

# Some corollaries of a theorem of Whitney on the chromatic polynomial\*

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## *Abstract*

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Let  $\mathcal{F}$  denote the family of simple undirected graphs on  $v$  vertices having  $e$  edges,  $P(G; \lambda)$  be the chromatic polynomial of a graph  $G$ . For the given integers  $v, e, \lambda$ , let  $f(v, e, \lambda) = \max\{P(G; \lambda) : G \in \mathcal{F}\}$ . In this paper we determine some lower and upper bounds for  $f(v, e, \lambda)$  provided that  $\lambda$  is sufficiently large. In some cases  $f(v, e, \lambda)$  is found and all graphs  $G$  for which  $P(G; \lambda) = f(v, e, \lambda)$  are described. Connections between these problems and some other questions from the extremal graph theory are analysed using Whitney's characterization of the coefficients of  $P(G; \lambda)$  in terms of the number of 'broken circuits' in  $G$ .

## 1. Introduction

The definitions in this paper are based on the books [3, 7]. All graphs we consider are undirected labelled graphs without loops and multiple edges.  $V(G)$  and  $E(G)$  denote the set of vertices and the set of edges of  $G$  respectively. The number of elements of a finite set  $A$  is denoted by  $|A|$ . We write  $v = v(G) = |V(G)|$  and  $e = e(G) = |E(G)|$ . By  $c(G)$  we denote the number of connected components of graph  $G$ . For any positive integer  $\lambda$ , a *proper  $\lambda$ -coloring* of a labelled graph  $G$  is a mapping of  $V(G)$  into the set  $\{1, 2, \dots, \lambda\}$  (the set of colors) such that no two adjacent vertices of  $G$  have the same image. Let  $P(\lambda) = P(G; \lambda)$  denote the number of proper  $\lambda$ -colorings of  $G$ . This function was introduced by Birkhoff [2] and it turned out to be a polynomial function of  $\lambda$ . The following problem was formulated by H.S. Wilf.

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**Problem.** Let  $\mathcal{F}$  be the family of all graphs having  $v$  vertices and  $e$  edges ( $(v, e)$ -graphs). Let  $\lambda$  be the number of colors. Denote by  $f(v, e, \lambda)$  the greatest number of proper  $\lambda$ -colorings that a  $(v, e)$ -graph can have, i.e.  $f(v, e, \lambda) = \max\{P(H; \lambda): H \in \mathcal{F}\}$ . Determine explicitly or find bounds for  $f(v, e, \lambda)$ . Describe, if possible, all the extremal graphs  $G$ , i.e. all  $G \in \mathcal{F}$  such that  $f(v, e, \lambda) = P(G; \lambda)$ .

In this paper we present some results on the problem in the case when  $\lambda \geq \lambda_0(e)$ , where  $\lambda_0(e)$  is a constant depending on  $e$  only. In Section 3 we concentrate on the case  $0 \leq e \leq v^2/4$ . Some of the methods developed there are used later in Section 4, which deals with the case when  $e > v^2/4$ . The main results are Theorems 3.1, 3.2, 3.6, 4.2, 4.3, 4.4.

Several other instances of the problem were considered by the author in [8–10]. In [8] the cases  $\lambda = 2$  and  $\lambda = 3$  are investigated in more detail. In [9] some new nontrivial upper bounds for the function  $f(v, e, \lambda)$  were found in the general case, i.e. in the case when the only restrictions on the integers  $v, e, \lambda$  are  $0 \leq e \leq v(v-1)/2, \lambda \geq 2$ . For large values of  $\lambda$  those bounds are worse than the ones we obtain in this article. The problem was motivated by the analysis of the running time of the backtrack algorithm for the graph coloring problem (see Wilf [20]; Bender and Wilf [1]). Another source of related problems is a paper of Wright [21], where an asymptotic approximation to the number  $M_{v,e}$  = the total number of proper  $\lambda$ -colorings of all  $(v, e)$ -graphs, was found for a fixed  $\lambda$ , large  $v$  and all  $e$ . This result allows us to obtain an asymptotic approximation to  $av(v, e, \lambda) = av(P(G; \lambda))$  = the average value of the chromatic polynomial of a graph, where average is understood to be taken over all  $(v, e)$ -graphs  $G$ . In relation to our problem, the role of the function  $av(v, e, \lambda)$  is the following: for large  $v$  a lower bound for  $f(v, e, \lambda)$  can be obtained from it easily. Tomescu [15–16] considered problems similar to ours but for different families of graphs (all graphs on  $v$  vertices whose chromatic number is equal to  $k$ ).

