Using eigenvalues to detect anomalies in the exterior of a cavity

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Abstract

We use modified near field operators and the generalized linear sampling method to investigate an inverse scattering problem for anisotropic media with data measured inside a cavity. The aim of this paper is to determine information on possible changes in the material properties of the surrounding medium under the assumption that the shape is known, and this will be accomplished by the introduction of a new class of eigenvalue problems for which the eigenvalues can be determined from the measured scattering data.

Key words. inverse scattering, nondestructive testing, modified transmission eigenvalues, non-selfadjoint eigenvalue problems

AMS subject classifications. 35J25, 35P05, 35P25, 35R30

1. Introduction

In this paper we propose to use modified near field operators [2, 8, 9, 10, 11] and the generalized linear sampling method [3, 6] to investigate an inverse scattering problem for anisotropic media with data measured inside a cavity. In this class of problems the objective is to determine the shape of a cavity and the material properties of the surrounding medium from the use of sources and measurements along a curve or surface inside the cavity. The problem of determining the shape of the cavity was considered in [7, 17] and hence in this paper we will consider the problem of determining information on possible changes in the material properties of the surrounding medium under the assumption that the shape is known. Such a problem arises, for example, in the nondestructive testing of the possibly anisotropic boundary of a container by placing receivers and transmitters inside the container. Since uniqueness in general does not hold for the inverse scattering problem for anisotropic media [13], the best that can be hoped for in general is to detect changes in the material properties of

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the medium. We will do this through the introduction of a new class of eigenvalue problems for which the eigenvalues can be determined from the measured scattering data.

The inverse scattering problem with measured data inside a cavity is of relatively recent origin [7, 16, 17, 18, 19, 22]. Here, as in [7], we are concerned with the case when the cavity is surrounded by inhomogeneous medium and it is desired to determine information about this medium using linear sampling methods. We are now assuming the shape is known and are concerned with determining information on the changes in the material properties of a surrounding anisotropic medium. In order to accomplish this we need to introduce a modified near field operator as in [11] where we considered the standard scattering problem with sources and receivers placed in the exterior of a bounded scattering obstacle. In both [11] and here, this leads to the problem of considering a modified transmission problem depending on an eigenparameter (called η in this paper) with the wave number k held fixed. Changes in the material properties of the medium are then detected by changes in the eigenparameter η. However, the modified *interior* transmission problem in [11] is now replaced by a modified *exterior* transmission problem and, as opposed to the interior problem, this exterior problem is no longer self-adjoint even when the index of refraction is real-valued. This leads to problems in computing the eigenvalues of the modified exterior transmission problem from the measured scattering data, since in contrast to the far field operator considered in [11] (see also [2]), the near field operator that arises in the current analysis does not have a symmetric factorization. A major part of our paper is devoted to resolving the above issues of the lack of self-adjointness of the modified exterior transmission problem and the failure of the near field operator to have a symmetric factorization (c.f. Sections 3 and 4).

The plan of our paper is as follows. In Section 2 we formulate the direct scattering problem corresponding to the scattering of a field due to a point source by the anisotropic penetrable boundary of a cavity. This leads us to the consideration of the modified exterior transmission problem. In the next two sections we then investigate this problem and its spectrum and show that the eigenvalues of the modified exterior transmission problem can be determined from the measured near field data (although the presence of eigenvalues can be clearly seen in numerical experiments, we have been unable to prove their existence mathematically). We conclude our paper with numerical examples showing that changes in the material properties of the anisotropic medium surrounding the cavity can be detected by corresponding shifts in the eigenvalues of the modified exterior transmission problem.

2. Scattering by a penetrable cavity

Let $D \subseteq \mathbb{R}^d$ $(d = 2, 3)$ be a bounded and simply connected Lipschitz domain with boundary ∂D and outward unit normal ν which contains the origin (see Figure 1). Let A be a symmetric $d \times d$ matrix with $L^\infty(\mathbb{R}^d)$ entries and $n \in$

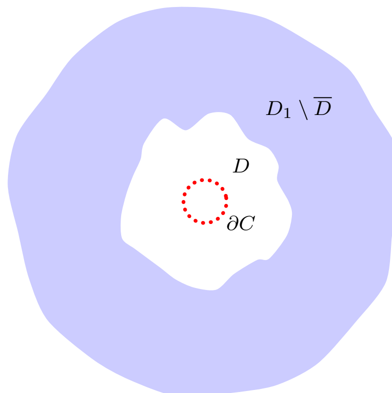


Figure 1: The anisotropic medium with a cavity. The sources and receivers are placed on the measurement manifold ∂C.

$L^\infty(\mathbb{R}^d)$ be such that

(i) $A = I$ and $n = 1$ in D;

(ii) there exists a bounded Lipschitz domain $D_1 \subset \mathbb{R}^d$ such that $A = I$ and $n = 1$ in $\mathbb{R}^d \setminus \overline{D}_1$ and $D \subset D_1$;

(iii) $\overline{\xi} \cdot \mathrm{Re}(A)\xi \geq \alpha\, |\xi|^2$, $\alpha > 0$, and $\overline{\xi} \cdot \mathrm{Im}(A)\xi \leq 0$ for all $\xi \in \mathbb{C}^d$;

(iv) $\mathrm{Re}(n) \geq n_* > 0$ and $\mathrm{Im}(n) \geq 0$ a.e. in $D_1 \setminus \overline{D}$.

We consider the problem of scattering by this medium of an incident field u^i generated from a point source inside D (e.g. $u^i = \Phi(\cdot, y)$ for given $y \in D$), which may be written in terms of the scattered field u^s in D and the total field u in $\mathbb{R}^d \setminus \overline{D}$ as follows: find $u^s \in H^1(D)$ and $u \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ such that

$$\nabla \cdot A\nabla u + k^2 n u = 0 \text{ in } \mathbb{R}^d \setminus \overline{D}, \tag{2.1a}$$

$$\Delta u^s + k^2 u^s = 0 \text{ in } D, \tag{2.1b}$$

$$u - u^s = u^i \text{ on } \partial D, \tag{2.1c}$$

$$\frac{\partial u}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \text{ on } \partial D, \tag{2.1d}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \tag{2.1e}$$

where the Sommerfeld radiation condition (2.1e) is assumed to hold uniformly in all directions, $\frac{\partial u}{\partial \nu_A} := (A\nabla u) \cdot \nu$ is the conormal derivative, and $k > 0$ is the wave number.

Now we formulate an auxiliary problem which depends on an artificial parameter and will lead to our modification of the near field operator. We begin by considering simply connected Lipschitz domains $B_1 \supseteq D_1$ and $B \subseteq D$ with boundaries ∂B_1 and ∂B, respectively, and outward unit normal ν such that each contains the origin, and we let A_0 be a $d \times d$ matrix with $L^\infty(\mathbb{R}^d)$ entries which shares the properties of A given above (with B_1 and B in the place of D_1 and D). Given a parameter $\eta \in \mathbb{C}$, we define the auxiliary refractive index $n_0 \in L^\infty(\mathbb{R}^d)$ as

$$n_0 := \begin{cases} \eta & \text{in } B_1 \setminus \overline{B}, \\ 1 & \text{elsewhere.} \end{cases}$$

The auxiliary problem we consider is to find $u_0^s \in H^1(B)$ and $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B})$ such that

$$\nabla \cdot A_0 \nabla u_0 + k^2 n_0 u_0 = 0 \text{ in } \mathbb{R}^d \setminus \overline{B}, \tag{2.2a}$$

$$\Delta u_0^s + k^2 u_0^s = 0 \text{ in } B, \tag{2.2b}$$

$$u_0 - u_0^s = u^i \text{ on } \partial B, \tag{2.2c}$$

$$\frac{\partial u_0}{\partial \nu_{A_0}} - \frac{\partial u_0^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \text{ on } \partial B, \tag{2.2d}$$

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_0}{\partial r} - iku_0 \right) = 0. \tag{2.2e}$$

We observe that (2.1a)–(2.1e) is well-posed given our assumptions on A and n, and the auxiliary problem (2.2a)–(2.2e) is well-posed provided that $\mathrm{Im}(\eta) \geq 0$ [6]. However, as η serves as our eigenparameter, we would like to allow it to attain values with negative imaginary part, and while the auxiliary problem (2.2a)–(2.2e) may not be well-posed for such choices of η, the following theorem shows that this set is discrete using analytic Fredholm theory [12]. We must first write the equivalent problem on a bounded domain as follows. We let B_R be a ball of radius R centered at the origin which strictly contains B_1 in its interior, and we define the exterior Dirichlet-to-Neumann map $T_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ by $T_k g = \left.\frac{\partial \psi}{\partial \nu}\right|_{\partial B_R}$, where ν is the outward unit normal to ∂B_R and $\psi \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B}_R)$ is the radiating solution of the exterior Dirichlet problem

$$\Delta \psi + k^2 \psi = 0 \text{ in } \mathbb{R}^d \setminus \overline{B}_R,$$
$$\psi = g \text{ on } \partial B_R.$$

