Chapter 6

Commodity Forwards and Futures

Question 6.1

The spot price of a widget is \$70.00. With a continuously compounded annual risk-free rate of 5%, we can calculate the annualized lease rates according to the formula:

$$F_{0,T} = S_0 \times e^{(r-\delta_l)\times T}$$

$$\Leftrightarrow \quad \frac{F_{0,T}}{S_0} = e^{(r-\delta_l)\times T}$$

$$\Leftrightarrow \quad \ln\left(\frac{F_{0,T}}{S_0}\right) = (r - \delta_l) \times T$$

$$\Leftrightarrow \quad \delta_l = r - \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right)$$

Time to expiration	Forward price	Annualized lease rate
3 months	\$70.70	0.0101987
6 months	\$71.41	0.0101147
9 months	\$72.13	0.0100336
12 months	\$72.86	0.0099555

The lease rate is less than the risk-free interest rate. The forward curve is upward sloping, thus the prices of exercise 6.1 are an example of contango.

Question 6.2

The spot price of oil is \$32.00 per barrel. With a continuously compounded annual risk-free rate of 2%, we can again calculate the lease rate according to the formula:

$$\delta_l = r - \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right)$$

Time to expiration	Forward price	Annualized lease rate
3 months	\$31.37	0.0995355
6 months	\$30.75	0.0996918
9 months	\$30.14	0.0998436
12 months	\$29.54	0.0999906

The lease rate is higher than the risk-free interest rate. The forward curve is downward sloping, thus the prices of exercise 6.2 are an example of backwardation.

Question 6.3

The question asks us to find the lease rate such that $F_{0,T} = S_0$. We take our pricing formula, $F_{0,T} = S_0 \times e^{(r-\delta_l)\times T}$, and immediately see that the sought equality is established if $e^{(r-\delta_l)\times T} = 1$, which is guaranteed for any $T > 0$ if and only if $r = \delta$.

If the lease rate were 3.5%, the lease rate would be higher than the risk-free interest rate. Therefore, a graph of forward prices would be downward sloping, and thus there would be backwardation.

Question 6.4

a) As we need to borrow a pound of copper to sell it short, we must pay the lender the lease rate for the time we borrow the asset, i.e., until expiration of the contract in one year. After one year, we have to pay back one pound of copper, which will cost us S_T, the uncertain future pound of copper price, plus the leasing costs: Total payment = $S_T + S_T \times e^{0.05} - 1 = S_T\, e^{0.05} = 1.05127 \times S_T$.

   It does not make sense to store a pound of copper in equilibrium, because even if we have an active lease market for pounds of copper, the lease rate is smaller than the risk-free interest rate. Lending money at 10 percent is more profitable than lending pounds of copper at 5 percent.

b) The equilibrium forward price is calculated according to our pricing formula:

$$F_{0,T} = S_0 \times e^{(r-\delta_l)\times T} = \$3.00 \times e^{(0.10-0.05)\times 1} = \$3.00 \times 1.05127 = \$3.1538,$$

   which is the price given in the exercise.

c) Copper need not be stored because there is a constant supply of copper. If the forward price was greater than 3.316, then a copper producer will find it profitable to sell forward production.

d) As the textbook points out, the reverse cash-and-carry arbitrage does not put a lower bound on the forward price. The lender will require a lease payment and can use the forward price to determine the lease rate.

Question 6.5

a) The spot price of gold is $300.00 per ounce. With a continuously compounded annual risk-free rate of 5 percent, and at a one-year forward price of 310.686, we can calculate the lease rate according to the formula:

$$\delta_l = r - \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right) = 0.05 - \ln\left(\frac{310.686}{300}\right) = 0.015$$

b) Suppose gold cannot be loaned. Then our cash and carry "arbitrage" is:

Transaction	Time 0	Time $T = 1$
Short forward	0	$310.686 - S_T$
Buy gold	−$300	S_T
Borrow @ 0.05	$300	−$315.38
Total	0	−4.6953

The forward price bears an implicit lease rate. Therefore, if we try to engage in a cash and carry arbitrage but we do not have access to the gold loan market and thus to the additional revenue on our long gold position, it is not possible for us to replicate the forward price. We incur a loss.

c) If gold can be loaned, we engage in the following cash and carry arbitrage:

Transaction	Time 0	Time $T = 1$
Short forward	0	$310.686 - S_T$
Buy tailed gold position, lend @ 0.015	−$295.5336	S_T
Borrow @ 0.05	$295.5336	−$310.686
Total	0	0

Therefore, we now just break even: Since the forward was fairly priced, taking the implicit lease rate into account, this result should not surprise us.

Question 6.6

a) The forward prices reflect a market for widgets in which seasonality is important. Let us look at two examples, one with constant demand and seasonal supply, and another one with constant supply and seasonal demand.

One possible explanation might be that widgets are extremely difficult to produce and that they need one key ingredient that is only available during July/August. However, the demand for the widget is constant throughout the year. In order to be able to sell the widgets throughout the year, widgets must be stored after production in August. The forward curve reflects the ever increasing storage costs of widgets until the next production cycle arrives. Once produced, widget prices fall again to the spot price.

