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# Closure planes

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**Abstract**

We introduce a simple algebraic method for constructing infinite affine (and projective) planes from an infinite set of finite planes of prime power order stemming from a "root" plane. The construction uses finite fields and infinite extensions of finite fields in a critical way. We obtain a classical-looking result which states that if the construction succeeds over the algbraic closure of a finite field, then both the infinite plane and the original root plane must be Desarguesian. The Lenz-Barlotti types for these planes are then linked to the Lenz-Barlotti type of the root plane. Examples are then given. These show that under suitable conditions, the method can yield infinitely many non-isomorphic infinite planes. These examples are of Lenz-Barlotti types II.1 and V.1.

## § 1. Introduction

Let $q$ be a power of the prime $p$. Throughout, the finite field of $q$ elements is denoted by $\mathbb{F}_q$ and its non-zero elements by $\mathbb{F}_q^\star$. Denote by $\mathbb{F}_q[X]$ the ring of polynomials in $X$ over $\mathbb{F}_q$. It is a well-known fact that every function on $\mathbb{F}_q$ can be uniquely represented by a polynomial of degree at most $q-1$, and that this may be extended to multivariate functions also. A polynomial $f \in \mathbb{F}_q[X]$ is called a *permutation* polynomial (PP) of some extension $\mathcal{K}$ of $\mathbb{F}_q$ if it induces a bijection of $\mathcal{K}$ under the evaluation map $c \mapsto f(c)$. From the large theory concerning permutation polynomials, the concept of exceptional polynomials is particular relevant to the present article – a polynomial $f \in \mathbb{F}_q[X]$ is *exceptional* if it permutes infinitely many finite extensions of $\mathbb{F}_q$, see the section titled "Exceptional polynomials" by M. Zieve from the *Handbook of Finite Fields* [17] for more information on these polynomials.

The method of coordinatisation was introduced in 1943 by Hall [11]. It is used to obtain algebraic objects – planar ternary rings (PTRs) – from affine and projective planes and vice versa. The method of

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Hall, coupled with later results of Pickert [19] showing specific flag-transitivities of the plane corresponded exactly with specific properties of the PTR, has been one of the most powerful tools available in the study of affine and projective planes ever since.

Throughout we shall follow the coordinatisation process outlined by Hughes and Piper in [13], Chapter 5. Let $\mathcal{A}$ be an affine plane and $\mathcal{P}$ it's projective closure, with the point and line at infinity being denoted by $(\infty)$ and $[\infty]$, respectively. The key initial step of the coordinatising method is the labelling of the quadrangle of points $\mathbf{O}, \mathbf{x}, \mathbf{y}, \mathbf{I}$ used in the method: we have $\mathbf{O} = (0,0)$, $\mathbf{x} = (0)$ and $\mathbf{y} = (\infty)$ on the line at infinity, and a final point $\mathbf{I} = (1,1)$. (Of course, from the perspective of coordinatising a projective plane, one first must designate the line at infinity.) If $\mathcal{A}$ had order $n$, then coordinatising $\mathcal{A}$ will produce a tri-variate function, called a *planar ternary ring (PTR)*, defined on a set of order $n$. When the order $n$ is a prime power, say $n = q$, then one can use the finite field $\mathbb{F}_q$ for the coordinatisation method, and use the $0$ and $1$ of the field to coincide with the $0$ and $1$ used to label $\mathbf{O}$, $\mathbf{x}$ and $\mathbf{I}$.

> *Throughout, we assume that any coordinatisation of a plane $\mathcal{A}$ of order $q$ is done by labelling the points of $\mathcal{A}$ using elements of $\mathbb{F}_q$ and where we label $\mathbf{O} = (0,0)$ and $\mathbf{I} = (1,1)$. After such a coordinatisation, $\mathbf{O} = (0,0)$, $\mathbf{I} = (1,1)$, $\mathbf{x} = (0)$ and $\mathbf{y} = (\infty)$.*

Following coordinatisation, the resulting PTR can be viewed as a reduced polynomial $T \in \mathbb{F}_q[X,Y,Z]$, which we call a *PTR polynomial*. We have the following important result, essentially due to Hall [11], specialised to our purposes; see also Hughes and Piper, [13], Theorem 5.1.

**Lemma 1** (Hall, [11], Theorem 5.4)**.** *Let $\mathcal{A}$ be an affine plane of prime power order $q$ and $T \in \mathbb{F}_q[X,Y,Z]$ be a PTR polynomial obtained from $\mathcal{A}$. Then $T(X,Y,Z)$ must satisfy the following properties:*

*(a)* $T(a,0,z) = T(0,b,z) = z$ *for all* $a,b,z \in \mathbb{F}_q$.

*(b)* $T(x,1,0) = x$ *and* $T(1,y,0) = y$ *for all* $x,y \in \mathbb{F}_q$.

*(c) If* $a,b,c,d \in \mathbb{F}_q$ *with* $a \neq c$, *then there exists a unique* $x$ *satisfying* $T(x,a,b) = T(x,c,d)$.

*(d) If* $a,b,c \in \mathbb{F}_q$, *then there is a unique* $z$ *satisfying* $T(a,b,z) = c$.

*(e) If* $a,b,c,d \in \mathbb{F}_q$ *with* $a \neq c$, *then there is a unique pair* $(y,z)$ *satisfying* $T(a,y,z) = b$ *and* $T(c,y,z) = d$.

*Conversely, any $T \in \mathbb{F}_q[X,Y,Z]$ which satisfies Properties (c) through (e) can be used to define an affine plane $\mathcal{A}_T$ of order $q$ as follows:*

- *the points of $\mathcal{A}$ are $(x,y)$, with $x,y \in \mathbb{F}_q$;*
- *the lines of $\mathcal{A}$ are the symbols $[m,a]$, with $m,a \in \mathbb{F}_q$, defined by*

$$[m,a] = \{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \,:\, a = T(m,x,y)\},$$