We will need several more definitions and preliminary results. Let  $\{x; y\}$  be an edge of  $G$ . Then by  $G - \{x; y\}$  we mean the graph obtained from  $G$  by deleting  $\{x; y\}$ . By  $K_v, \bar{K}_v, T_v$  and  $K_{m,n}$  we denote correspondingly the complete graph on  $v$  vertices (any two vertices are joined by an edge), the completely disconnected graph on  $v$  vertices (no edges at all), a tree on  $v$  vertices, and the complete bipartite graph whose vertex classes contain  $m$  and  $n$  vertices. By  $G + H$  we denote the disjoint union of graphs  $G$  and  $H$ . The following proposition describes some properties of  $P(G; \lambda)$  which we are going to use later. Proofs can be found in [7, p. 147].

**Theorem 1.1.** *Let  $G$  be a graph with  $v$  vertices and  $e$  edges, and let  $G_1, G_2, \dots, G_{c(G)}$  be all connected components of  $G$ . Let  $\lambda$  be an integer  $\geq 2$ . Then:*

$$(i) \quad P(G; \lambda) = P(G_1; \lambda) \cdot P(G_2; \lambda) \cdots P(G_{c(G)}; \lambda).$$

- (ii)  $P(\bar{K}_v; \lambda) = \lambda^v$ ,  $P(K_v; \lambda) = \lambda(\lambda - 1) \cdots (\lambda - v + 1)$ ,  $P(T_v; \lambda) = \lambda(\lambda - 1)^{v-1}$ .  
 (iii) For a connected graph  $G$ ,  $P(G; \lambda) \leq \lambda(\lambda - 1)^{v-1}$ .

The equality sign occurs if and only if  $G$  is a tree.

(iv) The coefficients in every chromatic polynomial alternate in sign, the leading coefficient is 1, and the degree is equal to  $|V(G)|$ . Therefore  $P(G; \lambda)$  can be written as

$$P(G; \lambda) = \lambda^v - a_1\lambda^{v-1} + a_2\lambda^{v-2} - \cdots + (-1)^{v-k}a_{v-k}\lambda^k, \quad a_i \geq 0.$$

We will use a characterization of the coefficients of  $P(G; \lambda)$  in terms of so called 'broken circuits' of graph  $G$ . It is due to Whitney [18]. First number all the edges of the graph  $G$  from 1 to  $e$  in some manner. Next, from each cycle  $C$  of  $G$  delete the edge of the highest index, obtaining, thereby, the *broken cycle*  $C'$ . Then we have the following theorem.

**Theorem 1.2** (Whitney's 'Broken circuits' theorem). *Let*

$$P(G; \lambda) = \lambda^v - a_1\lambda^{v-1} + a_2\lambda^{v-2} - \cdots + (-1)^{v-1}a_{v-1}\lambda.$$

The coefficient  $a_j$  is equal to the number of  $j$ -subsets of edges of the graph  $G$  which contain no broken cycles, for each  $j = 1, 2, \dots, v - 1$ .

## 2. A connection between our problem and the number of triangles in a graph

Let  $G \in \mathcal{F}$ . Then by Theorem 1.2

$$P(G; \lambda) = \lambda^v - a_1\lambda^{v-1} + a_2\lambda^{v-2} - a_3\lambda^{v-3} + \cdots + (-1)^{v-1}a_{v-1}\lambda \quad (2.1)$$

where  $a_j$  = the number of  $j$ -subsets of edges of  $G$  which contain no broken cycles, for each  $j = 1, 2, \dots, v - 1$ . We want to describe graphs from  $\mathcal{F}$  which chromatic polynomials take the greatest values provided that  $\lambda$  is large. More precisely, we are looking for a  $G \in \mathcal{F}$ , such that there exists a constant  $\lambda_0$  such that for any  $\lambda > \lambda_0$  and each  $H \in \mathcal{F}$ ,

$$P(G; \lambda) \geq P(H; \lambda).$$

The first two leading coefficients in (2.1) are fixed:  $a_0 = 1$ ,  $a_1 = e$  (Theorem 1.1(iv) and Theorem 1.2). Therefore the first natural step in our search for a graph  $G$  extremal for large  $\lambda$  is finding all  $G \in \mathcal{F}$  for which the positive coefficient  $a_2$  is the greatest. Let us denote the set of all such graphs by  $\mathcal{F}_1$ ,  $\mathcal{F}_1 \subset \mathcal{F}$ . Then among all the graphs from  $\mathcal{F}_1$  we choose ones for which  $a_3$  is the least. Suppose they form a subset  $\mathcal{F}_2$ ,  $\mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F}$ . Continuing this way one comes to the extremal graphs ( $\mathcal{F}$  is finite!).

As an immediate corollary from Theorem 1.2 (e.g. [12]) we get that  $a_2 = \binom{e}{2} - c_3$ , where  $c_3 = c_3(G)$  = the number of triangles in  $G$ . This relation shows that  $a_2$  is the greatest if and only if  $c_3$  is the least. Thus the question

“Which  $G \in \mathcal{F}_1$ ?” is equivalent to

$$\text{“Which graphs from } \mathcal{F} \text{ have the least number of triangles?”} \quad (2.2)$$

This is a well-known unsolved problem from extremal graph theory. It is discussed in a book by Bollobás [3, Ch. VI.1] where also some lower and upper bounds for the number of triangles in a graph are given. In the following two sections we continue the investigation considering cases depending on whether  $\mathcal{F}$  contains a graph without triangles (Section 3) or not (Section 4).