Then (2.2a)–(2.2e) is equivalent to finding $u_0^s \in H^1(B)$ and $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B})$ such that

$$\nabla \cdot A_0 \nabla u_0 + k^2 n_0 u_0 = 0 \text{ in } B_R \setminus \overline{B}, \tag{2.3a}$$
$$\Delta u_0^s + k^2 u_0^s = 0 \text{ in } B, \tag{2.3b}$$
$$u_0 - u_0^s = u^i \text{ on } \partial B, \tag{2.3c}$$
$$\frac{\partial u_0}{\partial \nu_{A_0}} - \frac{\partial u_0^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \text{ on } \partial B, \tag{2.3d}$$
$$\frac{\partial u_0}{\partial \nu} = T_k u_0 \text{ on } \partial B_R. \tag{2.3e}$$

Theorem 2.1. *The set of η for which the auxiliary problem* (2.2a)–(2.2e) *is not well-posed is discrete.*

Proof. Our strategy is to write an equivalent formulation of (2.2a)–(2.2e) as a compact perturbation of an invertible operator and apply the analytic Fredholm theorem [12, Theorem 8.26]. We first observe that in terms of the scattered field, problem (2.3a)–(2.3e) may be written as the equivalent variational problem of finding $u_0^s \in H^1(B_R)$ satisfying

$$a(u_0^s, \varphi) = \ell(\varphi) \quad \forall \varphi \in H^1(B_R), \tag{2.4}$$

where the sesquilinear form $a(\cdot,\cdot)$ is given by

$$a(\psi,\varphi) := (A_0 \nabla \psi, \nabla \varphi)_{B_R} - k^2 (n_0 \psi, \varphi)_{B_R} - \langle T_k \psi, \varphi \rangle_{\partial B_R} \quad \forall \psi, \varphi \in H^1(B_R)$$

and the antilinear functional ℓ contains the information about the point source u^i. Here we have denoted by $(\cdot,\cdot)_{\mathcal{O}}$ the $L^2(\mathcal{O})$ inner product and by $\langle \cdot,\cdot \rangle_{\partial \mathcal{O}}$ the $H^{-1/2}(\partial \mathcal{O}) \times H^{1/2}(\partial \mathcal{O})$ duality pairing for a given open set $\mathcal{O}$. We remark that there exists a mapping $T_0 : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ for which $T_0 - T_k$ is compact and $-\langle T_0 g, g \rangle_{\partial B_R} \geq 0$ for all $g \in H^{1/2}(\partial B_R)$ [6, Remark 1.37], and by means of the Riesz representation theorem we define the operators $\hat{\mathcal{A}}, \mathcal{A}_\eta : H^1(B_R) \to H^1(B_R)$ such that

$$(\hat{\mathcal{A}}\psi, \varphi)_{H^1(B_R)} = (A_0 \nabla \psi, \nabla \varphi)_{B_R} + k^2 (\psi, \varphi)_{B_R} - \langle T_0 \psi, \varphi \rangle_{\partial B_R}$$
$$(\mathcal{A}_\eta \psi, \varphi)_{H^1(B_R)} = -k^2 ((n_0 + 1)\psi, \varphi)_{B_R} + \langle (T_0 - T_k)\psi, \varphi \rangle_{\partial B_R}$$

for all $\psi, \varphi \in H^1(B_R)$. We see that

$$a(\psi, \varphi) = ((\hat{\mathcal{A}} + \mathcal{A}_\eta)\psi, \varphi)_{H^1(B_R)} \quad \forall \psi, \varphi \in H^1(B_R).$$

Since $\mathrm{Re}(A_0)$ is positive-definite and $-T_0$ is strictly coercive, it easily follows that $\hat{\mathcal{A}}$ is strictly coercive and hence invertible. Moreover, the compact embedding of $H^1(B_R)$ into $L^2(B_R)$ and compactness of the operator $T_0 - T_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ imply that $\mathcal{A}_\eta$ is compact, and we conclude that $\hat{\mathcal{A}} + \mathcal{A}_\eta$ is a Fredholm operator of index zero. Thus, the auxiliary problem satisfies the Fredholm property and existence follows from uniqueness of solutions.

Uniqueness is a direct consequence of Rellich's lemma and the unique continuation property whenever $\mathrm{Im}(\eta) \geq 0$ [6], but for $\mathrm{Im}(\eta) < 0$ the best we can hope for is that uniqueness fails only on a discrete set. However, for each $\psi \in H^1(B_R)$ the mapping $\eta \to \mathcal{A}_\eta \psi$ is clearly weakly analytic and hence the mapping $\eta \to \mathcal{A}_\eta$ is strongly analytic by Corollary 8.23 and Theorem 8.25 in [12]. Therefore, the analytic Fredholm theorem implies that the operator $\hat{\mathcal{A}}+\mathcal{A}_\eta$ is either invertible for no values of η or it is invertible for all η except for a discrete set. Since invertibility of this operator is equivalent to well-posedness of the auxiliary problem (2.2a)–(2.2e), which is well-posed whenever $\mathrm{Im}(\eta) \geq 0$, we obtain the desired result. □

Before we continue, we remark that we will require additional assumptions on A and A_0 in future sections; however, for purposes of generality we will state these assumptions only once they become necessary. We now describe how scattering data is measured for this problem. Let ∂C be a smooth $(d-1)$-manifold (called the measurement manifold) contained in B which encloses a bounded region C and satisfies the following assumption.

Assumption 2.2. *The measurement manifold ∂C is such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in the region C.*

The importance of this assumption lies in the following lemma, which follows from Assumption 2.2 and the unique continuation principle [15, Theorem 17.2.6]. We will observe that this lemma serves the same role as Rellich's lemma [6] in far field measurements: it allows us to obtain further information about a field given its measurement data.

Lemma 2.3. *If u_1 and u_2 are two solutions of the Helmholtz equation in B such that $u_1 = u_2$ on ∂C, then $u_1 = u_2$ in B.*

With a chosen measurement manifold ∂C satisfying Assumption 2.2, we define the *near field operator* $N : L^2(\partial C) \to L^2(\partial C)$ by

$$(Ng)(x) := \int_{\partial C} u^s(x,y)g(y)ds(y),\ x \in \partial C, \tag{2.5}$$

and we define the *single layer potential* u_g by

$$u_g(x) := \int_{\partial C} \Phi(x,y)g(y)ds(y),\ x \in \mathbb{R}^d \setminus \partial C, \tag{2.6}$$

for each $g \in L^2(\partial C)$. By linearity we see that if u^s is the scattered field in D arising from the incident field $u^i = \Phi(\cdot, y)$ $(y \in D)$ in (2.1a)–(2.1e), then Ng is the scattered field in D (evaluated on ∂C) arising from the incident field $u^i = u_g$ for some $g \in L^2(\partial C)$. Similarly, we define the *auxiliary near field operator* $N_0 : L^2(\partial C) \to L^2(\partial C)$ by

$$(N_0 g)(x) := \int_{\partial C} u_0^s(x,y)g(y)ds(y),\ x \in \partial C, \tag{2.7}$$

and we observe the same relationship between N_0 and solutions of (2.2a)–(2.2e). We now define the *modified near field operator* $\mathcal{N} : L^2(\partial C) \to L^2(\partial C)$ by $\mathcal{N} := N - N_0$, i.e.

$$(\mathcal{N}g)(x) := \int_{\partial C} \big[u^s(x,y) - u_0^s(x,y)\big] g(y) ds(y), \; x \in \partial C, \tag{2.8}$$

and in the following theorem we relate this operator to the eigenvalue problem that we will study for the remainder of the paper. The proof of this theorem follows by combining the ideas of the proof of Theorem 2.1 in [10] to the proof of Theorem 5.1 in [7].

Theorem 2.4. *The modified near field operator $\mathcal{N}$ is injective with dense range if and only if there does not exist a nontrivial solution $w, v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B})$ of the homogeneous* modified exterior transmission problem

$$\nabla \cdot A \nabla w + k^2 n w = 0 \; \text{in } \mathbb{R}^d \setminus \overline{B}, \tag{2.9a}$$

$$\nabla \cdot A_0 \nabla v + k^2 n_0 v = 0 \; \text{in } \mathbb{R}^d \setminus \overline{B}, \tag{2.9b}$$

$$w - v = 0 \; \text{on } \partial B, \tag{2.9c}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu_{A_0}} = 0 \; \text{on } \partial B, \tag{2.9d}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial w}{\partial r} - ikw \right) = 0, \tag{2.9e}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial v}{\partial r} - ikv \right) = 0, \tag{2.9f}$$

such that v is a generalized single layer potential*, i.e. it is of the form*

$$v = \int_{\partial C} u_0(\cdot, y) g(y) ds(y) \; \text{in } \mathbb{R}^d \setminus \overline{B} \tag{2.10}$$

for some $g \in L^2(\partial C)$, where $(u_0(\cdot,y), u_0^s(\cdot,y))$ satisfies (2.2a)–(2.2e) *with $u^i = \Phi(\cdot,y), y \in \partial C$.*

We call a value of η for which the modified exterior transmission problem (2.9a)–(2.9f) has nontrivial solutions a *modified exterior transmission eigenvalue*. We further investigate this problem and its spectrum in the next section, but we first address the question of whether these eigenvalues might coincide with values of η for which the auxiliary problem (2.2a)–(2.2e) is not well-posed. We previously showed that such values form a discrete set, and we further observe that whereas this set depends only on the choice of the domains B and B_1 and the matrix A_0, the modified exterior transmission eigenvalues also depend on the physical medium. Thus, we would find it quite unlikely that they would coincide.