*and the symbols $[c]$, with $c \in \mathbb{F}_q$, defined by*

$$[c] = \{(c,y) \,:\, y \in \mathbb{F}_q\}.$$

If $T \in \mathbb{F}_q[X,Y,Z]$ satisfies Properties (a) through (e) of Lemma 1 over some extension field $\mathcal{K}$ of $\mathbb{F}_q$, then we shall say $T$ *is a PTR polynomial over* $\mathcal{K}$. As the lemma states, a polynomial $T \in \mathbb{F}_q[X,Y,Z]$ satisfying Properties (c), (d) and (e) can be used to construct an affine plane – we shall call such a polynomial a *weak PTR*. If $T$ also satisfies Property (a) (resp. (b)), we shall call $T$ a *weak PTR with zero (resp. unity)*. Many such functions can be used to construct a plane isomorphic to a given affine plane; for examples one need only consider the theory for semifields and presemifields (see Knuth [14]) to note how weak PTRs with zero can play just as prominent a role as PTRs. With respect to coordinatisation using the elements of $\mathbb{F}_q$ for labelling, if we do not insist on labelling $\mathbf{I}$ with $(1,1)$, then the resulting $T \in \mathbb{F}_q[X,Y,Z]$

will be a weak PTR with zero and some element $c \in \mathbb{F}_q^\star$, $c \neq 1$, acting as unity. Likewise, if we do not insist on labelling $\mathbf{O}$ with $(0,0)$, then the resulting $T \in \mathbb{F}_q[X,Y,Z]$ will be a weak PTR with unity and some element $c \in \mathbb{F}_q^\star$, $c \neq 1$, acting as zero. In either case, we would be obtaining very special examples of weak PTRs; generally, weak PTRs do not necessarily have any element which behaves as a zero or unity. In deference to Properties (a) and (b) above, we define an addition $\oplus$ and multiplication $\odot$ by

$$x \oplus y = T(1,x,y),$$
$$x \odot y = T(x,y,0),$$

for all $x,y \in \mathcal{K}$. A PTR is called *linear* over $\mathcal{K}$ if $T(x,y,z) = (x \odot y) \oplus z$ for all $x,y,z \in \mathcal{K}$.

An important example of a PTR polynomial is the polynomial $T(X,Y,Z) = XY + Z$. It is easily checked that the polynomial $T$ is a linear PTR over any field $\mathcal{K}$; it defines the Desarguesian plane in every case. It cannot be over emphasised that the same plane can yield many different PTRs as choosing different quadrangles as the reference points $\mathbf{O}, \mathbf{x}, \mathbf{y}$ and $\mathbf{I}$, may yield very different PTRs. We discuss this further in Section 4.

In this article, the relevance of Lemma 1 stems from our desire to use finite planes to construct infinite planes. To this end, we shall be particularly interested in closures of fields, and more generally, closures of sets of fields.

**Definition 2.**

*(i) Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two fields of the same characteristic. The* closure *of $\mathcal{K}_1$ and $\mathcal{K}_2$, which we shall denote by $\overline{\mathcal{K}_1 \cup \mathcal{K}_2}$, is the intersection of all fields containing both $\mathcal{K}_1$ and $\mathcal{K}_2$.*

*(ii) Let $S$ be a set of fields of the same characteristic. The* closure *of $S$, denoted $\overline{S}$, is the intersection of all fields containing all of the fields in $S$.*

*(iii) A set $S$ of finite fields of the same characteristic is* closed *if whenever $\mathcal{K}_1, \mathcal{K}_2 \in S$, then $\overline{\mathcal{K}_1 \cup \mathcal{K}_2} \in S$.*

There are several points we should note. Clearly, if $\mathcal{K}_1$ and $\mathcal{K}_2$ have orders $p^n$ and $p^m$, then $\overline{\mathcal{K}_1 \cup \mathcal{K}_2}$ is the finite field of order $p^l$, with $l = \mathrm{lcm}(m,n)$. Additionally, if $S$ is of infinite order, so too is $\overline{S}$. Note also that if $S$ is the set of all finite extensions of some finite field $\mathbb{F}_q$, then $\overline{S}$ coincides with the algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$.

Our main result can be stated in the following form.

**Theorem 3.** *Let $\mathcal{A}$ be an affine plane of prime power order $q$ and $T \in \mathbb{F}_q[X,Y,Z]$ be a PTR representing $\mathcal{A}$. Let $E$ be the set of all finite extensions $\mathcal{K}$ of $\mathbb{F}_q$ for which $T(X,Y,Z)$ is a PTR over $\mathcal{K}$.*

*If $S \subseteq E$ is closed, then $T$ is a PTR on $\overline{S}$ and consequently can be used to construct an affine plane, denoted $\overline{\mathcal{A}_S}$ (or just $\overline{\mathcal{A}}$) and called the* closure plane *of $\mathcal{A}$ relative to $S$, whose order is equal to the order of $\overline{S}$.*

*This statement remains true if one replaces PTR everywhere with any of the terms weak PTR, weak PTR with zero, or weak PTR with unity.*

In the following section we give a short proof of Theorem 3. An immediate question is what types of closure planes can be obtained over the algebraic closure of $\mathbb{F}_q$. In Section 3 we prove the following classical-looking result.

**Theorem 4.** *Let $\mathcal{A}$ be an affine plane of order $q$ represented by a weak PTR with zero $T \in \mathbb{F}_q[X,Y,Z]$.*

*If $T$ is a weak PTR with zero over $\overline{\mathbb{F}_q}$, then $\mathcal{A}$ and $\overline{\mathcal{A}}$ are Desarguesian.*

For a more exact statement, see Theorem 8. We then consider the Lenz-Barlotti type of the plane $\overline{\mathcal{A}_S}$ in Section 4. The article ends with a section on examples, followed by some final comments.

## § 2. Proof of main result

We now move to establish Theorem 3; the proof relies on the following easy observation:

> Let $S$ be a closed set of fields containing some base field $\mathcal{F}$. If a polynomial defined over $\mathcal{F}$ exhibits some functional property over all fields $\mathcal{K} \in S$, then it must exhibit that same property on $\overline{S}$.

Suppose, then, that the conditions of Theorem 3 are met, with $T$ a PTR representing $\mathcal{A}$, and let $S$ be any closed subset of $E$. We need to prove that $T \in \mathbb{F}_q[X,Y,Z]$ is a PTR on $\overline{S}$. Note, first, that if $S$ is a finite set, then $\overline{S} \in S$, and so $T(X,Y,Z)$ is a PTR on $\overline{S}$ by assumption.

Now suppose that $S$ is infinite. If $T$ is a PTR on $\overline{S}$, then the order of the closure plane $\overline{\mathcal{A}_S}$ is clear by construction, so we need only establish that $T$ satisfies the properties of Lemma 1 on $\overline{S}$. Let us prove $T(X,Y,Z)$ satisfies Property (e) on $\overline{S}$. Choose any $a,b,c,d \in \overline{S}$ with $a \neq c$. Then there exists a finite field $\mathcal{K} \in S$ with $a,b,c,d \in \mathcal{K}$. Since $T(X,Y,Z)$ is a PTR over $\mathcal{K}$ by assumption, there exists a unique pair $(y,z) \in \mathcal{K} \times \mathcal{K} \subset \overline{S} \times \overline{S}$ for which $T(a,y,z) = b$ and $T(c,y,z) = d$. Suppose there is another pair $(y',z') \in \overline{S} \times \overline{S}$ which satisfies $T(a,y',z') = b$ and $T(c,y',z') = d$. Again, there must exist some field $\mathcal{K}' \in S$ where $a,b,c,d,y,z,y',z' \in \mathcal{K}'$. Since $T$ is a PTR on $\mathcal{K}'$, $T$ satisfies Property (e) on $\mathcal{K}'$ and so $y = y'$ and $z = z'$. Hence there is a unique pair $(y,z) \in \overline{S} \times \overline{S}$ satisfying $T(a,y,z) = b$ and $T(c,y,z) = d$, and so $T$ satisfies Property (e) on $\overline{S}$.