### 3. Case: $0 \leq e \leq v^2/4$

If  $e \leq v^2/4$ , then  $\mathcal{F}$  contains graphs without triangles (for example bipartite graphs). Let  $\mathcal{F}_1 = \{G: G \in \mathcal{F}, G \text{ has no triangles}\}$ . Chromatic polynomials of the graphs from  $\mathcal{F}_1$  have  $a_2 = \binom{e}{2}$  and we transfer our attention to  $a_3$ . It is easy to see (e.g. [12]) that for every  $G \in \mathcal{F}_1$ ,  $a_3 = \binom{e}{3} - c_4$ , where  $c_4 = c_4(G)$  = the number of quadrilaterals in  $G$ . This relation shows that  $a_3$  is the least if and only if  $c_4$  is the greatest. Thus the question “Which  $G \in \mathcal{F}_2$ ?” is equivalent to

$$\text{“Which graphs from } \mathcal{F} \text{ have no triangles and the greatest number of quadrilaterals?”} \quad (3.1)$$

We do not have a complete answer to (3.1) for arbitrary  $v, e, 0 \leq e \leq v^2/4$ . The following theorem gives upper and lower bounds for the number of cycles of length 4 in a triangle-free graph. It will imply the complete answer to the question (3.1) when  $e$  is a perfect square, say  $e = p^2$ .

**Theorem 3.1.** *Let  $e \leq v^2/4$  and  $c_4(v, e) = \max\{c_4(G): G \in \mathcal{F} \text{ and } G \text{ has no triangles}\}$ . Then*

$$\frac{e(e - 4\sqrt{e})}{4} < c_4(v, e) \leq \frac{e(\sqrt{e} - 1)^2}{4}.$$

The equality is attained only for the graph  $G_{v,p} = K_{p,p} + \bar{K}_{v-2p}$  when  $e = p^2$ ,  $v \geq 2p$ .

**Proof.** (i) *Lower bound.* Let  $p$  be a nonnegative integer such that  $p^2 \leq e < p^2 + p$ . Then  $c_4(K_{p,p}) \leq c_4(v, e)$  and  $p > \sqrt{e} - \frac{1}{2}$ . Therefore,

$$\begin{aligned} c_4(v, e) &\geq c_4(K_{p,p}) = \binom{p}{2} \binom{p}{2} = \frac{(p^2 - p)^2}{4} \\ &> \frac{1}{4} [(\sqrt{e} - \frac{1}{2})^2 - (\sqrt{e} - \frac{1}{2})]^2 > \frac{e(e - 4\sqrt{e})}{4}. \end{aligned}$$

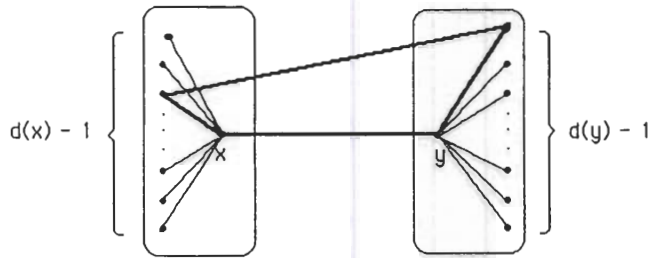


Fig. 1.  $\text{nbh}(x) \cap \text{nbh}(y) = \emptyset$ , since there are no triangles.

Let  $p^2 + p \leq e < (p + 1)^2$ . Then  $c_4(K_{p+1,p}) \leq c_4(v, e)$  and  $p > \sqrt{e} - 1$ . Therefore,

$$\begin{aligned} c_4(v, e) &\geq c_4(K_{p+1,p}) = \binom{p+1}{2} \binom{p}{2} = \frac{(p^2 - 1)p^2}{4} \\ &> \frac{(e - 2\sqrt{e})(\sqrt{e} - 1)^2}{4} > \frac{e(e - 4\sqrt{e})}{4}. \end{aligned}$$

(ii) *Upper bound.* Let  $G \in \mathcal{F}$  and  $\{x, y\}$  be an edge in  $G$ . Let  $d(a)$  and  $\text{nbh}(a)$  denote the degree of the vertex  $a$  and the set of all the neighbours of  $a$  in  $G$  ( $|\text{nbh}(a)| = d(a)$ ). It is easy to understand (Fig. 1) that the number of 4-cycles in  $G$  containing  $\{x, y\}$  does not exceed  $(d(x) - 1)(d(y) - 1)$ .

