+k-1}.$$

As $s \geq (2^k+k-1)/k$, this implies that

$$log(s) \geq k(2^{k-1}+k-1)/(2^k+k-1).$$

Finally, $log(s) < t+1$ by the definition of t and $s-1 < (2^k+k-1)/k$ by the definition of s, hence

$$\begin{aligned}(s-1)(k-t-1) &\leq \left(\frac{2^k+k-1}{k}\right)\Big(k - log(s)\Big)\\ &= 2^k+k-1-\left(\frac{(2^k+k-1)log(s)}{k}\right)\\ &\leq 2^k+k-1-(2^{k-1}+k-1)\\ &= 2^{k-1}\end{aligned}$$

as we wanted to show. ∎

Theorem 4.3

$s(Q_k) = \lceil \lambda(Q_k) \rceil$.

Proof. We will find an irregular assignment ω of strength $s = \lceil \lambda(Q_k) \rceil$ in which the vertex weights are $k, k+1, ..., 2^k+k-1$.

Let $S = \{\alpha \in V(Q_k) : \alpha_1 = 0\}$ and $T = V(Q_k) \setminus S$. Let us define an equivalence relation $\equiv$ on $V(Q_k)$ as follows: $\alpha \equiv \beta$ if and only if $\alpha_j = \beta_j$ for $2 \leq j \leq k-t$. Observe that there are 2^{k-t-1} equivalence classes, which we will denote by $R_1, ..., R_{k-t-1}$. Now set $S_i = S \cap R_i$ and $T_i = T \cap R_i$. Note that each S_i (and each T_i) induces a t-cube.

Now let d_i and e_i be as in Lemma 4.2. For each $i \in \{1, 2, ..., 2^{k-t-1}\}$ label the edges of the t-cube induced by S_i, using Lemma 4.1 and 4.2, so that the weight of every vertex in S_i is d_i. Also label the edges of the t-cube induced by T_i, for $1 \leq i \leq 2^{k-t-1}$, so that every vertex has weight e_i.

Next give the weight 1 to every edge of the subgraph induced by S which is not already labelled, and the weight s to every edge of the subgraph induced by T which is not already labelled. At this point every vertex of S_i has weight $d_i+k-t-1$, every vertex of T_i has weight $e_i + s(k-t-1)$, and only the edges from S to T remain to be labelled. Notice that these edges can be partitioned into sets $F_1, F_2, ..., F_{2^{k-t-1}}$, where F_i matches vertices of S_i to vertices of T_i. For each i, $|F_i| = 2^t$. Now, for every $i < 2^{k-t-1}$, arbitrarily label the edges of F_i so that the set of labels is $\{1, 2, ..., 2^t\}$. Label the edges of $F_{2^{k-t-1}}$ (arbitrarily) with the set $\{1, 2, ..., 2^t\}$ if $e_{2^{k-t-1}} = 2^k + st - (s-1)(k-1) - 2^t$. If $e_{2^{k-t-1}} = st$, label the edges of $F_{2^{k-t-1}}$ (arbitrarily) with the set $\{2^k - (s-1)(k-1) - (2^t-1), ..., 2^k - (s-1)(k-1)\}$. (Note that, in this case, the condition that $st \leq 2^k + st - (s-1)(k-1) + 2^t$ ensures that the smallest label in this set is at least 1. The definition of s guarantees that the largest label is at most s.) The set of weights of S_i is now $\{d_i + k - t - 1 + 1, ..., d_i + k - t - 1 + 2^t\} = \{k + (i-1)2^t, ..., k + i2^t - 1\}$. The set of weights of the vertices of T_i is $\{2^{k-1} + k + (i-1)2^t, ..., 2^{k-1} + k + i2^t - 1\}$. All told, $\{wt(v) : v \in V(Q_K)\} = \{k, k+1, ..., 2^k + k - 1\}$, as advertised. ■

5. A final remark

Let G be an r-regular graph on n vertices. It has recently been shown (see [5] and [7]) that $s(G) \leq \lceil n/2 \rceil + 2$ for r even, and $s(G) \leq n/2 + 9$ for r odd. Moreover, it was proved in [5] that if $G = C_t^{(k)}$ is the k-fold "explosion" of the t-cycle C_t (by vertex multiplication), then $s(C_t) = \lceil n/r \rceil + 1$ for any $t \geq 3$ and $k \geq 2$. This provides some support for the conjecture mentioned in the introduction, that $s(G) \leq n/r + c$ for all r-regular graphs on n vertices, where c is a constant. In this same vein, we look in this section at graphs which may be obtained by taking the product of an arbitrary regular graph with the d-dimensional cube Q_d. In particular, we show that $s(G \times Q_d) \leq \alpha(n_d/r_d)$ for sufficiently large d, where α is a constant depending only on G. Here n_d and r_d are the order and regularity, respectively, of $G \times Q_d$.

Let G be any r-regular graph on n vertices with strength s, and assume that an irregular assignment of G with strength s has been made. Let $G_d = G \times Q_d$ for $d \geq 1$, and $G_0 = G$. Clearly G_d is a regular graph with regularity $r_d = r + d$ and order $n_d = n2^d$. We let s_d denote the strength of G_d. Thinking of $G_d = G_{d-1} \times K_2$ as consisting of two copies of G_{d-1} joined by a matching M, we label the edges of G_d recursively as follows. Label the first copy of G_{d-1} with an irregular assignment of strength s_{d-1} with vertex weights among the elements of $\{\delta_{d-1}, \delta_{d-1}+1, ..., \Delta_{d-1}\}$. Next label each edge of the second copy of G_{d-1} by adding p_{d-1} to the label of the corresponding edge in the first copy of G_{d-1}, where

$$p_{d-1} = \frac{\Delta_{d-1} - \delta_{d-1} + 1}{r_{d-1}}.$$

This ensures that the vertex weights in the second copy of G_{d-1} are distinct from one another, as well as from the vertex weights in the first copy of G_{d-1}. Finally, label each edge of the matching M with 1, thereby obtaining an irregular assignment of G_d.