Similar arguments can be given to prove $T(X,Y,Z)$ satisfies any of the remaining properties from Lemma 1 on $\overline{S}$ provided it satisfies them on all $\mathcal{K} \in S$, and so $T(X,Y,Z)$ is a PTR on $\overline{S}$. This proves Theorem 3 in full.

## § 3. Closure planes on the algebraic closure of $\mathbb{F}_q$

In this section, we resolve one obvious question concerning our construction: What planes can we construct over the algebraic closure? We shall prove that in such cases we can only obtain the classical plane. Our proof relies on a classical result of Leonard Carlitz.

Carlitz' result concerns the form of a polynomial which permutes every finite extension of a finite field; he classified such polynomials in [4].

**Lemma 5** ([4], Theorem 4)**.** *A polynomial $f \in \mathbb{F}_q[X]$ is a permutation polynomial of all finite extensions of $\mathbb{F}_q$ if and only if it is of the form $f(X) = aX^{p^h} + b$, where $a \neq 0$, $p$ is the characteristic of $\mathbb{F}_q$, and $h$ is a non-negative integer.*

In a recent article [5], the first author initiated the study of how Properties (a) through (e) can impact the form of the resulting PTR polynomial $T \in \mathbb{F}_q[X,Y,Z]$. In particular, the following two statements will prove useful.

**Lemma 6** ([5], Theorem 6)**.** *Suppose $T \in \mathbb{F}_q[X,Y,Z]$ satisfies Property (a). Then*

$$T(X,Y,Z) = Z + XYZ\, M_1(X,Y,Z) + M_2(X,Y), \tag{1}$$

*with $XYZ\, M_1(X,Y,Z)$ and $M_2(X,Y)$ reduced polynomials and where $XY | M_2(X,Y)$. In particular,*

$$x \odot y = T(x,y,0) = M_2(x,y)$$

*for all $x,y \in \mathbb{F}_q$.*

**Lemma 7** ([5], Theorem 7)**.** *Let $T \in \mathbb{F}_q[X,Y,Z]$. The following statements hold.*

*(i) Suppose $T$ satsfies Properties (a) and (c). Then $T(X,y,z)$ is a PP in $X$ for every choice of $(y,z) \in \mathbb{F}_q^\star \times \mathbb{F}_q$.*

*(ii) Suppose $T$ satsfies Properties (a) and (e). Then $T(x, Y, z)$ is a PP in $Y$ for every choice of $(x, z) \in \mathbb{F}_q^\star \times \mathbb{F}_q$.*

*(iii) Suppose $T$ satisfies Property (d). Then $T(x, y, Z)$ is a PP in $Z$ for every choice of $(x, y) \in \mathbb{F}_q \times \mathbb{F}_q$.*

We now use the above results to classify closure planes obtained from the algebraic closure of $\mathbb{F}_q$.

**Theorem 8.** *Fix a prime $p$ and set $q = p^e$ for some $e \in \mathbb{N}$. Let $S$ be the set of all finite extensions of $\mathbb{F}_q$. Suppose $\mathcal{A}$ is an affine plane of order $q$ for which $\overline{\mathcal{A}_S}$ exists for some weak PTR with zero, $T \in \mathbb{F}_q[X, Y, Z]$, that represents $\mathcal{A}$. Then $\mathcal{A}$ and $\overline{\mathcal{A}_S}$ are Desarguesian, and $T(X, Y, Z) = Z + wX^{p^m}Y^{p^n}$ for some $w \in \mathbb{F}_q^\star$ and non-negative integers $m, n$.*

*Proof.* By assumption, $T \in \mathbb{F}_q[X, Y, Z]$ satisfies Properties (a), (c), (d) and (e) of Lemma 1 over any finite extension $\mathcal{K}$ of $\mathbb{F}_q$. We first prove the statement under the assumption that $T$ is reduced over $\mathbb{F}_q$. Since $T$ satisfies Property (a), by Lemma 6 we have

$$T(X, Y, Z) = Z + XYZ\, M_1(X, Y, Z) + M_2(X, Y), \tag{2}$$

for reduced $M_1, M_2 \in \mathbb{F}_q[X, Y, Z]$.

Now choose any finite extension $\mathcal{K}$ of $\mathbb{F}_q$. Fixing $x, y \in \mathbb{F}_q$, Lemma 7 (iii) tells us $T(x, y, Z)$ is a PP over $\mathcal{K}$ for all finite extensions of $\mathbb{F}_q$. Combining Lemma 5 with (2), we find $T(X, Y, Z)$ is necessarily linear in $Z$, and we can write $T(X, Y, Z)$ as

$$T(X, Y, Z) = Z + XY(ZM_1'(X, Y) + M_2'(X, Y)),$$

for reduced $M_1', M_2' \in \mathbb{F}_q[X, Y]$.

Next, by Lemma 7 (ii), $T(x, Y, z)$ is a PP over every finite extension of $\mathbb{F}_q$ for all $x, z \in \mathbb{F}_q$, $x \neq 0$. Again, by appealing to Lemma 5, we find only one power of $Y$ can occur in $T(X, Y, Z)$, and this is necessarily of the form $Y^{p^n}$ for some non-negative integer $n < e$. So

$$T(X, Y, Z) = Z + Y^{p^n}(Zf(X) + h(X)),$$

for reduced $f, h \in \mathbb{F}_q[X]$. Note $f(0) = h(0) = 0$ by Property (a). Suppose $f(X) \neq 0$. Then there exists some $x \in \mathbb{F}_q^\star$ for which $f(x) = c \neq 0$. Setting $y = -c^{-p^{e-n}}$, we find $T(x, y, Z) = Z - Z + h(x) = h(x)$, which is not a PP, contradicting Lemma 7 (iii). Thus $f(X) = 0$ is forced. Hence $T(X, Y, Z) = Z + h(X)Y^{p^n}$.

Lemma 7 (i) now tells us $T(X, 1, 0) = h(X)$ is a PP over every finite extension $\mathcal{K}$ of $\mathbb{F}_q$. Carlitz' result now implies $h(X) = wX^{p^m}$ with $w \in \mathbb{F}_q[X]$ and $m < e$ some non-negative integer.