Suppose  $G$  has no isolated vertices. Then,

$$\begin{aligned} c_4(G) &= \frac{1}{4} \sum_{(x,y) \in E(G)} (\# \text{ of 4-cycles having } \{x,y\} \text{ as an edge}) \\ &\leq \frac{1}{4} \sum_{(x,y) \in E(G)} (d(x) - 1)(d(y) - 1). \end{aligned}$$

Since  $G$  has no isolated vertices, the last expression is equal to

$$\frac{1}{4} \cdot \frac{1}{2} \left[ \sum_{x \in V(G)} (d(x) - 1) \sum_{y \in \text{nbh}(x)} (d(y) - 1) \right] \leq \frac{1}{8} \left[ \sum_{x \in V(G)} (d(x) - 1)(e - d(x)) \right]$$

because  $G$  contains no triangles. Expanding the obtained sum and using the fact that the quadratic average is not less than the arithmetic average, we obtain

$$\begin{aligned} &\frac{1}{8} \left[ (e + 1) \sum_{x \in V(G)} d(x) - e \sum_{x \in V(G)} 1 - \sum_{x \in V(G)} (d(x))^2 \right] \\ &= \frac{1}{8} \left[ (e + 1)2e - ev - \sum_{x \in V(G)} (d(x))^2 \right] \\ &\leq \frac{1}{8} \left[ (e + 1)2e - ev - \left( \sum_{x \in V(G)} d(x) \right)^2 / v \right] \\ &= \frac{1}{8} \left[ (e + 1)2e - ev - \frac{4e^2}{v} \right] \leq \frac{1}{8} [2e^2 + 2e - 4e\sqrt{e}] = \frac{e(\sqrt{e} - 1)^2}{4}. \end{aligned}$$

Clearly, the equality in the last ' $\leq$ ' sign occurs if and only if  $v^2 = 4e$ . So, in the case of equality,  $e = p^2$ , for some integer  $p$  and  $v = 2p$ . Since  $G$  has no triangles,

then Turán's theorem implies that  $G = K_{p,p}$ . From the other side it is easy to see that for  $K_{p,p}$  the upper bound is achieved. In order to finish the proof we consider the case when  $G$  has  $q \geq 1$  isolated vertices. Then  $G = G_1 + \bar{K}_q$ , where  $v(G_1) = v - q$ ,  $e(G_1) = e(G) = e$  and  $G_1$  has no isolated vertices. But  $c_4(G) = c_4(G_1)$ . Therefore Theorem 3.1 is proved.  $\square$

The result just obtained allows us to determine a graph which possesses the greatest number of colorings for large  $\lambda$  in a particular case, namely  $e = p^2$  and  $v \geq 2p$ . It will lead us to upper bounds for the values of the chromatic polynomials in a more general case when  $e \leq v^2/4$  and  $e$  is not necessarily a square.

**Theorem 3.2.** (i) Let  $v \geq 2p$ ,  $e = p^2 \geq 5$  and  $G_{v,p} = K_{p,p} + \bar{K}_{v-2p}$ . Then for  $\lambda \geq e^4/12$ ,

$$P(G_{v,p}; \lambda) > P(G; \lambda) \quad \text{for all } G \in \mathcal{F}, G \neq G_{v,p}.$$

(ii) Let  $v = 2p$ ,  $e = p^2 \geq 5$ . Then for  $\lambda \geq e^{2.5}$ ,

$$P(K_{p,p}; \lambda) > P(G; \lambda) \quad \text{for all } G \in \mathcal{F}, G \neq K_{p,p}.$$

**Proof.** As it is seen from the proof of Theorem 3.1 for the given values of  $v$  and  $e$ , the attempt of maximizing  $a_2$  and, this being done, minimizing  $a_3$  leads to the complete and unique characterization of the graph. Therefore Theorem 3.2 is proved for sufficiently large  $\lambda$ , i.e. for all  $\lambda \geq \lambda_0$ , where  $\lambda_0$  is some constant. To get an estimate for  $\lambda_0$  we find an upper bound  $R$  for the absolute values of the roots of the polynomial

$$H(\lambda) = P(G_{v,p}; \lambda) - P(G; \lambda).$$

Then for all  $\lambda > R$ ,  $H(\lambda) > 0$ .

In order to compute  $R$  in terms of the coefficients of the polynomial we use the following lemma due to Fujiwara [6]; for a reference in English see Wilf [19]):

**Lemma 3.3.** All the roots of the polynomial  $f(z) = b_0 z^n + b_1 z^{n-1} + \dots + b_n$  lie in the circle

$$|z| \leq R = 2 \max\{|b_i/b_0|^{1/i}; 1 \leq i \leq n\}.$$

Let  $P(G_{v,p}; \lambda) = \lambda^v - k_1 \lambda^{v-1} + k_2 \lambda^{v-2} - \dots$  and  $P(G; \lambda) = \lambda^v - g_1 \lambda^{v-1} + g_2 \lambda^{v-2} - \dots$ . Then  $H(\lambda) = P(G_{v,p}; \lambda) - P(G; \lambda) = h_2 \lambda^{v-2} - h_3 \lambda^{v-3} \dots$ , where  $h_i = k_i - g_i$ ,  $i = 2, 3, \dots, v-1$  ( $h_0 = 1 - 1 = 0$ ,  $h_1 = e - e = 0$ ). The leading coefficient  $h_2 = k_2 - g_2 = e(e-1)/2 - (e(e-1)/2 - c_3(G)) = c_3(G)$ . If  $c_3(G) \neq 0$ , then the direct application of Lemma 3.3 gives