Recalling that $s_d = s(G_d)$, we have $s_d \leq s_{d-1} + p_{d-1}$ and thus $s_d \leq s + p_0 + p_1 + ... + p_{d-1}$. Moreover, the vertex weights for the above irregular assignment of G_d are from the set $\{\delta_d, \delta_d + 1, ..., \Delta_d\}$, where $\delta_d = \delta_{d-1} + 1$ and $\Delta_d = \Delta_{d-1} + p_{d-1}r_{d-1} + 1$.

Using $r_{d-1} = r + d - 1$, we immediately obtain the recurrence

$$\begin{aligned}
p_d &= \left\lceil \frac{\Delta_d - \delta_d + 1}{r_d} \right\rceil \\
&= \left\lceil \frac{(\Delta_0 - \delta_0 + 1) + p_0 r + p_1(r+1) + \ldots + p_{d-1}(r+d-1)}{r+d} \right\rceil \\
&= \left\lceil \frac{(\Delta_0 - \delta_0 + 1) + p_0 r + p_1(r+1) + \ldots + p_{d-2}(r+d-2)}{r+d-1} \cdot \frac{r+d-1}{r+d} \right. \\
&\qquad\qquad \left. + \frac{p_{d-1}(r+d-1)}{r+d} \right\rceil \\
&\leq \left\lceil p_{d-1}\left(\frac{r+d-1}{r+d}\right) + p_{d-1}\left(\frac{r+d-1}{r+d}\right) \right\rceil \\
&\leq 2\left(\frac{r+d-1}{r+d}\right) p_{d-1} + 1.
\end{aligned}$$

Solving the recurrence, we obtain

$$p_d \leq 2^d \left(\frac{r}{r+d}\right) p_0 + \frac{1}{r+d}\left[2^{d-1}(r+1+2^{d-2}(r+2)+\ldots+2^0(r+d)\right].$$

Using the fact that $\sum_{i=1}^{m} i/2^i = (2^{m+1} - m - 2)/2^m$, we obtain

$$\begin{aligned}
p_d &\leq 2^d \left(\frac{r}{r+d}\right) p_0 + \frac{1}{r+d}\left[(r+2)2^d - (r+d+2)\right] \\
&\leq \frac{(rp_0 + r + 2)2^d}{r+d} - 1.
\end{aligned}$$

Finally, using

$$\frac{2^{m+1}}{m-1} \leq \sum_{i=1}^{m} \frac{2^i}{i} \leq \frac{2^{m+1}}{m-2}$$

for $m \geq 4$, we have

$$\begin{aligned}
s_d &\leq s + p_o + p_1 + \ldots + p_{d-1} \\
&\leq s - d + \frac{rp_0 + r + 2}{2^r} \sum_{i=r}^{r+d-1} \frac{2^i}{i} \\
&\leq s - d + (rp_0 + r + 2)\left(\frac{2^d}{d+r-3} - \frac{1}{r-2}\right),
\end{aligned}$$

provided $r \geq 5$. But

$$p_0 = \left\lceil \frac{\Delta_0 - \delta_0 + 1}{r} \right\rceil \leq \frac{rs - r + 1}{r} + 1 = s + \frac{1}{r},$$

hence

$$s_d \leq (rp_0 + r + 2)\binom{2^d}{d+r-3} \leq \binom{rs+r+3}{r+d-3}2^d$$

whenever $d \geq s$. As $n_d = 2^d n$ and $r_d = r + d$, we have

$$s_d \leq \binom{rs+r+4}{n}\binom{n_d}{r_d} \qquad (*)$$

provided

$$\binom{rs+r+3}{r+d-3} \leq \binom{rs+r+4}{r+d}.$$

Since the latter condition is equivalent to $d \geq r(3s+2)+12$ and since this clearly implies $d \geq s$, we have inequality $(*)$ holding whenever $d \geq r(3s+2)+12$ so long as $r \geq 5$. Direct computation shows this is also true for $r = 2, 3, 4$.

Now let $\alpha = (rs+r+4)/n$ be the constant multiple of n_d/r_d used in the above upper bound for $s_d = s(G_d)$. Notice that this constant depends only on the graph $G_0 = G$ and not on d. If r is even, then $s \leq \lceil n/2 \rceil + 2$ from [5] and hence $\alpha \leq r/2+4$. If r is odd then $s \leq n/2+9$ (from [7]) and we have $\alpha \leq r/2+10$. In the special case when $G = C_t^{(k)}$ (see [5]), then $s = \lceil n/r \rceil + 1$, where $n = tk$ and $r = 2k$. Thus in this case $\alpha \leq 1 + 5/t + 4/(tk)$ and therefore for any $\epsilon > 0$, we have $\alpha \leq 1+\epsilon$ for t sufficiently large. In any case, since $r_d \to \infty$ as $d \to \infty$, the upper bound on $s(G_d)$ given in $(*)$ is considerably better for large d than the known general bound of $n_d/2$ plus a constant.

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