We now know $T(X, Y, Z) = Z + wX^{p^m}Y^{p^n}$. The plane can now by recoordinatised by the PTR $T'(x, y, z) = T(x^{p^{-m}}, y^{p^{-n}}, z) = z + wxy$, which defines a Desarguesian plane over every finite extension of $\mathbb{F}_q$ and so $\overline{\mathcal{A}_S}$ contains a finite Desarguesian subplane of order $q^i$ for every $i \in \mathbb{N}$. In particular, $\mathcal{A}$ is Desarguesian.

This also proves that $\overline{\mathcal{A}_S}$ is Desarguesian, for suppose that the plane is not Desarguesian. Then without loss of generality, there must be two triangles in perspective axially which are not in perspective centrally. However, these two triangles must have coordinates lying in some finite extension $\mathcal{K}$ of $\mathbb{F}_q$, and by the above we know that Desargues' Theorem holds for $\mathcal{A}_\mathcal{K}$, so that the two triangles must be in perspective centrally, a contradiction. Hence $\overline{\mathcal{A}_S}$ is Desarguesian.

To extend the result to non-reduced $T$, we note that since $S$ is infinite, there must be some smallest field $\mathcal{K} \in S$ with $|\mathcal{K}|$ greater than the degree of $T$, and we can view $T$ as reduced over $\mathcal{K}$. The argument above on the form of $T$ could then be made over $\mathcal{K}$ instead, proving $T(X, Y, Z) = Z + wX^{p^m}Y^{p^n}$ as before. Since $T \in \mathbb{F}_q[X, Y, Z]$, the remainder of the argument still holds, and so we arrive at the same conclusions. □

Part of the above proof yields the following corollary.

**Corollary 9.** *Let $T \in \mathbb{F}_q[X,Y,Z]$ be a weak PTR with zero representing the Desarguesian plane of order $q$ and let $E$ be the set of all finite extensions of $\mathbb{F}_q$ for which $T$ is a weak PTR representing the Desarguesian plane. Then for any closed subset $S$ of $E$, $\overline{\mathcal{A}_S}$ is Desarguesian.*

That infinite examples of such planes exist is clear: Choose any two exceptional linearised polynomials $L, M \in \mathbb{F}_q[X]$, and set $T(X,Y,Z) = Z + L(X)M(Y)$. Then $T$ is a weak PTR with zero over $\mathbb{F}_q$, and potentially over infinitely many extensions of $\mathbb{F}_q$. If $L$ and $M$ were chosen so that they were both PPs over all odd extensions of $\mathbb{F}_q$, say, then $E$ would be the set of all odd natural numbers, and this set contains infinitely many distinct closed subsets. We discuss more examples in Section 5.

## § 4. On the Lenz-Barlotti type of closure planes

We now want to consider the Lenz-Barlotti classification of projective planes when restricted to closure planes. To this end, we give a brief introduction to the classification method.

Let $\mathcal{P}$ be a projective plane and denote the full collineation group of $\mathcal{P}$ by $\Gamma$. A *central collineation* is any collineation in $\Gamma$ that fixes a line $\mathcal{L}$ pointwise and a point $\mathbf{p}$ linewise. In such case, $\mathcal{L}$ and $\mathbf{p}$ are called the *axis* and *center* of the collineation, respectively. It is well known that the center and axis of a central collineation is necessarily unique. Denote by $\Gamma(\mathbf{p}, \mathcal{L})$ be the subgroup of $\Gamma$ consisting of all central collineations of $\mathcal{P}$ with center $\mathbf{p}$ and axis $\mathcal{L}$. The plane $\mathcal{P}$ is said to be $(\mathbf{p}, \mathcal{L})$*-transitive* if for every two distinct points $\mathbf{q}, \mathbf{r}$ that are (a) collinear with $\mathbf{p}$ but not equal to $\mathbf{p}$, and (b) not on $\mathcal{L}$, there exists a necessarily unique collineation $\gamma \in \Gamma(\mathbf{p}, \mathcal{L})$ which maps $\mathbf{q}$ to $\mathbf{r}$. Now let $\mathcal{M}$ be a second line of $\mathcal{P}$, not necessarily distinct from $\mathcal{L}$. If $\mathcal{P}$ is $(\mathbf{p}, \mathcal{L})$-transitive for all $\mathbf{p} \in \mathcal{M}$, then $\mathcal{P}$ is said to be $(\mathcal{M}, \mathcal{L})$*-transitive*; the concept of $(\mathbf{p}, \mathbf{q})$*-transitivity* is defined dually. If $\mathcal{P}$ is $(\mathcal{L}, \mathcal{L})$-transitive, then $\mathcal{L}$ is called a *translation line* and $\mathcal{P}$ is called a *translation plane* with respect to the line $\mathcal{L}$. The definitions of *translation point* and *dual translation plane* are defined dually also.

The Lenz-Barlotti (LB) classification for projective planes was developed by Lenz [15] and refined by Barlotti [3], and is based on the possible sets

$$\mathcal{T} = \{(\mathbf{p}, \mathcal{L}) \,:\, \mathcal{P} \text{ is } (\mathbf{p}, \mathcal{L})\text{-transitive}\}$$

of point-line transitivities that $\Gamma$ can exhibit. It has a heirarchy of types, starting with little to no point-line transitivities in types I and II, through to type VII.2, which represents the Desarguesian plane and where $\mathcal{T}$ consists of every possible point-line flag. The question of existence of finite projective planes of LB types I.2, I.3, I.4 and II.2 remains unresolved. For an affine plane $\mathcal{A}$, one can talk of its LB type by examining $\mathcal{T}$ for the projective closure $\mathcal{P}$ of $\mathcal{A}$ – here $\mathcal{A} = \mathcal{P}^{[\infty]}$.

There is also a corresponding structural heirarchy on the algebraic properties of possible PTRs representing the planes as one ascends through the LB types, provided the coordinatisation is done in an optimal manner. The way in which this can be done was made explicit in [5]. In particular, when viewing the PTRs as polynomials over finite fields (as we do in this paper), we have the following result.