$$R = 2 \max\{|h_{2+i}/h_2|^{1/i}; 1 \leq i \leq v-3\}.$$

If  $h_2 = c_3(G) = 0$ , then

$$-h_3 = -k_3 + g_3 = -\left[\binom{e}{3} - c_4(G_{v,p})\right] + \left[\binom{e}{3} - c_4(G)\right] = c_4(G_{v,p}) - c_4(G).$$

By Theorem 3.1,  $|h_3| \geq 1$ . Applying Lemma 3.3, we obtain

$$R = 2 \max\{|h_{3+i}/h_3|^{1/i} : 1 \leq i \leq v-4\}. \quad (3.2)$$

In order to find an upper bound for  $R$  we use the following lemma.

**Lemma 3.4.** Let  $1 \leq A \leq \binom{e}{2}$ ,  $e \geq v \geq 5$ . Let  $c_i = [\binom{e}{2+i}/A]^{1/i}$ ,  $1 \leq i \leq v-2$ . Let  $d_i = [\binom{e}{3+i}/A]^{1/i}$ ,  $1 \leq i \leq v-3$ . Then:

- (i)  $\{c_i\}$  is monotone decreasing and  $\max\{c_i : 1 \leq i \leq v-2\} = c_1 = \binom{e}{3}/A$ .
- (ii)  $\{d_i\}$  is monotone decreasing and  $\max\{d_i : 1 \leq i \leq v-3\} = d_1 = \binom{e}{4}/A$ .

**Proof.** It is easy to show (see [10]) that  $(c_{i+1}/c_i)^{i(i+1)} < 1$ , for all  $i$ ,  $1 \leq i \leq v-3$ , and  $(d_{i+1}/d_i)^{i(i+1)} < 1$  for all  $i$ ,  $1 \leq i \leq v-4$ .  $\square$

Now we are ready to find an upper bound for  $R$ . Due to Theorem 1.2,  $|h_m| = |k_m - g_m| \leq \binom{e}{m}$ ,  $m = 1, \dots, v-1$ . This makes  $h_2 \leq \binom{e}{2}$ . We consider the following two cases.

Case 1: Let  $h_2 = c_3(G) \geq 1$ .

Setting  $A = h_2$  in Lemma 3.4 (i) (the condition  $1 \leq A \leq \binom{e}{2}$  is met), we get

$$R = 2 \max\{|h_{2+i}/h_2|^{1/i}\} \leq 2 \max\left\{\left[\frac{\binom{e}{2+i}}{h_2}\right]^{1/i}\right\} = 2 \binom{e}{3}/h_2 \leq 2 \binom{e}{3}.$$

Case 2: Let  $h_2 = 0$ .

Then by Theorem 3.1,

$$1 \leq |h_3| = c_4(G_{v,p}) - c_4(G) \leq c_4(G_{v,p}) \leq e(\sqrt{e}-1)^2/4 < \binom{e}{2}.$$

Setting  $A = |h_3|$  in Lemma 3.4 (ii) (the condition  $1 \leq A \leq \binom{e}{2}$  is met), and dropping the first term we obtain

$$R = 2 \max\{|h_{3+i}/h_3|^{1/i}\} \leq 2 \max\left\{\left[\frac{\binom{e}{3+i}}{|h_3|}\right]^{1/i}\right\} = 2 \binom{e}{4}/|h_3| \leq 2 \binom{e}{4}.$$

In both cases  $R < e^4/12$  for  $e \geq 5$ . This proves Theorem 3.2(i).

If  $v = 2p$ , then it is possible to obtain a better upper bound for  $R$ . We use a particular case of a result of Erdős [4]:

**Lemma 3.5.** Let  $v(G) = v = 2p$ ,  $e(G) = p^2$  and  $G \neq K_{p,p}$ . Then  $G$  contains at least  $p-1$  triangles, i.e.  $c_3(G) \geq p-1$ .