Another story that is consistent with the observed prices of widgets is that widgets are in particularly high demand during the summer months. The storage of widgets may be costly, which means that widget producers are reluctant to build up inventory long before the summer. Storage occurs slowly over the winter months and inventories build up sharply just before the highest demand during the summer months. The forward prices reflect those storage cycle costs.

b) Let us take the December Year 0 forward price as a proxy for the spot price in December Year 0. We can then calculate with our cash and carry arbitrage tableau:

Transaction	Time 0	Time $T = 3/12$
Short March forward	0	$3.075 - S_T$
Buy December Forward (= Buy spot)	−3.00	S_T
Pay storage cost	−0.03	
Total	−3.00	3.045

We can calculate the annualized rate of return as:

$$\frac{3.045}{3.00} = e^{(r)\times T}$$

$$\Leftrightarrow \ln\left(\frac{3.045}{3.00}\right) = r \times 3/12$$

$$r = 0.05955$$

which is the prevailing risk-free interest rate of 0.06. This result seems to make sense.

c) Let us again take the December Year 0 forward price as a proxy for the spot price in December Year 0. We can then calculate with our cash and carry arbitrage tableau:

Transaction	Time 0	Time $T = 9/12$
Short Sep forward	0	$2.75 - S_T$
Buy spot	−3.00	S_T
Pay storage cost Sep		−0.03
FV(Storage Jun)		−0.0305
FV(Storage Mar)	−0.0309	
Total	−3.00	2.6586

We can calculate the annualized rate of return as:

$$\frac{2.6586}{3.00} = e^{(r)\times T} \Leftrightarrow \ln\left(\frac{2.6586}{3.00}\right) = r \times 9/12$$

$$r = -0.16108$$

This result does not seem to make sense. We would earn a negative annualized return of 16 percent on such a cash and carry arbitrage. Therefore, it is likely that our naive calculations do not capture an important fact about the widget market. In particular, we will buy and hold the widget through a time where the forward curve indicates that there is a significant convenience yield attached to widgets.

It is tempting, although premature, to conclude that a reverse cash and carry arbitrage may make a positive 16 percent annualized return. Question 6.7 deals with this aspect.

Question 6.7

a) Let us take the December Year 0 forward price as a proxy for the spot price in December Year 0. We can then calculate a reverse cash and carry arbitrage tableau:

Transaction	Time 0	Time $T = 3/12$
Long March forward	0	$S_T - 3.075$
Short widget	+3.00	$-S_T$
Lend money	−3.00	+3.045
Total	0	−0.03

We need to receive 0.03 as a compensation from the lender of the widget. This cost reflects the storage cost of the widget that the lender does not need to pay. The lease rate is negative.

b) Let us take the December Year 0 forward price as a proxy for the spot price in December Year 0. We can then calculate a reverse cash and carry arbitrage tableau:

Transaction	Time 0	Time $T = 9/12$
Long Sept forward	0	$S_T - 3.075$
Short widget	+3.00	$-S_T$
Lend money	−3.00	+3.13808
Total	0	0.06308

Although we free the lender of the widget of the burden to pay three times storage costs, we still need to pay him 0.06308. This reflects the fact that we hold the widget through the period in which widgets are valuable (as reflected by the forward contracts) and are returning it at a time it is worth less. The lender is only willing to do so if we compensate him for the opportunity cost.

Question 6.8

a) The first possibility is a simple cash and carry arbitrage:

Transaction	Time 0	Time $T = 3/12$
Short March forward	0	$3.10 - S_T$
Buy December forward (= Buy spot)	−3.00	S_T
Borrow @ 6%	+3.00	−3.045
Pay storage cost	−0.03	
Total	0	+0.025

The second possibility involves using the June futures contract. It is a forward cash-and-carry strategy:

Transaction	Time = $T(1) = 3/12$	Time = $T(2) = 6/12$
Short March forward	$3.10 - S_{T(1)}$	$-S_{T(2)}$
Buy June forward	0	$S_{T(2)} - 3.152$
Lend @ 6%	−3.10	3.1469
Receive storage cost		+0.03
Total	0	+0.02485

We can use the June futures in our calculations and claim to receive storage costs because it is easy to show that the value of it is reflecting the negative lease rate of the storage costs.

b) It is not possible to undertake an arbitrage with the futures contracts that expire prior to September Year 1. A decrease in the September futures value means that we would need to buy the September futures contract, and any arbitrage strategy would need some short position in the widget. However, the drop in the futures price in September indicates that there is a significant convenience yield factored into the futures price over the period June–September. As we have no information about it, it is not possible for us to guarantee that we will find a lender of widgets at a favorable lease rate to follow through our arbitrage trading program. The decrease in the September futures may in fact reflect an increase in the opportunity costs of widget owners.