**Lemma 10** ([5], Theorem 16)**.** *Let $\mathcal{A}$ be a projective plane of order $q = p^e$ for some prime $p$ which is $((\infty), [\infty])$-transitive and where $\Gamma((\infty), [\infty])$ is elementary abelian. Suppose $T \in \mathbb{F}_q[X,Y,Z]$ is a PTR polynomial obtained from coordinatising $\mathcal{A}$ optimally, so that the resulting additive loop is field addition.*

*(i) If $\mathcal{A}$ is strictly LB type II.1, then*

$$T(X,Y,Z) = M_2(X,Y) + Z, \tag{3}$$

*where $M_2(X,Y)$ is as in (1).*

*(ii) If $\mathcal{A}$ is strictly LB type II.2, then $T \in \mathbb{F}_q[X,Y,Z]$ is of the shape (3) and where*

$$M_2(x, M_2(y,z)) = M_2(M_2(x,y), z)$$

*for all $x, y, z \in \mathbb{F}_q$.*

*(iii) If $\mathcal{A}$ is a translation plane of LB type at least IV, then $T \in \mathbb{F}_q[X,Y,Z]$ is of the shape (3) and where*

$$M_2(X,Y) = \sum_{i=0}^{e-1} \sum_{j=1}^{q-1} c_{ij} X^{p^i} Y^j. \tag{4}$$

*(iv) If $\mathcal{A}$ is a dual translation plane of LB type at least IV, then $T \in \mathbb{F}_q[X,Y,Z]$ is of the shape (3) and where*

$$M_2(X,Y) = \sum_{i=1}^{q-1} \sum_{e=0}^{e-1} c_{ij} X^{i} Y^{p^j}. \tag{5}$$

*(v) If $\mathcal{A}$ is LB type at least V, then $T \in \mathbb{F}_q[X,Y,Z]$ is of the shape (3) and where*

$$M_2(X,Y) = \sum_{i=0}^{e-1} \sum_{j=0}^{e-1} c_{ij} X^{p^i} Y^{p^j}. \tag{6}$$

We need one further result concerning subplanes. Recall that a *subplane* of a projective (resp. affine) plane is a subset $\mathcal{S}$ of points and lines which is itself a projective (resp. affine) plane, relative to the incidence relation defining the plane. The following result is a consequence of a result of Lüneburg, [16].

**Lemma 11** ([16], Hilfssatz (page 446–7))**.** *Let $\mathcal{P}$ be a projective plane and $\mathcal{S}$ be a projective subplane of $\mathcal{P}$. Choose any point $\mathbf{p} \in \mathcal{S}$ and line $\mathcal{L}$ of $\mathcal{P}$ intersecting $\mathcal{S}$ non-trivially. If $\mathcal{P}$ is $(\mathbf{p},\mathcal{L})$-transitive, then $\mathcal{S}$ is $(\mathbf{p},\mathcal{L})$-transitive.*

If we restrict the lemma to affine planes, so that the line at infinity is made explicit, then we obtain the following.

**Corollary 12.** *Let $\mathcal{A}$ be an affine plane and let $\mathcal{S}$ be any subplane of $\mathcal{A}$ that contains the triangle $\triangle(0,0)(0)(\infty)$. Then the specific Lenz-Barlotti type of $\mathcal{S}$ is greater than or equal to the specific Lenz-Barlotti type of $\mathcal{A}$. That is, $\mathcal{T}_{\mathcal{A}} \subseteq \mathcal{T}_{\mathcal{S}}$.*

In terms of the coordinatisation, the corollary tells us that the algebraic structure of the PTR of a plane $\mathcal{A}$ is at least as rich when the PTR is restricted to a subplane $\mathcal{S}$ of $\mathcal{A}$.

We now use the above statements to restrict the possible LB types of closure planes.

**Theorem 13.** *Let $\mathcal{A}$ be a plane of order $q = p^e$ for some prime $p$ which is $((\infty),[\infty])$-transitive and where $\Gamma((\infty),[\infty])$ is elementary abelian. Suppose $T \in \mathbb{F}_q[X,Y,Z]$ is a PTR polynomial obtained from coordinatising $\mathcal{A}$ optimally, so that the resulting additive loop is field addition. Let $E$ be the set of all finite extensions $\mathcal{K}$ of $\mathbb{F}_q$ for which $T(X,Y,Z)$ is a PTR over $\mathcal{K}$ and let $S \subseteq E$ be any closed set. The following statements hold.*

*(i) If $\mathcal{A}$ is strictly LB type II.1, then $\overline{\mathcal{A}}$ is LB type II.1.*

*(ii) If $\mathcal{A}$ is strictly LB type II.2, then $\overline{\mathcal{A}}$ is LB type II.1 or II.2.*

*(iii) (Quasifield case) If $\mathcal{A}$ is LB type IV, then $\overline{\mathcal{A}}$ is the same LB type IV as $\mathcal{A}$.*

*(iv) (Semifield case) If $\mathcal{A}$ is LB type V.1, then $\overline{\mathcal{A}}$ is LB type V.1.*

*(v) (Desarguesian case) If $\mathcal{A}$ is LB type VII.2, then $\overline{\mathcal{A}}$ is LB type V.1 or VII.2.*

*Proof.* Let $\mathcal{A}$ be a LB type II plane with $\Gamma((\infty),[\infty])$ elementary abelian. By Lemma 10, we know $T(X,Y,Z) = M(X,Y) + Z$. Choose a closed set $S \subset E$ and consider $T$ over $\overline{S}$. Clearly $T$ is a linear PTR over $\overline{S}$. Furthermore, $\oplus$ remains field addition, and so we get $\Gamma((\infty),[\infty])$ is elementary abelian of the same cardinality as $\overline{S}$, which means $\overline{\mathcal{A}}$ is $((\infty),[\infty])$-transitive. As $\mathcal{A}$ is assumed to not be a LB type IV plane or more, we know $T$ cannot be linearised in $X$ or $Y$, and so $\overline{\mathcal{A}}$ cannot be LB type IV or more.

We need first to show that $\overline{\mathcal{A}}$ cannot be LB type III. Suppose it were. Then $\overline{\mathcal{A}}$ would be $(\mathbf{q}, \mathbf{p}\,\mathbf{q})$-transitive for all $\mathbf{q} \in \mathcal{L}$ with $(\mathbf{p}, \mathcal{L})$ a non-incident point-line flag. If $\mathcal{L} = [\infty]$, then it intersects every subplane $\mathcal{A}_{\mathcal{K}}$, $\mathcal{K} \in S$, non-trivially by construction. Likewise, if $\mathbf{p} = (\infty)$, then it lies in every subplane $\mathcal{A}_K$, $\mathcal{K} \in S$. If either $\mathcal{L} \neq [\infty]$ or $\mathbf{p} \neq (\infty)$, then $\mathcal{L}$ or $\mathbf{p}$ must have coordinates in some $\mathcal{K} \in S$. Consequently, regardless of what the point-line flag $(\mathbf{p}, \mathcal{L})$ is, there must exist some $\mathcal{K} \in S$ for which $\mathbf{p}$ is in the closure of $\mathcal{A}_{\mathcal{K}}$ and $\mathcal{L}$ intersects $\mathcal{A}_{\mathcal{K}}$ non-trivially. By Lemma 11, $\mathcal{A}_{\mathcal{K}}$ must be a finite LB type III plane, which contradicts the fact no finite LB type III planes can exist. Hence $\overline{\mathcal{A}}$ can be LB type II.1 or II.2. If $\mathcal{A}$ was LB type II.1, then the multiplication defined by the PTR would be non-associative over $\mathbb{F}_q$. Consequently, $\overline{\mathcal{A}}$ cannot be LB type II.2. This completes the proof of the first two parts.