Having  $h_2 \geq p - 1 > 0$  we get

$$R \leq 2 \binom{e}{3} / h_2 \leq p^2(p^2 - 1)(p^2 - 2) / 3(p - 1).$$

It implies that  $R < p^5$  for all  $p > 1$ . This proves Theorem 3.2(ii).  $\square$

Combining all the results we state and prove the main theorem of this section:

**Theorem 3.6.** *Let  $4 < e \leq v^2/4$  and  $P(G; \lambda) = \max\{P(H; \lambda) \mid H \in \mathcal{F}\}$ . Let  $p = \lfloor \sqrt{e} \rfloor$ . Then the following bounds for  $P(G; \lambda)$  are valid: for  $v = 2p$ ,  $\lambda \geq p^5$ ,*

$$(1 - p/2\lambda)^{2p} \cdot \lambda^v < P(G; \lambda) < [(1 - p/2\lambda)^{2p} + \frac{1}{4}(p^3 - p^2)\lambda^{-2}] \cdot \lambda^v;$$

for  $v = 2p + 1$ ,  $\lambda \geq p^5$ ,

$$(1 - p/\lambda)(1 - p/2\lambda)^{2p} \cdot \lambda^v < P(G; \lambda) < [(1 - p/2\lambda)^{2p} + \frac{1}{4}(p^3 - p^2)\lambda^{-2}] \cdot \lambda^v;$$

for  $v \geq 2p + 2$ ,  $\lambda \geq e^4/12$

$$(1 - (p + 1)/2\lambda)^{2p+2} \cdot \lambda^v < P(G; \lambda) < [(1 - p/2\lambda)^{2p} + \frac{1}{4}(p^3 - p^2)\lambda^{-2}] \cdot \lambda^v.$$

**Proof.** The idea of the proof is the following: Let  $G_1$  and  $G_2$  be two graphs on  $v$  vertices having  $e_1$  and  $e_2$  edges respectively, where  $e_1 \leq e \leq e_2$ . Then

$$P(G_2; \lambda) \leq P(G; \lambda). \quad (3.3)$$

Indeed, deleting any  $e_2 - e$  edges from  $G_2$  we obtain a graph  $H_2$  such that  $P(G_2; \lambda) \leq P(H_2; \lambda)$ .  $H_2 \in \mathcal{F}$  and from the definition of  $G$  we have  $P(H_2; \lambda) \leq P(G; \lambda)$ . This implies the inequality (3.3). Despite the fact that in general the inequality

$$P(G; \lambda) \leq P(G_1; \lambda) \quad (3.4)$$

is false, it is valid if  $G_1$  is taken to be a graph on  $v$  vertices with  $e_1$  edges which possesses the greatest number of colorings among all the graphs with the same number of vertices and edges. Then the proof of (3.4) is identical to the one for (3.3).

Graphs  $G_1$  and  $G_2$  will be taken as unions of some complete bipartite graphs and isolated vertices, i.e.  $K_{n,m} + \bar{K}_{v-(n+m)}$ . A chromatic polynomial

$$P(K_{n,m} + \bar{K}_{v-(n+m)}; \lambda) = P(K_{n,m}; \lambda) \cdot \lambda^{v-(n+m)}$$

can be found explicitly, since  $P(K_{n,m}; \lambda)$  is known (e.g. Swenson [14]). But the expression one obtains this way is rather complicated. Therefore bounds for  $P(G_i; \lambda)$ ,  $i = 1, 2$ , will be found and used along with (3.3) and (3.4) in order to get bounds for  $P(G; \lambda)$ . The following lemma will serve our needs.

**Lemma 3.7.** (i) *Let  $K_{p,p}$  be a complete bipartite graph. Then for any  $\lambda \geq 3p^4$ ,  $p > 2$ .*

$$(\lambda - p/2)^{2p} \leq P(K_{p,p}; \lambda) < (\lambda - p/2)^{2p} + \frac{1}{4}(p^3 - p^2)\lambda^{2p-2}. \quad (3.5)$$



(ii) Let  $G_{v,p} = K_{p,p} + \bar{K}_{v-2p}$ . Then for any  $\lambda \geq 3p^4$ ,  $p > 2$ ,

$$(1 - p/2\lambda)^{2p} \cdot \lambda^v \leq P(G_{v,p}; \lambda) < [(1 - p/2\lambda)^{2p} + \frac{1}{4}(p^3 - p^2)\lambda^{-2}] \cdot \lambda^v. \quad (3.6)$$

**Proof of Lemma 3.7.** (i) The proof is based on Lemma 3.3 and Lemma 3.4, and is very similar to the one of Theorem 3.2 (Case 1). For details see [10].

(ii) By Theorem 1.1(i),

$$P(G_{v,p}; \lambda) = P(K_{p,p}; \lambda) \cdot P(\bar{K}_{v-2p}; \lambda) = P(K_{p,p}; \lambda) \cdot \lambda^{v-2p}.$$

Using (3.5), we obtain

$$(\lambda - p/2)^{2p} \cdot \lambda^{v-2p} \leq P(G_{v,p}; \lambda) < [(\lambda - p/2)^{2p} + \frac{1}{4}(p^3 - p^2)\lambda^{2p-2}] \cdot \lambda^{v-2p}.$$

But this can be rewritten as (3.6).  $\square$

Now we finish the proof of Theorem 3.6. Let  $p = \lfloor \sqrt{e} \rfloor$ . Take  $G_1 = K_{p,p}$  and  $G_2$  given by:

$$G_2 = \begin{cases} K_{p,p} & \text{if } v = 2p; \\ K_{p+1,p} & \text{if } v = 2p + 1; \\ G_{v,p+1} & \text{if } v \geq 2p + 2. \end{cases}$$