3. The modified exterior transmission problem

In order to investigate the solvability of the modified exterior transmission problem, we introduce the following nonhomogeneous version. Given $\ell_1, \ell_2 \in L^2(B_1 \setminus$

$\overline{B}$) that are extended by zero outside B_1, $f \in H^{1/2}(\partial B)$, and $h \in H^{-1/2}(\partial B)$, we seek $w, v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B})$ satisfying

$$\nabla \cdot A \nabla w + k^2 n w = \ell_1 \text{ in } \mathbb{R}^d \setminus \overline{B}, \tag{3.1a}$$

$$\nabla \cdot A_0 \nabla v + k^2 n_0 v = \ell_2 \text{ in } \mathbb{R}^d \setminus \overline{B}, \tag{3.1b}$$

$$w - v = f \text{ on } \partial B, \tag{3.1c}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu_{A_0}} = h \text{ on } \partial B, \tag{3.1d}$$

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial w}{\partial r} - ikw \right) = 0, \tag{3.1e}$$

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial v}{\partial r} - ikv \right) = 0. \tag{3.1f}$$

If we choose $\varphi \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B})$ to be the unique radiating solution of the non-homogeneous exterior Dirichlet problem

$$\Delta \varphi + k^2 \varphi = 0 \text{ in } \mathbb{R}^d \setminus \overline{B},$$
$$\varphi = f \text{ on } \partial B,$$

then we may write $v_0 = v + \varphi$ and obtain an equivalent problem to (3.1a)–(3.1f) with $f = 0$ and all other right-hand sides modified accordingly. Thus, we may assume that $f = 0$ in (3.1a)–(3.1f).

In order to write an equivalent problem posed on a bounded domain, we let B_R be a ball of radius R centered at the origin which strictly contains B_1, and we define the exterior Dirichlet-to-Neumann map $T_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ as we did in the proof of Theorem 2.1. Then an equivalent problem to (3.1a)–(3.1f) is to find $w, v \in H^1(B_R \setminus \overline{B})$ satisfying

$$\nabla \cdot A \nabla w + k^2 n w = \ell_1 \text{ in } B_R \setminus \overline{B}, \tag{3.2a}$$

$$\nabla \cdot A_0 \nabla v + k^2 n_0 v = \ell_2 \text{ in } B_R \setminus \overline{B}, \tag{3.2b}$$

$$w - v = 0 \text{ on } \partial B, \tag{3.2c}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu_{A_0}} = h \text{ on } \partial B, \tag{3.2d}$$

$$\frac{\partial w}{\partial \nu} = T_k w \text{ on } \partial B_R, \tag{3.2e}$$

$$\frac{\partial v}{\partial \nu} = T_k v \text{ on } \partial B_R, \tag{3.2f}$$

where we have used that $A = A_0 = I$ near ∂B_R [6]. We now develop an equivalent variational formulation of this problem in the space

$$\mathcal{H} := \{(w, v) \in H^1(B_R \setminus \overline{B}) \times H^1(B_R \setminus \overline{B}) \mid w - v = 0 \text{ on } \partial B\},$$

which is equipped with the inner product

$$((w, v), (w', v'))_{\mathcal{H}} := (w, w')_{H^1(B_R \setminus \overline{B})} + (v, v')_{H^1(B_R \setminus \overline{B})}$$

and the associated norm $\|\cdot\|_{\mathcal{H}}$. Given a test function pair $(w', v') \in \mathcal{H}$, we multiply $\overline{w}'$ and $\overline{v}'$ by (3.2a) and (3.2b), respectively, integrate by parts over $B_R \setminus \overline{B}$, and apply the boundary conditions on ∂B_R to obtain

$$\begin{aligned} -(A\nabla w, \nabla w')_{B_R\setminus\overline{B}} + k^2(nw, w')_{B_R\setminus\overline{B}} - \left\langle \frac{\partial w}{\partial \nu_A}, w' \right\rangle_{\partial B} \\ + \langle T_k w, w'\rangle_{\partial B_R} = (\ell_1, w')_{B_R\setminus\overline{B}}, \quad (3.3) \\ -(A_0\nabla v, \nabla v')_{B_R\setminus\overline{B}} + k^2(n_0 v, v')_{B_R\setminus\overline{B}} - \left\langle \frac{\partial v}{\partial \nu_{A_0}}, v' \right\rangle_{\partial B} \\ + \langle T_k v, v'\rangle_{\partial B_R} = (\ell_2, v')_{B_R\setminus\overline{B}}. \quad (3.4) \end{aligned}$$

Subtracting (3.3) from (3.4) and enforcing the boundary conditions on ∂B yields the equivalent variational equation of finding $(w, v) \in \mathcal{H}$ satisfying

$$a_\eta((w, v), (w', v')) = \ell(w', v') \quad \forall (w', v') \in \mathcal{H}, \tag{3.5}$$

where the bounded sesquilinear form a_η is defined by

$$\begin{aligned} a_\eta((w, v), (w', v')) := (A\nabla w, \nabla w')_{B_R\setminus\overline{B}} - (A_0\nabla v, \nabla v')_{B_R\setminus\overline{B}} - k^2(nw, w')_{B_R\setminus\overline{B}} \\ + k^2(n_0 v, v')_{B_R\setminus\overline{B}} - \langle T_k w, w'\rangle_{\partial B_R} + \langle T_k v, v'\rangle_{\partial B_R} \quad (3.6) \end{aligned}$$

for all $(w, v), (w', v') \in \mathcal{H}$, and the bounded antilinear functional ℓ is defined by

$$\ell(w', v') := -(\ell_1, w')_{B_R\setminus\overline{B}} + (\ell_2, v')_{B_R\setminus\overline{B}} - \langle h, v'\rangle_{\partial B} \tag{3.7}$$

for all $(w', v') \in \mathcal{H}$. Conversely, if (w, v) is a solution of the variational problem (3.5), then it follows from choosing $w' \in C_0^\infty(B_1 \setminus \overline{B})$ and $v' = 0$ that w satisfies (3.1a), and similar reasoning implies that v satisfies (3.1b). Choosing $(w', v') \in \mathcal{H}$ with $w' = v' = 0$ on ∂B_R provides that (w, v) satisfies (3.1d). Finally, choosing $(w', 0) \in \mathcal{H}$ implies that w satisfies (3.1e), and similar reasoning implies that v satisfies (3.1f). Therefore, the modified exterior transmission problem (3.1a)–(3.1f) and the variational problem (3.5) are equivalent.

We now define the operator $\mathbb{A}_\eta : \mathcal{H} \to \mathcal{H}$ by means of the Riesz representation theorem such that

$$(\mathbb{A}_\eta(w, v), (w', v'))_{\mathcal{H}} = a_\eta((w, v), (w', v')) \quad \forall (w, v), (w', v') \in \mathcal{H}.$$

In order to establish the Fredholm property of (3.5) and hence of (3.1a)–(3.1f), it suffices to write $\mathbb{A}_\eta$ as a compact perturbation of an invertible operator. To this end, we define the operators $\hat{\mathbb{A}}, \mathbb{B}_\eta : \mathcal{H} \to \mathcal{H}$ by means of the Riesz representation theorem such that

$$\begin{aligned} (\hat{\mathbb{A}}(w, v), (w', v'))_{\mathcal{H}} &= (A\nabla w, \nabla w')_{B_R\setminus\overline{B}} - (A_0\nabla v, \nabla v')_{B_R\setminus\overline{B}} + k^2\alpha(w, w')_{B_R\setminus\overline{B}} \\ &\quad - k^2\beta(v, v')_{B_R\setminus\overline{B}} - \langle T_0 w, w'\rangle_{\partial B_R} + \langle T_0 v, v'\rangle_{\partial B_R}, \\ (\mathbb{B}_\eta(w, v), (w', v'))_{\mathcal{H}} &= -k^2((n + \alpha)w, w')_{B_R\setminus\overline{B}} + k^2((n_0 + \beta)v, v')_{B_R\setminus\overline{B}} \\ &\quad + \langle (T_0 - T_k)w, w'\rangle_{\partial B_R} - \langle (T_0 - T_k)v, v'\rangle_{\partial B_R} \end{aligned}$$

for all $(w,v),(w',v') \in \mathcal{H}$, where $T_0 : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is a mapping for which $T_0 - T_k$ is compact and $-\langle T_0 g, g\rangle \geq 0$ for all $g \in H^{1/2}(\partial B_R)$ [6, Remark 1.37]. The constants $\alpha, \beta > 0$ will be determined later. We observe that $\mathbb{A}_\eta = \hat{\mathbb{A}} + \mathbb{B}_\eta$, and we will show that $\hat{\mathbb{A}}$ is invertible and $\mathbb{B}_\eta$ is compact. Due to the different signs of the gradient terms in the definition of $\hat{\mathbb{A}}$, we apply the idea of T-coercivity [5, 6]. However, we first require an assumption on A and A_0 in order to proceed.