Now suppose $\mathcal{A}$ is LB type IV or V.1. Then over every extension $\mathcal{K}$ of $\mathbb{F}_q$, including $\overline{S}$, the PTR $T$ will display precisely the same distributive laws as it has over $\mathbb{F}_q$. Consequently, if $\mathcal{A}$ is a LB type IV plane, then $\overline{\mathcal{A}}$ must also be the same type of LB type IV plane. If $\mathcal{A}$ is LB type V.1, then it's possible $\overline{\mathcal{A}}$ is LB type V.1 or VII.2. Now every subplane of a Desarguesian plane is Desarguesian, so since $\mathcal{A}$ is a subplane of $\overline{\mathcal{A}}$, $\overline{\mathcal{A}}$ cannot possibly be Desarguesian. Similarly, if $\overline{\mathcal{A}}$ were a LB VII.1 plane, then it would be flag transitive for every incident point-line flag. Lemma 11 now implies every subplane of $\overline{\mathcal{A}}$ is at least type VII.1 also, a contradiction. Thus $\overline{\mathcal{A}}$ must be LB type V.1 if $\mathcal{A}$ is.

Finally, suppose $\mathcal{A}$ is Desarguesian. In a Desarguesian plane, every point is a translation point and every line is a translation line. So, regardless of which two points are chosen for $\mathbf{x}$ and $\mathbf{y}$, the conditions of Lemma 10 (iv) must be met, and so $T(X,Y,Z) = M(X,Y) + Z$ with $M$ linearised in both $X$ and $Y$. Thus the PTR will display both distributive laws over $\overline{S}$, so that $\overline{\mathcal{A}}$ must be LB type V.1, VII.1 or VII.2. However, the above argument that eliminated type VII.1 holds here also, and the proof is complete. □

For Desarguesian planes, the statement is best possible. In the following section, we give an example of a Desarguesian plane of order 27 which can be represented by a weak PTR that produces infinitely many infinite planes of LB type V.

We end the section with a result concerning when two closure planes are isomorphic.

**Theorem 14.** *Let $\mathcal{A}$ be an affine plane of prime power order $q$ and $T \in \mathbb{F}_q[X,Y,Z]$ be a PTR obtained from $\mathcal{A}$ via coordinatisation. Let $E$ be the set of all finite extensions $\mathcal{K}$ of $\mathbb{F}_q$ for which $T(X,Y,Z)$ is a PTR over $\mathcal{K}$. Let $S_1$ and $S_2$ be two closed subsets of $E$. Then $\overline{\mathcal{A}_{S_1}}$ and $\overline{\mathcal{A}_{S_2}}$ are isomorphic if and only if $S_1 = S_2$.*

*Proof.* If $S_1 = S_2$, there is nothing to prove. If $S_1 \neq S_2$, then $\overline{S_1} \neq \overline{S_2}$ (because they are assumed to be closed). In particular, and without loss of generality, there must exist a $\mathcal{K} \in S_1 \setminus S_2$. Consequently, $\overline{\mathcal{A}_{S_1}}$ must contain a subplane $\mathcal{A}_{\mathcal{K}}$ which $\overline{\mathcal{A}_{S_2}}$ does not, and so the two closure planes cannot be isomorphic. □

For more on the isomorphism issue, see Section 5.1.1, and specifically the discussion of Example 2.

## § 5. Examples

In this last section, we provide various examples of closure planes. These examples are not intended as exhaustive in any sense. In the main, we shall rely on planar functions, as they formed the original motivation for the closure plane construction, though we also provide an example which cannot be obtained from a planar function. As shall be seen, we construct infinite numbers of non-isomorphic planes of infinite order and Lenz-Barlotti type II.1, V.1 or VII.2. It remains an open question as to whether or not it is possible to construct infinite closure planes of LB type IV, though we openly admit we have made no attempt to trawl through the immense number of examples of translation planes to find a suitable base plane.

### § 5.1. Examples from planar functions

Planar functions were introduced by Dembowski and Ostrom [8] to describe affine planes with specific collineation groups. In their original definition, it can be shown they do not exist over any group containing an involution; in particular, they do not exist over fields of even characteristic. However, a variant of the definition was recently given by Zhou [22] specifically for fields of even characteristic. Consequently, when talking about planar functions over finite fields, there are distinct definitions based on the characteristic being even or odd.

### 5.1.1 In odd characteristic

Throughout this section, we assume $q$ is an odd prime power.

**Definition 15.** *Let $q$ be an odd prime power. A polynomial $f \in \mathbb{F}_q[X]$ is* planar *over $\mathbb{F}_q$ if the difference operator $\Delta_f(X,a) = f(X+a) - f(X) - f(a)$ is a PP over $\mathbb{F}_q$ for all $a \in \mathbb{F}_q^\star$.*

Let $f \in \mathbb{F}_q[X]$ be a planar polynomial over $\mathbb{F}_q$ with $f(0) = 0$. This can be assumed without loss of generality, as the addition of a constant does not affect the planarity of a function, nor the type of plane defined by it. Results from [6] show that the polynomial $T = T_f \in \mathbb{F}_q[X,Y,Z]$ given by $T(X,Y,Z) = Z + \Delta_f(X,Y)$ is a weak PTR with zero over $\mathbb{F}_q$ which generates an affine plane $\mathcal{A}_T$ isomorphic to that generated by the planar polynomial $f(X)$. Thus, for every planar polynomial, we have a corresponding weak PTR with zero.

There are several well-known classes of planar polynomials; we shall concentrate on only three examples.

1. The polynomial $f(X) = X^{p^\alpha+1}$ is planar over $\mathbb{F}_{p^e}$ if and only if $e/\gcd(\alpha,e)$ is odd, see [7], Theorem 3.3. Further, results from [7] and [6] show $f(X)$ describes the Desarguesian plane if $\alpha \equiv 0 \bmod e$, and otherwise describes a Lenz-Barlotti type V.1 plane.

   Fix $\alpha$, and set $E_\alpha$ to be the set of all finite extensions $\mathcal{K}$ of $\mathbb{F}_p$ for which $f(X)$ is planar over $\mathcal{K}$. By definition, the corresponding $T_f \in \mathbb{F}_p[X,Y,Z]$ is a PTR over every $\mathcal{K} \in E_\alpha$. It is immediate from the planarity condition for $f(X)$ that $E_\alpha$ is infinite. Moreover, there are infinitely many infinite closed subsets $S$ of $E_\alpha$. Note, also, that $E_\alpha$ is itself closed.