By using (3.3) and (3.4) for these pairs of  $G_1$  and  $G_2$ , and (3.5), (3.6) for the bounds of  $P(G_1; \lambda)$  and  $P(G_2; \lambda)$ , we obtain the inequalities of Theorem 3.6 for all  $\lambda \geq \max\{p^5, 3p^4\} = p^5$  (since  $p > 2$ ) when  $v = 2p$  and for all  $\lambda \geq \max\{p^5, 3p^4, e^4/12\} = e^4/12$  when  $v \geq 2p + 2$ . If  $v = 2p + 1$  we use the simple observation that  $P(K_{p+1,p}; \lambda) > (\lambda - p)P(K_{p,p}; \lambda)$  for  $\lambda > p$ .

#### 4. Case: $e > v^2/4$

If  $e > v^2/4$ , then Turán's Theorem implies that each graph from  $\mathcal{F}$  contains triangles.

One of the lower bounds for the number of cycles of length 3 in a graph of order  $v$  and size  $e$  was discovered by Nordhaus and Stewart [13]. We present a proof of it given in the book of Lovász [11, p. 396], since its analysis will be used later in this section. Let  $c_3(v, e)$  denote the least number of triangles a graph with  $v$  vertices and  $e$  edges may have.

**Theorem 4.1.** Let  $G \in \mathcal{F}$ . Then

$$c_3(G) \geq c_3(v, e) \geq (e/3v)(4e - v^2). \quad (4.1)$$

**Proof.** Let  $\{x, y\} \in E(G)$ . It is clear that the number of triangles of  $G$  having the edge  $\{x, y\}$  as a side is not less than  $d(x) + d(y) - v$ . Therefore for the total

number of triangles of  $G$  we have:

$$\begin{aligned} c_3(G) &= \frac{1}{3} \sum_{\{x,y\} \in E(G)} (\# \text{ of 3-cycles having } \{x,y\} \text{ as an edge}) \\ &\geq \frac{1}{3} \sum_{\{x,y\} \in E(G)} (d(x) + d(y) - v). \end{aligned}$$

Since  $d(x)$  occurs in this sum exactly  $d(x)$  times, the sum can be written as

$$\begin{aligned} \frac{1}{3} \left[ \sum_{x \in V(G)} d(x)^2 - ve \right] &\geq \frac{1}{3} \left[ \frac{1}{v} \left( \sum_{x \in V(G)} d(x) \right)^2 - ve \right] \\ &= \frac{1}{3} \left( \frac{4e^2}{v^2} - e \right) = (e/3v)(4e - v^2). \end{aligned}$$

The last inequality sign is due to the fact that the quadratic average is not less than the arithmetic average.  $\square$

**Remarks.** The proof above shows that '=' occurs if and only if:

- (1) all  $d(x)$  are equal to  $d = 2e/v$ , i.e.  $G$  is a regular graph;
- (2) every edge  $\{x; y\}$  is a side of  $2d - v = (4e - v^2)/v$  triangles.

Unfortunately Theorem 4.1 does not give an answer to the question (2.2) for arbitrary  $v$  and  $e$ ,  $e > v^2/4$ . Nevertheless it allows us to find a graph which has the least number of triangles for some particular values of  $v$  and  $e$ ,  $e > v^2/4$ . For a positive integer  $q$ , by  $T_q(v)$  we denote a unique complete  $q$ -partite graph on  $v$  vertices whose vertex classes are as equal as possible. If  $v = pq$ , for some integer  $p$ , then each vertex class of  $T_q(v)$  contains  $p$  vertices. We formulate and prove the following theorem.

**Theorem 4.2.** *Let  $p$  and  $q$  be two positive integers,  $q \geq 3$ . Let  $v = pq$  and  $e = t_q(v) = \binom{q}{2} p^2$ . Then  $T_q(v)$  is the unique graph having the least number of triangles among all the graphs of order  $v$  and size  $e$ .*

**Proof.** (i) First we check that  $c_3(T_q(v)) = (e/3v)(4e - v^2)$ . In order to do this we notice that  $T_q(v)$  is regular with  $d = d(x) = 2e/v = p(q-1)$ . Let  $\{x; y\}$  be an edge of  $T_q(v)$  joining two vertices from the partition classes  $V_x$  and  $V_y$ . To form a triangle with one side  $\{x; y\}$  we choose any vertex  $z$  from the remaining  $q-2$  partition classes of  $T_q(v)$ . Since each partition class contains  $p$  vertices, we have  $p(q-2)$  triangles. But  $p(q-2) = 2d - v$ !. So the statement follows from the Remarks (1) and (2) to Theorem 4.1.