Assumption 3.1. *Assume that there exists a neighborhood of ∂B in which $\mathrm{Im}(A) = 0$ and $\mathrm{Im}(A_0) = 0$, and denote by Ω its intersection with $B_1 \setminus \overline{B}$.*

For $\xi \in \mathbb{C}^d$, denote

$$A_* := \inf_{x\in\Omega} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi \quad \text{and} \quad A^* := \sup_{x\in\Omega} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi.$$

We first remark that A_0 may be chosen freely (subject to the aforementioned requirements) to satisfy this assumption, and for simplicity we choose $A_0 = \gamma I$ in Ω for some $\gamma > 0$. In addition to Assumption 3.1, we will see that we must choose either $0 < \gamma < A_*$ or $\gamma > A^*$ in order to establish invertibility of $\hat{\mathbb{A}}$, and in the proof of the following lemma we only include the case $0 < \gamma < A_*$. We note that Assumption 3.1 is automatically satisfied for A by the neighborhood $\Omega = D \setminus \overline{B}$ if B is chosen such that $\overline{B} \subsetneq D$. In this case we observe that $A_* = A^* = 1$ and consequently we must choose $\gamma \neq 1$.

Lemma 3.2. *If $0 < \gamma < A_*$ or $\gamma > A^*$, then the operator $\hat{\mathbb{A}} : \mathcal{H} \to \mathcal{H}$ is invertible.*

Proof. We consider the case $0 < \gamma < A_*$. Choose a smooth cutoff function χ which equals 1 in a neighborhood of ∂B with support in Ω (in particular $\chi = 0$ near ∂B_R), and define $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ by $\mathcal{T}(w,v) := (w, -v + 2\chi w)$, which is an isomorphism since $\mathcal{T}^2 = I$. We will show that the sesquilinear form $\hat{a}^{\mathcal{T}}(\cdot,\cdot)$ defined by

$$\hat{a}^{\mathcal{T}}((w,v),(w',v')) := (\hat{\mathbb{A}}(w,v), \mathcal{T}(w,v))_{\mathcal{H}} \quad \forall (w,v),(w',v') \in \mathcal{H}$$

is coercive and conclude invertibility of $\hat{\mathbb{A}}$ from this result. We see that

$$\begin{aligned}
\hat{a}^{\mathcal{T}}((w,v),(w,v)) &= (\hat{\mathbb{A}}(w,v), (w, -v + 2\chi w))_{\mathcal{H}} \\
&= (A\nabla w, \nabla w)_{B_R\setminus\overline{B}} + (A_0 \nabla v, \nabla v)_{B_R\setminus\overline{B}} - 2(A_0\nabla v, \nabla(\chi w))_{B_R\setminus\overline{B}} \\
&\quad + k^2\alpha(w,w)_{B_R\setminus\overline{B}} + k^2\beta(v,v)_{B_R\setminus\overline{B}} - 2k^2\beta(v,\chi w)_{B_R\setminus\overline{B}} \\
&\quad - \langle T_0 w, w\rangle_{\partial B_R} - \langle T_0 v, v\rangle_{\partial B_R} + 2\langle T_0 v, \chi w\rangle_{\partial B_R}
\end{aligned}$$

for all $(w,v) \in \mathcal{H}$. For all $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ it follows from the product rule, the triangle inequality, and Young's inequality that

$$\begin{aligned}
2\left|(\nabla v, \nabla(\chi w))_{B_R\setminus\overline{B}}\right| &\leq (\epsilon_1 + \epsilon_2)(\nabla v, \nabla v)_\Omega + \epsilon_1^{-1}(\nabla w, \nabla w)_\Omega \\
&\quad + \epsilon_2^{-1}(\nabla(\chi) w, \nabla(\chi) w)_\Omega,
\end{aligned}$$

and

$$2\left|(v,\chi w)_{B_R\setminus\overline{B}}\right| \le \epsilon_3(v,v)_\Omega + \epsilon_3^{-1}(w,w)_\Omega.$$

By definition of the cutoff function χ it follows that $\langle T_0 v, \chi w\rangle_{\partial B_R} = 0$. Noting that $A_0 = \gamma I$ in Ω, we obtain

$$\begin{aligned}
\operatorname{Re}\hat{a}^{\mathcal{T}}((w,v),(w,v)) \ge{}& \operatorname{Re}(A\nabla w,\nabla w)_{(B_R\setminus\overline{B})\setminus\overline{\Omega}} + \operatorname{Re}(A_0\nabla v,\nabla v)_{(B_R\setminus\overline{B})\setminus\overline{\Omega}} \\
&+ k^2\alpha(w,w)_{(B_R\setminus\overline{B})\setminus\overline{\Omega}} + k^2\beta(v,v)_{(B_R\setminus\overline{B})\setminus\overline{\Omega}} \\
&+ (A_* - \gamma\epsilon_1^{-1})(\nabla w,\nabla w)_\Omega + \gamma(1-\epsilon_1-\epsilon_2)(\nabla v,\nabla v)_\Omega \\
&+ \left[k^2(\alpha - \beta\epsilon_3^{-1}) - \gamma\epsilon_2^{-1}\sup_\Omega|\nabla\chi|^2\right](w,w)_\Omega \\
&+ k^2\beta(1-\epsilon_3)(v,v)_\Omega.
\end{aligned}$$

We observe that choosing $\gamma A_*^{-1} < \epsilon_1 < 1$, $0 < \epsilon_2 < 1-\epsilon_1$, $0 < \epsilon_3 < 1$, and α sufficiently large yields coercivity of $\hat{a}^{\mathcal{T}}$, and hence the Lax-Milgram lemma and the fact that $\mathcal{T}$ is an isomorphism imply that $\hat{\mathbb{A}} : \mathcal{H} \to \mathcal{H}$ is invertible. A similar computation with $\mathcal{T}(w,v) := (w - 2\chi v, -v)$ and β taken to be sufficiently large establishes the result for $\gamma > A^*$. $\square$

In view of Lemma 3.2, we assume for the remainder of the discussion that $\gamma > 0$ is chosen such that $\gamma < A_*$ or $\gamma > A^*$. We establish compactness of $\mathbb{B}_\eta$ in the following lemma.

Lemma 3.3. *The operator $\mathbb{B}_\eta : \mathcal{H} \to \mathcal{H}$ is compact.*

Proof. We see from the Cauchy-Schwarz inequality and the trace theorem that there exists a constant $c > 0$ such that

$$\begin{aligned}
\|\mathbb{B}_\eta(w,v)\|_{\mathcal{H}} &= \sup_{(w',v')\neq 0} \frac{|(\mathbb{B}_\eta(w,v),(w',v'))_{\mathcal{H}}|}{\|(w',v')\|_{\mathcal{H}}} \\
&\le c\Big(\|(w,v)\|_{L^2(B_R\setminus\overline{B})\times L^2(B_R\setminus\overline{B})} + \|(T_0 - T_k)w\|_{H^{-1/2}(\partial B_R)} \\
&\qquad + \|(T_0 - T_k)v\|_{H^{-1/2}(\partial B_R)}\Big)
\end{aligned}$$

for all $(w,v) \in \mathcal{H}$. Thus, compactness of $\mathbb{B}_\eta$ follows from compactness of $T_0 - T_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ and the compact embedding of $\mathcal{H}$ into $L^2(B_R\setminus\overline{B}) \times L^2(B_R\setminus\overline{B})$. $\square$

Combining Lemmas 3.2 and 3.3 implies that $\mathbb{A}_\eta = \hat{\mathbb{A}} + \mathbb{B}_\eta$ is a Fredholm operator of index zero, and we conclude that the variational problem (3.5) and hence the nonhomogeneous modified exterior transmission problem (3.1a)–(3.1f) satisfies the Fredholm property. In particular, the problem (3.1a)–(3.1f) is well-posed provided that η is not a modified exterior transmission eigenvalue.

We may use the above results to investigate the discreteness of modified exterior transmission eigenvalues, which we show in the following theorem using analytic Fredholm theory in a manner similar to the proof of Theorem 2.1. Unlike that case, we are unable to find a value of η for which the modified exterior problem is well-posed for a given fixed k, and as a result we instead choose k for which such a value of η exists. By similar reasoning to Theorem 3.5 (see also Lemma 3.1) in [7], the problem (2.9a)–(2.9f) is well-posed for all k except in a discrete set in the complex plane whenever either i) $\gamma < A_*$ and $\eta < n_*$, or ii) $\gamma > A^*$ and $\eta > n^*$, where n_* and n^* are the infimum and supremum of n in Ω, respectively. Thus, the assumption in the following theorem is valid. We remark that if B is chosen to be a ball strictly contained in D, then the choice $\Omega = D \setminus \overline{B}$ implies that this statement holds whenever $1-\gamma$ and $1-\eta$ have the same sign.