Question 6.9

If the February corn forward price is \$2.80, the observed forward price is too expensive relative to our theoretical price of \$2.7273. We will therefore sell the February contract short, and create a synthetic long position, engaging in cash and carry arbitrage:

Transaction	Nov	Dec	Jan	Feb
Short Feb forward	0			$2.80 - S_T$
Buy spot	−2.50			S_T
Borrow purchasing cost	+2.50			−2.57575
Pay storage cost Dec,		−0.05		−0.051005
borrow storage cost		+0.05		
Pay storage cost Jan,			−0.05	−0.0505
borrow storage cost			+0.05	
Pay storage cost Feb				−0.05
Total	0	0	0	0.072745

We made an arbitrage profit of 0.07 dollar.

If the February corn forward price is $2.65, the observed forward price is too low relative to our theoretical price of $2.7273. We will, therefore, buy the February contract and create a synthetic short position, engaging in reverse cash and carry arbitrage:

Transaction	Nov	Dec	Jan	Feb
Long Feb forward	0			$S_T - 2.65$
Sell spot	2.50			$-S_T$
Lend short sale proceeds	−2.50			+2.57575
Receive storage cost		+0.05		+0.051005
Dec, lend them		−0.05		
Receive storage cost			+0.05	+0.0505
Jan, lend them			−0.05	
Receive cost Feb				+0.05
Total	0	0	0	0.077255

We made an arbitrage profit of $0.08. It is important to keep in mind that we ignored any convenience yield that there may exist to holding the corn. We assumed the convenience yield is zero.

Question 6.10

Our best bet for the current spot price is the first available forward price, properly discounted by taking the interest rate and the lease rate into account, and by ignoring any storage cost and convenience yield (because we do not have any information on it):

$$F_{0,T} = S_0 \times e^{(r-\delta_l)\times T}$$

$$\Leftrightarrow S_0 = F_{0,T} \times e^{-(r-\delta_l)\times T}$$

$$\Leftrightarrow S_0 = 313.81 \times e^{-(0.06-0.015)\times 1} = 313.81 \times 0.956 = 300.0016$$

Question 6.11

There are 42 gallons of oil in one barrel. For each of these production ratios, we have to buy oil and sell gasoline and heating oil.

We calculate for the

2:1:1 split: Profit = \$2/gallon × 1 gallon + \$1.80/gallon × 1 gallon – 2 × \$80/42 = –0.0095

3:2:1 split: Profit = \$2/gallon × 2 gallons + \$1.80/gallon × 1 gallon – 3 × \$80/42 = 0.0857

5:3:2 split: Profit = \$2/gallon × 3 gallons + \$1.80/gallon × 2 gallons – 5 × \$80/42 = 0.07619

The 3:2:1 split maximizes profits

With these prices, we calculate for the

2:1:1 split: Profit = \$1.80/gallon × 1 gallon + \$2.10/gallon × 1 gallon – 2 × \$80/42 = 0.0905

3:2:1 split: Profit = \$1.80/gallon × 2 gallons + \$2.10/gallon × 1 gallon – 3 × \$80/42 = –0.0143

5:3:2 split: Profit = \$1.80/gallon × 3 gallons + \$2.10/gallon × 2 gallons – 5 × \$80/42 = 0.0762

Now, the 2:1:1 split maximizes profit.

You would expect the heating oil to be the most expensive in winter, and hence the 2:1:1 split to be most profitable in winter. People drive more and need less heating oil during the summer, so then you would choose a split with the highest fraction of gasoline, the 3:2:1.

Question 6.12

a) Widgets do not deteriorate over time and are costless to store; therefore, the lender does not save anything by lending me the widget. On the other hand, there is a constant demand and flexible production—there is no seasonality. Therefore, we should expect that the convenience yield is very close to the risk-free rate, merely compensating the lender for the opportunity cost.

b) Demand varies seasonally, but the production process is flexible. Therefore, we would expect that producers anticipate the seasonality in demand, and adjust production accordingly. Again, the lease rate should not be much higher than the risk-free rate.

c) Now we have the problem that the demand for widgets will spike up without an appropriate adjustment of production. Let us suppose that widget demand is high in June, July, and August. Then, we will face a substantial lease rate if we were to borrow the widget in May and return it in September: We would deprive the merchant of the widget when he would need it most (and could probably earn a significant amount of money on it), and we would return it when the season is over. We most likely pay a substantial lease rate.

   On the other hand, suppose we want to borrow the widget in January and return it in June. Now we return the widget precisely when the merchant needs it and have it over a time where demand is low, and he is not likely to sell it. The lease rate is likely to be very small.

   However, those stylized facts are weakened by the fact that the merchant can costlessly store widgets, so the smart merchant has a larger inventory when demand is high, offsetting the above effects at a substantial amount.

d) Suppose that production is very low during June, July, and August. Let us think about borrowing the widget in May and returning it in September. We do again deprive the merchant of the widget when he needs it most, because with a constant demand, less production means widgets become a comparably scarce resource and increase in price. Therefore, we pay a higher lease rate. The opposite effects can be observed for a widget-borrowing from January to June.

   Again, these stylized facts are offset by the above mentioned inventory considerations.

e) If widgets cannot be stored, the seasonality problems become very severe, leading to larger swings in the lease rate due to the impossibility of managing inventory.