(ii) Now we are going to prove that if  $c_3(G) = c_3(T_q(v))$  for some  $G \in \mathcal{F}$ , then  $G \cong T_q(v)$ .

Having the least number of triangles, graph  $G$  satisfies the conditions (1) and (2) of the Remark to Theorem 4.1. Therefore  $G$  is regular with  $d = d(x) = 2e/v = p(q-1)$  and each edge of  $G$  belongs to  $2d - v = p(q-2)$  triangles. Let

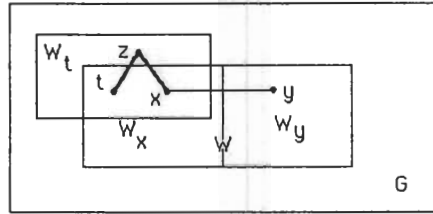


Fig. 2.

$\{x; y\} \in E(G)$  and  $W = V(G) - \text{nbh}(x) \cap \text{nbh}(y)$ .  $|W| = v - p(q - 2) = 2p$ . Since  $v - d(x) = v - d(y) = pq - p(q - 1) = p$ , the sets  $W_x = V(G) - \text{nbh}(x) = W \cap \text{nbh}(y)$  and  $W_y = V(G) - \text{nbh}(y) = W \cap \text{nbh}(x)$  form a disjoint partition of  $W$ ,  $|W_x| = |W_y| = p$ . This also shows that

$$\text{any vertex adjacent to } x \text{ is connected to every vertex from } W_x. \quad (4.2)$$

We claim that  $W_x$  is an independent set of vertices, i.e. no edge of  $G$  joins two points from  $W_x$ . In order to show this we prove that  $W_t = W_x$  for all  $t \in W_x$ . Indeed, take  $t \in W_x, t \neq x$ . If  $W_t = W_x$  for all  $t \in W_x$ , then  $W_x$  is an independent set. Suppose there exists a vertex  $t \in W_x$  such that  $W_t \neq W_x$  (Fig. 2).

Having the same number of elements and being different, the sets  $W_t$  and  $W_x$  cannot be subsets of one another. So there is  $z \in W_t$  which does not belong to  $W_x$ . Then  $z \in \text{nbh}(x)$  and  $\{z; x\}$  is an edge. Since  $t \in W_x$ , the statement (4.2) implies that  $z$  and  $t$  are connected. This contradiction proves that  $W_t = W_x$  for all  $t \in W_x$  and  $W_x$  is an independent set of points. Let's summarize the facts:

- (a) each vertex  $x \in V(G)$  is a member of some maximal independent set of vertices  $W_x$  having  $p$  elements;
- (b) for any edge  $\{x; y\}$  of  $G$ ,  $W_y \cap W_x = \emptyset$ ;
- (c) any vertex which is not in  $W_x$  is adjacent to all vertices from  $W_x$ .

It is clear that (a), (b), (c) imply  $G \cong T_q(v)$ .  $\square$

An immediate corollary of Theorem 4.2 is the fact that  $P(T_q(v); \lambda) > P(H; \lambda)$  for all  $H \in \mathcal{F}, H \neq T_q(v)$ , provided that  $\lambda$  is sufficiently large. The precise statement and a lower bound for  $\lambda$  will be given in Theorem 4.4.

Let  $C_3 = \max\{c_3(G); G \in \mathcal{F}\}$ . Contrary to  $c_3(v, e)$ ,  $C_3$  can be determined explicitly for any  $v$  and  $e$ . It turns out that it depends only on  $e$ . It follows from a more general result of Erdős and Hanani [5], that  $C_3 = C_3(e) = \binom{s}{3} + \binom{t}{2}$ , where  $e = \binom{s}{2} + t, 0 < t \leq s$ . We formulate a theorem which gives bounds for the values of  $P(G; \lambda), G \in \mathcal{F}$ . The proof is similar to the one of Theorem 3.2 (Case 1) and can be found in [10].

**Theorem 4.3.** Let  $G$  be a graph having  $v \geq 4$  vertices and  $e$  edges,  $e > v^2/4, \lambda \geq 2\binom{e}{3}$ . Then

$$(\lambda - e/v)^v + k_1 \lambda^{v-2} \leq P(G; \lambda) \leq (\lambda - e/v)^v + k_2 \lambda^{v-2} \quad (4.3)$$

where

$$k_1 = C_3(e) - (e/2)[(e/v) - 1] + 1, \quad k_2 = (e/2)[(e/v) - 1] - c_3(v, e) + 1.$$

Applying Theorems 4.2 and 4.3 we get the following theorem.

**Theorem 4.4.** *Let  $p$  and  $q$  be two positive integers,  $q \geq 3$ . Let  $v = pq$ ,  $e = t_q(v) = \binom{2}{3}p^2$  and  $\lambda \geq 2\binom{5}{3}$ . Then  $T_q(v)$  is the unique graph which possesses the greatest number of proper  $\lambda$ -colorings among all the graphs of order  $v$  and size  $e$ .*

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