Theorem 3.4. *Given η_0 such that either i) $\gamma < A_*$ and $\eta_0 < n_*$, or ii) $\gamma > A^*$ and $\eta_0 > n^*$, assume that $k > 0$ is chosen such that the problem* (2.9a)–(2.9f) *with $\eta = \eta_0$ is well-posed. Then the set of modified exterior transmission eigenvalues is discrete.*

Proof. We recall from Lemmas 3.2 and 3.3 that $\hat{\mathbb{A}}$ is invertible and $\mathbb{B}_\eta$ is compact, and we observe from the definition of $\mathbb{B}_\eta$ and similar reasoning to the proof of Theorem 2.1 that the mapping $\eta \mapsto \mathbb{B}_\eta$ is analytic. It follows from the analytic Fredholm theorem [12, Theorem 8.26] applied to the operator $\mathbb{A}_\eta = \hat{\mathbb{A}} + \mathbb{B}_\eta$ that the set of modified exterior transmission eigenvalues is discrete provided that some η exists for which $\mathbb{A}_\eta$ is invertible, which holds for $\eta = \eta_0$ by assumption. $\square$

In the case of real-valued A, n this result implies A_η is invertible for some real η, but we unfortunately cannot relate the modified exterior transmission eigenvalues to the spectrum of a self-adjoint operator as in [11] for the case of far field measurements due to the fact that the Dirichlet-to-Neumann map T_k is not symmetric. Thus, existence of modified exterior transmission eigenvalues remains an open question, and there may exist eigenvalues with nonzero imaginary part. Moreover, the variational formulation (3.5) does not exclude the possibility of modified exterior transmission eigenvalues with negative imaginary part, which differs considerably from the eigenvalue problems considered in [8], [9] and [11]. Though existence results have been obtained for the eigenvalue problems considered in [8] and [11] using Agmon's theory of non-selfadjoint elliptic equations [1], we do not find it likely that such techniques may be applied here due to the unusual structure of the modified exterior transmission problem and the observation that the auxiliary index of refraction n_0 is non-smooth in $\mathbb{R}^d \setminus \overline{B}$.

4. Determination of modified exterior transmission eigenvalues from internal measurements

In this section, we will establish that it is indeed possible to compute modified exterior transmission eigenvalues, if they exist, from internal measurements using the recently developed generalized linear sampling method [3, 6]. We first develop a factorization of the modified near field operator $\mathcal{N}$. Define $H : L^2(\partial C) \to H^1_{loc}(\mathbb{R}^d \setminus \overline{B})$ by $Hg := v_g$, where (v_g, v_g^s) is the solution of the auxiliary problem (2.2a)–(2.2e) with $u^i = u_g$, and define $G : \overline{R(H)} \subset H^1_{loc}(\mathbb{R}^d \setminus \overline{B}) \to L^2(\partial C)$ by $G\psi := w^*|_{\partial C}$, where $w^* \in H^1_{loc}(\mathbb{R}^d)$ is the radiating solution of

$$\nabla \cdot A\nabla w^* + k^2 n w^* = \nabla \cdot (A_0 - A)\nabla\psi + k^2(n_0 - n)\psi \text{ in } \mathbb{R}^d. \tag{4.1}$$

We observe that $\mathcal{N} = GH$, and we now establish a result on the range of H.

Lemma 4.1. *If B is chosen to be a ball and we define*

$$\mathcal{V}_0(B) := \left\{ v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B}) \,\middle|\, \nabla \cdot A_0\nabla v + k^2 n_0 v = 0 \text{ in } \mathbb{R}^d \setminus \overline{B}, \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} - ikv\right) = 0 \right\},$$

then the closure of the range of $H : L^2(\partial C) \to H^1_{loc}(\mathbb{R}^d \setminus \overline{B})$ is given by

$$\overline{R(H)} = \mathcal{V}_0(B).$$

Proof. We follow similar reasoning as for the standard single layer potential given in Lemma 5.1 of [7], but we must split the proof into two steps. First, we will show that the result holds with B replaced by a Lipschitz domain B' satisfying $\overline{C} \subset B' \subseteq \overline{B'} \subset B$ which contains the origin and H replaced by the operator $\tilde{H} : L^2(\partial C) \to H^1_{loc}(\mathbb{R}^d \setminus \overline{B'})$ defined by $\tilde{H}g := \tilde{v}_g$, where $\tilde{v}_g$ is the extension of v_g as $\tilde{v}_g = v_g^s + u_g$ in $B \setminus \overline{B'}$. We will then provide a simple extension to obtain the result for B. Indeed, consider B' with the properties listed above. We remark that if the exterior Dirichlet problem of finding $\tilde{v} \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B'})$ satisfying

$$\nabla \cdot A_0\nabla\tilde{v} + k^2 n_0 \nabla\tilde{v} = 0 \text{ in } \mathbb{R}^d \setminus \overline{B'} \tag{4.2}$$

$$\tilde{v} = h \text{ on } \partial B', \tag{4.3}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial\tilde{v}}{\partial r} - ik\tilde{v}\right) = 0, \tag{4.4}$$

is well-posed for any given $h \in H^{1/2}(\partial B')$, then it follows that for each compact subset $K \subset \mathbb{R}^d$ with $K \supset B'$ there exists a constant $c > 0$ for which

$$\|\tilde{v}\|_{H^1(K\setminus\overline{B'})} \le c\,\|h\|_{H^{1/2}(\partial B')}$$

and consequently we need only show that the set $\{\tilde{v}_g|_{\partial B'} \mid g \in L^2(\partial C)\}$ of Dirichlet data is dense in $H^{1/2}(\partial B')$. The problem (4.2)–(4.4) is well-posed for $\mathrm{Im}(\eta) \geq 0$, and we may apply the same technique as in the proof of Theorem 2.1 to conclude that the set of η for which it is not well-posed is discrete and hence may be easily avoided. As stated previously, when B, B_1, and C are chosen to be balls centered at the origin, these values of η may be computed by separation of variables, and we assume for the remainder of the proof that η is such that (4.2)–(4.4) is well-posed and hence it suffices to prove the reduced density result proposed above. To this end, let $f \in H^{-1/2}(\partial B')$ satisfy

$$\int_{\partial B'} f(x)\tilde{v}_g(x)ds(x) = 0 \tag{4.5}$$

for all $g \in L^2(\partial C)$. Using the fact that

$$\tilde{v}_g(x) = \int_{\partial C} u_0(x,y)g(y)ds(y), x \in \mathbb{R}^d \setminus \partial C,$$

by linearity (where we have also extended $u_0 = u_0^s + \Phi$ in $B \setminus \partial C$), we may substitute $\tilde{v}_g$ into (4.5) and interchange the order of integration to obtain

$$\int_{\partial C} \left(\int_{\partial B'} u_0(x,y)f(x)ds(x) \right) g(y)ds(y) = 0$$

for all $g \in L^2(\partial C)$. It follows that

$$w_f(y) := \int_{\partial B'} u_0(x,y)f(x)ds(x) = 0 \quad \forall y \in \partial C.$$

The proof of Theorem 5.1 in [7] further implies that $u_0^s(x,y) = u_0^s(y,x)$ for $x, y \in B$, and hence $u_0(x,y) = u_0(y,x)$ for all $x, y \in B$, $x \neq y$, by symmetry of Φ. We remark that this symmetry property is the reason for first establishing the result for B' rather than B; our formulation of the problem does not even allow us to write $u_0(y,x)$ for $x \in \partial B$. Since $\partial B' \subset B$, we may apply this symmetry in order to write

$$w_f(y) = \int_{\partial B'} u_0(y,x)f(x)ds(x),\ y \in \mathbb{R}^d \setminus \partial B',$$

and we observe that w_f satisfies the Helmholtz equation in B'. Since $w_f = 0$ on ∂C, Lemma 2.3 implies that $w_f = 0$ in B', and we obtain

$$\left.\begin{array}{rcl} w_f^- & = & 0 \\ \frac{\partial w_f^-}{\partial \nu} & = & 0 \end{array}\right\} \text{ on } \partial B', \tag{4.6}$$

where we let $+$ and $-$ refer to the trace and normal derivative from outside and inside the domain, respectively. Since $u_0^s(\cdot, x)$ is an $H^1(B)$-solution of the

Helmholtz equation for each $x \in \partial B'$, we see that w_f possesses the same jump conditions as the standard single layer potential on $\partial B'$, namely

$$\left.\begin{array}{rcl} w_f^- &=& w_f^+ \\ \frac{\partial w_f^-}{\partial \nu} &=& \frac{\partial w_f^+}{\partial \nu} + f \end{array}\right\} \text{ on } \partial B'. \tag{4.7}$$

Combining the first lines of (4.6) and (4.7) implies that $w_f^+ = 0$ on $\partial B'$, and hence well-posedness of the exterior Dirichlet problem (4.2)–(4.4) implies that $w_f = 0$ in $\mathbb{R}^d \setminus \overline{B'}$. In particular, we obtain $\frac{\partial w_f^+}{\partial \nu} = 0$ on $\partial B'$, and combining the second lines of (4.6) and (4.7) yields $f = 0$. Observing that the set $\mathcal{V}_0(B')$ is closed in $H^1_{loc}(\mathbb{R}^d \setminus \overline{B'})$, the result is established for B' in place of B.