   When $\alpha = 0$, $E_\alpha$ contains all finite extensions of $\mathbb{F}_q$, and so Theorem 8 shows $\mathcal{A}_T$ is Desarguesian, as is $\overline{\mathcal{A}_S}$ for any closed subset $S$ of $E_\alpha$.

   Now suppose $\alpha > 0$ and that the largest power of 2 dividing $\alpha$ is $2^a$. Then $E_\alpha = \{\mathbb{F}_{p^e} : e = 2^a m, m \in \mathbb{N} \text{ odd}\}$. Choose any sequence of pairwise-relatively prime odd prime powers $q_1, q_2, \dots$ and set

   $$S = \{\mathbb{F}_{p^e} : e = 2^a \prod_{i=1} q_i^{e_i}, \text{ where only finitely many } e_i \text{ are non-zero}\}.$$

   Clearly $S$ is an infinite closed subset of $E_\alpha$, and so we obtain the closure plane $\overline{\mathcal{A}}$ relative to $S$. Moreover, by the above categorisation of the possible Lenz-Barlotti types generated by $f(X)$, we see that $\overline{\mathcal{A}}$ is necessarily Lenz-Barlotti type V.1, unless the sequence of prime powers has finite length $k$ and $2^a \prod_{i=1}^k q_i$ divides $\alpha$, in which case we again obtain an infinite Desarguesian plane.

2. The polynomial $f_a(X) = X^{10} + aX^6 - a^2X^2 \in \mathbb{F}_{3^e}[X]$ with $a \neq 0$ is planar over $\mathbb{F}_{3^e}$ if $e$ is odd or $e = 2$ and $a = \pm 1$, see [7] and [9]. Let $q = 3^e$ for some odd $e$ and $a \in \mathbb{F}_q^\star$. Set $E$ to be the set of all odd extensions of $\mathbb{F}_q$. Then $f_a(X)$ is planar over $\mathcal{K}$ for every $\mathcal{K} \in E$, and necessarily defines a Lenz-Barlotti type V.1 plane unless $\mathcal{K} = \mathbb{F}_3$ (so $e = 1$), or $\mathbb{F}_{3^3}$ (so $e \in \{1,3\}$) and $a \in \mathbb{F}_q^\star$ is a non-square, in which case we get the Desarguesian plane, see [6]. Clearly, $E$ is infinite, is itself closed, and contains infinitely many infinite closed subsets. Thus we can construct infinitely many infinite closure planes from any $f_a(X)$, all of which will necessarily be Lenz-Barlotti type V.1. Note in particular, that if we start with $a \in \mathbb{F}_{27}$ a non-square, then the weak PTR with zero given from $T(X,Y,Z) = f_a(X+Y) - f_a(X) - f_a(Y) + Z$ is a reduced polynomial over $\mathbb{F}_{27}$ that defines a Desarguesian plane $\mathcal{A}$ over $\mathbb{F}_{27}$ and for which all other $\mathcal{K} \in E$ describes a LB type V plane over $\mathcal{K}$. Thus for any closed subset $S$ of $E$ with $S \neq \{\mathbb{F}_{27}\}$ we get $\overline{\mathcal{A}_S}$ is LB type V, while $\mathcal{A}$ is Desarguesian; see Theorem 13.

   There is a further interesting point regarding this example. The set $E$ is the same for $a = 1$ or $a = -1$. Set $T_+$ and $T_-$ to be the weak PTRs representing these two cases, respectively. Over

$\mathbb{F}_3$, both $T_+$ and $T_-$ represent the Desarguesian plane. However, for any closed subset $S$ of $E$ with $\overline{S} \neq \mathbb{F}_3$, $T_+$ and $T_-$ yield non-isomorphic closure planes. Thus it is entirely possible to have two distinct weak PTRs with zero which represent the same base plane but which yield non-isomorphic closure planes for some closed set $S$ applicable to both PTRs.

3. The polynomial $f(X) = X^{(3^{\alpha+1}+1)/2}$ is planar over $\mathbb{F}_{3^e}$ if and only if $\gcd(\alpha, 2e) = 1$, see [7], Theorem 4.1. Further, results from [7] and by J.C.D.S. Yaqub (private correspondence, see the added in proof section of [7], and also [20]) show $f(X)$ describes the Desarguesian plane if $\alpha \equiv \pm 1 \bmod 2e$, and otherwise describes a Lenz-Barlotti type II.1 plane.

   As with our first planar example, fix $\alpha$, and set $E_\alpha$ to be the set of all finite extensions $\mathcal{K}$ of $\mathbb{F}_p$ for which $f(X)$ is planar over $\mathcal{K}$, so that the corresponding $T_f \in \mathbb{F}_p[X, Y, Z]$ is a weak PTR with zero over every $\mathcal{K} \in E_\alpha$. It is immediate from the planarity condition for $f(X)$ that $E_\alpha$ is infinite. Moreover, there are infinitely many infinite closed subsets $S$ of $E$. Any closed subset $S$ for which $|\overline{S}| > \frac{3^\alpha+1}{2}$ will now yield a closure plane $\overline{\mathcal{A}_S}$ of Lenz-Barlotti type II.1.

### 5.1.2 In even characteristic

Throughout this section, we assume $q$ is a power of 2. As mentioned previously, the classical definition of planar function does not allow examples in characteristic two. However, Zhou [22] gave the following variant of the definition for characteristic two.

**Definition 16.** *Let $q$ be a power of two. A polynomial $f \in \mathbb{F}_q[X]$ is* planar *over $\mathbb{F}_q$ if the polynomial $f(X+a) + f(X) + aX$ is a PP over $\mathbb{F}_q$ for all $a \in \mathbb{F}_q^\star$.*

Zhou also provided in [22] a form for the PTR in this case, though not in a true polynomial form. In fact, the form given there, see (7) or (14) of [22], is dependent on the extension of $\mathbb{F}_2$ under consideration. It is easily verified that a weak PTR with zero for a planar polynomial over $\mathbb{F}_{2^e}$ is given by

$$T(X, Y, Z) = Z + f(X+Y) + f(X) + f(Y) + XY. \tag{7}$$

Schmidt and Zhou [21] and then Müller and Zieve [18] consider monomials over $\mathbb{F}_{2^e}$ that are planar over infinitely many finite extensions of $\mathbb{F}_{2^e}$. The latter pair prove the following.

**Lemma 17** (Müller & Zieve, [18], Theorem 1)**.** *Let $t$ be a positive integer such that $t^4 \leq 2^e$, and let $a \in \mathbb{F}_{2^e}^\star$. The monomial $aX^t$ is planar over $\mathbb{F}_{2^e}$ if and only if $t$ is a power of 2.*

Clearly, one can use this result of Müller and Zieve to construct infinite closure planes. However, as Zhou notes in [22], every such polynomial defines a finite Desarguesian plane, and the closure planes with respect to these monomials are necessarily Desarguesian as well. At present, there are no known planar functions in characteristic two which do not define at least a Lenz-Barlotti type V plane, so any infinite closure planes arising from the known examples will also be either LB type V or Desarguesian.