Now we let $v \in \mathcal{V}_0(B)$, and we consider the extension $\tilde{v}$ of v as a solution of the Helmholtz equation in $B \setminus \overline{B'}$. This extension is possible since B is a ball and B' contains the origin, which may be seen by representing v in terms of spherical wave functions. By our first result, there exists a sequence $\{g_j\}$ in $L^2(\partial C)$ such that $\tilde{v}_{g_j} \to \tilde{v}$ in $H^1_{loc}(\mathbb{R}^d \setminus \overline{B'})$ as $j \to \infty$. Since $\mathbb{R}^d \setminus \overline{B} \subseteq \mathbb{R}^d \setminus \overline{B'}$, we clearly obtain that $v_{g_j} \to v$ in $H^1_{loc}(\mathbb{R} \setminus \overline{B})$, and the final result is established. □

The proof of the above lemma clearly holds for any such domain B for which the utilized extension property holds. For simplicity, we will assume that B is a ball contained within D for the remainder of the paper. We now provide two theorems which relate modified exterior transmission eigenvalues to the range of G, and we first remark that $G : \overline{R(H)} \to L^2(\partial C)$ is compact by the compactness of the trace operator into $L^2(\partial C)$.

Theorem 4.2. *If η is not a modified exterior transmission eigenvalue, then $\Phi(\cdot, z) \in R(G)$ whenever $z \in B_1 \setminus \overline{B}$.*

Proof. Since η is not a modified exterior transmission eigenvalue, the Fredholm property of the modified exterior transmission problem implies the existence of a unique pair (w_z, v_z) such that w_z and v_z satisfy the radiation condition and

$$\nabla \cdot A \nabla w_z + k^2 n w_z = 0 \text{ in } \mathbb{R}^d \setminus \overline{B}, \tag{4.8a}$$

$$\nabla \cdot A_0 \nabla v_z + k^2 n_0 v_z = 0 \text{ in } \mathbb{R}^d \setminus \overline{B}, \tag{4.8b}$$

$$w_z - v_z = \Phi(\cdot, z) \text{ on } \partial B, \tag{4.8c}$$

$$\frac{\partial w_z}{\partial \nu_A} - \frac{\partial v_z}{\partial \nu_{A_0}} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \text{ on } \partial B. \tag{4.8d}$$

We extend $w_z^* := w_z - v_z$ by $\Phi(\cdot, z)$ in B. We see from Lemma 4.1 that $v_z \in \overline{R(H)}$, and it follows that $w_z^* \in H^1_{loc}(\mathbb{R}^d)$ satisfies (4.1) with $\psi = v_z$. By construction we observe that $G v_z = w_z^*|_{\partial C} = \Phi(\cdot, z)|_{\partial C}$, and we conclude that $\Phi(\cdot, z) \in R(G)$. □

Theorem 4.3. *If η is a modified exterior transmission eigenvalue, then the set of points $z \in B_1 \setminus \overline{B}$ such that $\Phi(\cdot, z) \in R(G)$ is nowhere dense in $B_1 \setminus \overline{B}$.*

Proof. Suppose that η is a modified exterior transmission eigenvalue, and suppose to the contrary that $\Phi(\cdot, z) \in R(G)$ for z in a dense subset of a ball $B_\rho \subseteq B_1 \setminus \overline{B}$. It follows that for each such z there exists $v_z \in \overline{R(H)}$ such that $Gv_z = \Phi(\cdot, z)$, and if $w_z^* \in H^1_{loc}(\mathbb{R}^d)$ is the solution of (4.1) with $\psi = v_z$, then by definition of G we have that $w_z^*|_{\partial C} = Gv_z = \Phi(\cdot, z)|_{\partial C}$. Since both w_z^* and $\Phi(\cdot, z)$ satisfy the Helmholtz equation in B, it follows from Lemma 2.3 that $w_z^* = \Phi(\cdot, z)$ in B. Thus, we see from Lemma 4.1 that $(w_z, v_z) := (w_z^* + v_z, v_z)$ satisfies the modified interior transmission problem (4.8a)–(4.8d). Now we choose an eigenfunction pair (w_η, v_η) of the modified exterior transmission problem corresponding to η. We see from an application of Green's second identity over $B_R \setminus \overline{B}$ that

$$\int_{\partial B} \left(w_z \frac{\partial w_\eta}{\partial \nu_A} - w_\eta \frac{\partial w_z}{\partial \nu_A} \right) ds = \int_{B_R \setminus \overline{B}} \Big(w_z \nabla \cdot A \nabla w_\eta - w_\eta \nabla \cdot A \nabla w_z \Big) dx + \int_{\partial B_R} \left(w_z \frac{\partial w_\eta}{\partial \nu_A} - w_\eta \frac{\partial w_z}{\partial \nu_A} \right) ds.$$

Since w_z and w_η are both radiating solutions to the Helmholtz equation in $\mathbb{R}^d \setminus \overline{B}_R$, we see that the integral over ∂B_R vanishes. Moreover, as w_z and w_η satisfy the same equation in $B_R \setminus \overline{B}$, we observe that the integral over $B_R \setminus \overline{B}$ vanishes as well. These results along with similar computations for v_z and v_η yield

$$\int_{\partial B} \left(w_z \frac{\partial w_\eta}{\partial \nu_A} - w_\eta \frac{\partial w_z}{\partial \nu_A} \right) ds = 0, \tag{4.9a}$$

$$\gamma \int_{\partial B} \left(v_z \frac{\partial v_\eta}{\partial \nu} - v_\eta \frac{\partial v_z}{\partial \nu} \right) ds = 0. \tag{4.9b}$$

By subtracting (4.9a) and (4.9b) and enforcing the boundary conditions, we see that

$$v_\eta^i(z) := \int_{\partial B} \left[\gamma \Phi(\cdot, z) \frac{\partial v_\eta}{\partial \nu} - v_\eta \frac{\partial \Phi(\cdot, z)}{\partial \nu} \right] ds, \; z \in \mathbb{R}^d \setminus \overline{B},$$

vanishes in a dense subset of B_ρ, and by analyticity it follows that v_η^i is identically zero in $\mathbb{R}^d \setminus \overline{B}$. We further define

$$v_\eta^s(y) := - \int_{\partial B} \left[\gamma \Phi(\cdot, y) \frac{\partial v_\eta}{\partial \nu} - v_\eta \frac{\partial \Phi(\cdot, y)}{\partial \nu} \right] ds, \; y \in B,$$

and we observe that (v_η, v_η^s) satisfies (2.2a)–(2.2e) with $u^i = v_\eta^i = 0$. Since this problem is well-posed, we conclude that $v_\eta = 0$ in $\mathbb{R}^d \setminus \overline{B}$. A similar argument shows that $w_\eta = 0$ as well, which contradicts the choice of (w_η, v_η) as an eigenpair associated with η. □

4.1. Factorization of the auxiliary near field operator

We now provide a characterization of the range of G in terms of an indicator function which, when combined with the above two theorems, provides a method to compute eigenvalues from measured scattering data. In particular, we construct a suitable indicator function using the recently developed generalized linear sampling method (GLSM). However, unlike previous results which allow for the computation of eigenvalues for the case of far field measurements [2], we will see that the auxiliary near field operator N_0 does not have a symmetric factorization. It is possible to define the near field operator using non-physical incident fields (i.e. $u^i = \overline{\Phi(\cdot, y)}$ for $y \in \partial C$) as in [17] in order to obtain a symmetric factorization of the near field operator, but the formulation of the resulting exterior transmission problem is not entirely clear and resists standard techniques of study. Thus, we embrace the nonsymmetric factorization described below and introduce a new cost functional similar to that developed in [4] in the context of shape reconstruction.

We first derive a certain factorization of the near field operator which is essential to the application of GLSM, and for convenience we define the product space $\mathcal{L} := L^2(B_1 \setminus \overline{B}) \times L^2(B_1 \setminus \overline{B})$ with the usual inner product $(\cdot,\cdot)_{\mathcal{L}}$ and induced norm $\|\cdot\|_{\mathcal{L}}$. We begin by defining the single layer potential operator $S : L^2(\partial C) \to \mathcal{L}$ by $Sg := (\nabla u_g, u_g)$, where u_g is the single layer potential given by (2.6). We also define the solution operator $G_0 : \overline{R(S)} \to L^2(\partial C)$ by $G_0(\varphi, \psi) := v^*|_{\partial C}$, where $v^* \in H^1_{loc}(\mathbb{R}^d)$ is the radiating solution of

$$\nabla \cdot A_0 \nabla v^* + k^2 n_0 v^* = \nabla \cdot (I - A_0)\nabla\varphi + k^2(1 - n_0)\psi \text{ in } \mathbb{R}^d, \tag{4.10}$$

and we observe that we have the factorization $N_0 = G_0 S$. We see from the definition of S that its adjoint is given by

$$(S^*(\varphi, \psi))(x) = \int_{B_1\setminus\overline{B}} \left[\nabla_y \overline{\Phi(y,x)} \cdot \varphi(y) + \overline{\Phi(y,x)}\psi(y)\right] dy,\ x \in \partial C, \tag{4.11}$$

for all $(\varphi, \psi) \in \mathcal{L}$. Applying Green's formula to the solution v^* of (4.10) yields the representation

$$\begin{aligned} v^*(x) = \int_{B_1\setminus\overline{B}} \Big[&\nabla_y \Phi(y,x) \cdot (I - A_0)(\nabla v^*(y) + \varphi(y)) \\ &+ \Phi(y,x) \cdot k^2(n_0 - 1)(v^*(y) + \psi(y))\Big] dy,\ x \in \partial C. \end{aligned} \tag{4.12}$$

Due to the presence of the complex conjugates in (4.11), we see that we will not be able to obtain a symmetric factorization of N_0. However, if we define the operator $\overline{S} : L^2(\partial C) \to \mathcal{L}$ as for S but with $\Phi(x, y)$ replaced with $\overline{\Phi(x,y)}$ in the definition of the single layer potential, then we obtain the factorization $N_0 = \overline{S}^* TS$, where the middle operator $T : \mathcal{L} \to \mathcal{L}$ is defined by

$$T(\varphi, \psi) := ((I - A_0)(\nabla v^* + \varphi), k^2(n_0 - 1)(v^* + \psi)) \tag{4.13}$$

with v^* the solution of (4.10). We first establish a necessary coercivity property of T in the following lemma, the proof of which requires the following assumption. The reason for including the space $P_1(\overline{R(S)})$ will become clear in Remark 4.6.

Assumption 4.4. *Define the operator $P_1 : \overline{R(S)} \to H^1(B_1 \setminus \overline{B})$ such that $P_1(\nabla u, u) = u$, and assume that k, A_0, and n_0 are such that the nonstandard interior transmission problem of finding $(w, u, v) \in H^1(B_1 \setminus \overline{B}) \times P_1(\overline{R(S)}) \times H^1(B)$ which satisfy*

$$\nabla \cdot A_0 \nabla w + k^2 n_0 w = 0 \text{ in } B_1 \setminus \overline{B}, \tag{4.14a}$$

$$\Delta u + k^2 u = 0 \text{ in } B_1 \setminus \overline{B}, \tag{4.14b}$$

$$\Delta v + k^2 v = 0 \text{ in } B, \tag{4.14c}$$

$$w - u = 0 \text{ on } \partial B_1, \tag{4.14d}$$

$$\frac{\partial w}{\partial \nu_{A_0}} - \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_1, \tag{4.14e}$$

$$w - u = v \text{ on } \partial B, \tag{4.14f}$$

$$\frac{\partial w}{\partial \nu_{A_0}} - \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \partial B, \tag{4.14g}$$

has only the trivial solution.

Lemma 4.5. *If $\gamma \neq 1$ and Assumption 4.4 is satisfied, then the operator $T : \mathcal{L} \to \mathcal{L}$ defined in* (4.13) *is coercive on $\overline{R(S)}$.*

Proof. The first part of the proof follows in a manner similar to Theorem 2.42 in [6]. We first remark from [7, Lemma 5.1] that the closure of the range of S in $\mathcal{L}$ is given by

$$\overline{R(S)} = \left\{ (\nabla u, u) \,\middle|\, u \in H^1(B_1 \setminus \overline{B}),\ \Delta u + k^2 u = 0 \text{ in } B_1 \setminus \overline{B} \right\}. \tag{4.15}$$

For $(\nabla u, u) \in \overline{R(S)}$ and $v^* \in H^1_{loc}(\mathbb{R}^d)$ the solution of (4.10) with $(\varphi, \psi) = (\nabla u, u)$, we have

$$(T(\nabla u, u), (\nabla u, u))_{B_1 \setminus \overline{B}} = \int_{B_1 \setminus \overline{B}} \left[(I - A_0)\nabla(v^* + u) \cdot \nabla \overline{u} + k^2 (n_0 - 1)(v^* + u)\overline{u} \right] dx$$

by definition of T. From (4.10) we see that

$$\nabla v^* + k^2 v^* = \nabla \cdot (I - A_0)\nabla(v^* + u) + k^2(1 - n_0)(v^* + u) \text{ in } \mathbb{R}^d,$$

and after multiplying both sides by $\overline{v^*}$ and integrating by parts over a ball $B_R \supset B_1$ we obtain

$$\int_{B_1 \setminus \overline{B}} \left[(I - A_0)\nabla(v^* + u) \cdot \nabla \overline{v^*} + k^2 (n_0 - 1)(v^* + u)\overline{v^*} \right] dx$$
$$= \int_{B_R} \left[|\nabla v^*|^2 - k^2 |v^*|^2 \right] dx - \int_{\partial B_R} \frac{\partial v^*}{\partial r} \overline{v^*} ds.$$

Taking the imaginary part of this expression and letting $R \to \infty$ gives

$$\mathrm{Im} \int_{B_1 \setminus \overline{B}} \left[(I - A_0)\nabla(v^* + u) \cdot \nabla\overline{v^*} + k^2(n_0 - 1)(v^* + u)\overline{v^*} \right] dx = -k \int_{\mathbb{S}^{d-1}} |v^*_\infty|^2 \, ds,$$

where v^*_∞ is the far field pattern of the radiating field v^*. The identities

$$\begin{aligned} (v^* + u)\overline{u} &= |v^* + u|^2 - (v^* + u)\overline{v^*} \\ (I - A_0)\nabla(v^* + u) \cdot \nabla\overline{u} &= (I - A_0)\nabla(v^* + u) \cdot \nabla(\overline{v^* + u}) \\ &\quad - (I - A_0)\nabla(v^* + u) \cdot \nabla\overline{v^*} \end{aligned}$$

then provide the result

$$\begin{aligned} \mathrm{Im}(T(\nabla u, u), (\nabla u, u))_{B_1 \setminus \overline{B}} = &- \int_{B_1 \setminus \overline{B}} \mathrm{Im}(A_0)\nabla(v^* + u) \cdot \nabla(\overline{v^* + u}) ds \\ &+ k^2 \int_{B_1 \setminus \overline{B}} \mathrm{Im}(n_0) \, |v^* + u|^2 \, dx \\ &+ k \int_{\mathbb{S}^{d-1}} |v^*_\infty|^2 \, ds. \end{aligned} \tag{4.16}$$

We are now ready to show the desired coercivity property of T. Suppose to the contrary that there exists a sequence $\{(\nabla u_j, u_j)\}$ in $\overline{R(S)}$ such that $\|u_j\|_{H^1(B_1 \setminus \overline{B})} = 1$ for all $j \in \mathbb{N}$ and

$$(T(\nabla u_j, u_j), (\nabla u_j, u_j))_{B_1 \setminus \overline{B}} \to 0 \text{ as } j \to \infty. \tag{4.17}$$

For each $j \in \mathbb{N}$, let $v^*_j \in H^1_{loc}(\mathbb{R}^d)$ be the solution of (4.10) with $(\varphi, \psi) = (\nabla u_j, u_j)$, from which elliptic regularity implies that the sequence $\{v^*_j\}$ is bounded in $H^2(K \setminus \overline{B}_1)$ for any bounded domain $K \supset B_1$, and consequently, up to changing the initial sequence, we may assume that $\{u_j\}$ converges weakly to some u in $H^1(B_1 \setminus \overline{B})$ and $\{v^*_j\}$ converges weakly to some v^* in $H^1_{loc}(\mathbb{R}^d) \cap H^2_{loc}(\mathbb{R}^d \setminus \overline{B}_1)$. We immediately observe that v^* satisfies (4.10) with $(\varphi, \psi) = (\nabla u, u)$ and that u satisfies the Helmholtz equation in $B_1 \setminus \overline{B}$. In view of the representation (4.16), assumption (4.17) implies that $v^*_{j,\infty} \to 0$ in $L^2(\mathbb{S}^{d-1})$ as $j \to \infty$ and consequently $v^*_\infty = 0$. It follows from Rellich's lemma that $v^* = 0$ in $\mathbb{R}^d \setminus \overline{B}_1$. Upon labeling $w = v^* + u$ in $B_1 \setminus \overline{B}$ and $v = v^*$ in B for convenience, we see that $(w, u, v) \in H^1(B_1 \setminus \overline{B}) \times P_1(\overline{R(S)}) \times H^1(B)$ satisfies (4.14a)–(4.14g), from which Assumption 4.4 implies that $w = u = 0$ and consequently $v^* = 0$ as well. Finally, assuming that $\gamma \neq 1$ we may follow the rest of the proof of Theorem 2.42 in [6] exactly to obtain $u_j \to 0$ in $H^1(B_1 \setminus \overline{B})$, a contradiction. Therefore, the operator T is coercive on $\overline{R(S)}$. □