## § 5.2. Other examples from Lenz-Barlotti type V

Lenz-Barlotti type V.1 planes have a translation line and a translation point. They are necessarily of order $p^e$ for some prime $p$. Set $q = p^e$. By Lemma 10, if the translation line is used as the line at infinity in the coordinatisation process, then any weak PTR with zero $T \in \mathbb{F}_q[X, Y, Z]$ obtained from such a plane will have the shape

$$T(X, Y, Z) = Z + \sum_{i,j=0}^{e-1} a_{ij} X^{p^i} Y^{p^j}.$$

LB type V planes come in two varieties: commutative or non-commutative presemifield planes. It has been shown, see [6], that any commutative semifield plane of odd order is equivalent to a planar DO polynomial.

On the other hand, any non-commutative semifield plane which is non-isomorphic to a commutative semifield plane cannot be constructed in such a fashion. A classical class of examples of presemifields are Albert's generalised twisted fields [1, 2]. Let $q = p^e$ for some prime $p$, $q > 2$, and $\phi, \psi \in Aut(\mathbb{F}_q)$. Choose $c \in \mathbb{F}_q$, and define a multiplication $\odot$ by

$$x \odot y = xy - cx^{\phi}y^{\psi}.$$

Provided $c$ is chosen so that the condition

$$x \odot y = 0 \text{ if and only if } xy = 0 \tag{8}$$

is satisfied, then $\odot$, along with field addition defines a presemifield $\mathcal{R}_q(\phi, \psi, c)$. The corresponding weak PTR with zero is given by $T(X, Y, Z) = Z + XY - cX^{\phi}Y^{\psi}$. It is easy to select $c \in \mathbb{F}_q$, $\phi$ and $\psi$ so that (8) is satisfied over $\mathbb{F}_q$ and the set of $E$ of all finite extensions of $\mathbb{F}_q$ for which (8) is satisfied is infinite. (Indeed, it may be that any selection of $c \in \mathbb{F}_q$, $\phi$ and $\psi$ for which (8) is satisfied over $\mathbb{F}_q$ leads to an infinite set $E$.) By Theorem 3, any closed subset of $E$ will define a closure plane via $T$. Since some generalised twisted field planes are not isomorphic to any commutative semifield plane, we know some of these closure planes cannot be isomorphic to closure planes obtained from planar functions.

## § 6. Some final comments

In the above examples, we rely heavily on examples of planar functions or on semifield examples closely related to semifields (the generalised twisted fields). It would be very interesting to find examples of PTR polynomials not related to planar functions that yield infinite closure planes.

The first problem one encounters when approaching this construction is that practically all known PTRs defined over finite fields have forms dependent on the particular order of the field. Two examples of this nature that immediately come to mind are the Hughes planes [12] and the Figueroa planes [10]. Both of these examples are LB type I.1 planes and consequently cannot be represented by linear PTRs. In this sense, these examples point at an additional problem of interest. Are there any non-linear PTR polynomials that can be used to produce infinite closure planes? We suspect they cannot exist, but have no supporting evidence.

Finally, in regards to Example 2, it would be interesting to know of other examples of two weak PTRs with zero having exactly the same set $E$ and producing infinitely many non-isomorphic closure planes, especially if the PTR polynomials had very similar forms like those in that example.

## References


[1] A.A. Albert, *Generalized twisted fields*, Pacific J. Math. **11** (1961), 1–8.

[2] A.A. Albert, *Isotopy for generalized twisted fields*, An. Acad. Brasil. Ci. **33** (1961), 265–275.

[3] A. Barlotti, *Le possibili configurazioni del sistema delle coppie punto-retta $(A, a)$ per cui un piano grafico risulta $(A, a)$-transitivo*, Boll. Un. Mat. Ital. **12** (1957), 212–226.

[4] L. Carlitz, *Permutations in finite fields*, Acta Sci. Math. Szeged **24** (1963), 196–203.

[5] R.S. Coulter, *On coordinatising planes of prime power order using finite fields*, preprint.

[6] R.S. Coulter and M. Henderson, *Commutative presemifields and semifields*, Adv. Math. **217** (2008), 282–304.

[7] R.S. Coulter and R.W. Matthews, *Planar functions and planes of Lenz-Barlotti class II*, Des. Codes Cryptogr. **10** (1997), 167–184.

[8] P. Dembowski and T.G. Ostrom, *Planes of order $n$ with collineation groups of order $n^2$*, Math. Z. **103** (1968), 239–258.

[9] C. Ding and J. Yuan, *A family of skew Hadamard difference sets*, J. Combin. Theory Ser. A **113** (2006), 1526–1535.

[10] R. Figueroa, *A family of not $(v, l)$-transitive projective planes of order $q^3$, $q \not\equiv 1 \bmod 3$ and $q > 2$*, Math. Z. **181** (1982), 471–479.

[11] M. Hall, *Projective planes*, Trans. Amer. Math. Soc **54** (1943), 229–277.

[12] D.R. Hughes, *A class of non-Desarguesian projective planes*, Canad. J. Math. **9** (1957), 378–388.

[13] D.R. Hughes and F.C. Piper, *Projective Planes*, Graduate Texts in Mathematics, vol. 6, Springer-Verlag, New York, Heidelberg, Berlin, 1973.

[14] D.E. Knuth, *Finite semifields and projective planes*, J. Algebra **2** (1965), 182–217.

[15] H. Lenz, *Zur Begründung der analytischen Geometrie*, S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. (1954), 17–72.

[16] H. Lüneburg, *Charakterisierungen der endlichen desarguesschen projektiven Ebenen*, Math. Z. **85** (1964), 419–450.

[17] G.L. Mullen and D. Panario, *Handbook of Finite Fields*, Discrete Mathematics and Its Applications, vol. 78, Chapman and Hall, CRC Press, 2013.

[18] P. Müller and M.E. Zieve, *Low-degree planar monomials in characteristic two*, J. Algebraic Combin. **42** (2015), 695–699.

[19] G. Pickert, *Projektive Ebenen*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXX, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

[20] D. Pierce and M.J. Kallaher, *A note on planar functions and their planes*, Bull. Inst. Combin. Appl. **42** (2004), 53–76.

[21] K-U. Schmidt and Y. Zhou, *Planar functions over fields of characteristic two*, J. Algebraic Combin. **40** (2014), 503–526.

[22] Y. Zhou, *$(2^n, 2^n, 2^n, 1)$-relative difference sets and their representations*, J. Combin. Des. **21** (2013), 563–584.