Remark 4.6. A requirement such as Assumption 4.4 is common in order to establish coercivity of this type of middle operator T (c.f. Theorem 2.42 in [6]), and it may indeed be possible to apply analytic Fredholm theory to the nonstandard interior transmission problem (4.14a)–(4.14g) in order to show that the problem is well-posed for all k except in a discrete set as has been done for other such problems. Unfortunately, we have not been able to obtain this result due to the unusual structure of this problem, which resists standard techniques utilizing T-coercivity (such as that used in the proof of Lemma 3.2). However, we now show that the required coercivity property holds for any $k > 0$ provided A_0 is chosen such that $\mathrm{Im}(A_0)$ is negative definite in some connected relative neighborhood Γ of ∂B_1 in $\overline{B}_1$ and $\eta \neq 0$. Indeed, following the proof of Lemma 4.5 with this choice of A_0, the assumption (4.17) and the representation (4.16) imply that $\nabla(v_j^* + u_j) \to 0$ in $L^2(\Gamma)$ as $j \to \infty$ since $\mathrm{Im}(A_0)$ is negative definite by assumption. Thus, it follows that $\nabla(v^* + u) = 0$ in Γ, or equivalently that $\nabla w = 0$ in Γ. Since Γ is connected, we see that $w = c$ in Γ for some constant c, and (4.14a) implies that $c = 0$ since $n_0 \neq 0$ in Γ. From the unique continuation principle we obtain $w = 0$ in $B_1 \setminus \overline{B}$, and in particular we see that the Cauchy data of w vanishes on both ∂B_1 and ∂B. The boundary conditions (4.14f)–(4.14g) on ∂B imply that $-u = v$ and $-\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}$ on ∂B, and consequently

$$w_0 := \begin{cases} -u & \text{in } B_1 \setminus \overline{B}, \\ v & \text{in } B, \end{cases}$$

is an $H^1(B_1)$-solution of the Helmholtz equation in B_1. From the boundary conditions (4.14d)–(4.14e) on ∂B_1 we see that the Cauchy data of w_0 on ∂B_1 vanishes, and an application of Green's formula implies that $w_0 = 0$ in B_1. In particular we obtain $u = 0$ and $v^* = 0$, and the remainder of the proof follows as above.

4.2. The nonsymmetric generalized linear sampling method

We now show that a nonsymmetric version of GLSM similar to that studied in [4] may be applied in order to compute eigenvalues from measured scattering data. For some $\xi \in (0,1)$ and $\phi \in L^2(\partial C)$, consider the cost functional $J_\alpha(\phi;\cdot) : L^2(\partial C) \times L^2(\partial C) \to \mathbb{R}$ defined as

$$J_\alpha(\phi; g) := \alpha \left|(N_0 g_2, g_1)_{\partial C}\right| + \alpha^{1-\xi} \left\|Sg_2 - \overline{S}g_1\right\|_{\mathcal{L}}^2 + \left\|\mathcal{N}g_2 - \phi\right\|_{\partial C}^2, \tag{4.18}$$

where $g = (g_1, g_2)$. Compared to the cost functional found in [2] used for far field measurements (and hence a symmetric factorization), we have added the extra term $\left\|Sg_2 - \overline{S}g_1\right\|_{\mathcal{L}}^2$ in order to formally ensure that $Sg_2 \approx \overline{S}g_1$, which then results in a penalty term of the form

$$\left|(N_0 g_2, g_1)_{\partial C}\right| = \left|(TSg_2, \overline{S}g_1)_{\mathcal{L}}\right| \approx \left|(TSg_2, Sg_2)_{\mathcal{L}}\right|$$

as in the symmetric case. Though this cost functional may not have a unique minimizer, due to nonnegativity we may define

$$j_\alpha(\phi) := \inf_{g \in L^2(\partial C)} J_\alpha(\phi; g). \tag{4.19}$$

The following theorem will be necessary for the proof of the main theorem of GLSM, and its proof follows in a similar manner to Lemma 3.4 of [4].

Lemma 4.7. *If $\mathcal{N}$ has dense range, then $j_\alpha(\phi) \to 0$ as $\alpha \to 0$ for each $\phi \in L^2(\partial C)$.*

With the notation $g = (g_1, g_2)$ and

$$\mathcal{B}(g) := |(N_0 g_2, g_1)_{\partial C}| + \alpha^{-\xi} \left\| S g_2 - \overline{S} g_1 \right\|_{\mathcal{L}}^2,$$

we establish the following lemma. We first remark that $\overline{R(S)} = \overline{R(\overline{S})}$.

Lemma 4.8. *If T is coercive on $\overline{R(S)}$, then the following relationship between $\mathcal{B}$ and H holds.*

(a) If $\{\mathcal{B}(g^\alpha)\}$ is bounded as $\alpha \to 0$, then $\{H g_2^\alpha\}$ is bounded as $\alpha \to 0$, where $g^\alpha = (g_1^\alpha, g_2^\alpha)$.

(b) If $\{H g_2^\alpha\}$ is bounded as $\alpha \to 0$, then there exists a sequence $\{g_1^\alpha\}$ for which $\{\mathcal{B}(g^\alpha)\}$ is bounded as $\alpha \to 0$, where $g^\alpha = (g_1^\alpha, g_2^\alpha)$.

Proof. For part (a), suppose that $\{\mathcal{B}(g^\alpha)\}$ is bounded by a constant $M > 0$ independent of α as $\alpha \to 0$. From the coercivity property of T (say with coercivity constant μ) and the definition of $\mathcal{B}(g)$ we see that

$$\begin{aligned}
\mu \left\| S g_2^\alpha \right\|_{\mathcal{L}}^2 &\le |(T S g_2^\alpha, S g_2^\alpha)_{\mathcal{L}}| \\
&\le \left|(T S g_2^\alpha, \overline{S} g_1^\alpha)_{\mathcal{L}}\right| + \left|(T S g_2^\alpha, S g_2^\alpha - \overline{S} g_1^\alpha)_{\mathcal{L}}\right| \\
&\le |(N_0 g_2^\alpha, g_1^\alpha)_{\partial C}| + \|T\| \left\| S g_2^\alpha \right\|_{\mathcal{L}} \left\| S g_2^\alpha - \overline{S} g_1^\alpha \right\|_{\mathcal{L}} \\
&\le M + \sqrt{M} \alpha^{\xi/2} \|T\| \left\| S g_2^\alpha \right\|_{\mathcal{L}},
\end{aligned}$$

which implies that $\{S g_2^\alpha\}$ is bounded in $\mathcal{L}$ as $\alpha \to 0$. By the definition of S we see that the sequence $\{u_{g_2^\alpha}\}$ of single layer potentials is bounded in $H^1(B_1 \setminus \overline{B})$ as $\alpha \to 0$, and well-posedness of (2.2a)–(2.2e) implies that $\{H g_2^\alpha\}$ is bounded in $\mathcal{L}$ as $\alpha \to 0$.

For part (b), suppose that $\{H g_2^\alpha\} = \{v_{g_2^\alpha}\}$ is bounded as $\alpha \to 0$. Since each single layer potential $u_{g_2^\alpha}$ is a radiating solution of the Helmholtz equation in $\mathbb{R}^d \setminus \overline{B}$, we may apply Green's formula along with the boundary conditions (2.2c)–(2.2d) to obtain

$$\begin{aligned}
u_{g_2^\alpha}(x) = &\int_{\partial B} \left[v_{g_2^\alpha}(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial v_{g_2^\alpha}(y)}{\partial \nu_{A_0}} \Phi(x,y) \right] ds(y) \\
&- \int_{\partial B} \left[v_{g_2^\alpha}^s(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial v_{g_2^\alpha}^s(y)}{\partial \nu} \Phi(x,y) \right] ds(y)
\end{aligned}$$

for $x \in B_1 \setminus \overline{B}$. Moreover, each $v_{g_2^\alpha}^s$ satisfies the Helmholtz equation in B, and an application of Green's second identity implies that the second integral vanishes. Thus, it follows from our assumption that $\{u_{g_2^\alpha}\}$ is bounded in $H^1(B_1 \setminus \overline{B})$ as $\alpha \to 0$, and equivalently $\{S g_2^\alpha\}$ is bounded in $\mathcal{L}$ as $\alpha \to 0$. Since $\overline{R(S)} = \overline{R(\overline{S})}$, we may choose $g_1^\alpha \in L^2(\partial C)$ such that

$$\left\| S g_2^\alpha - \overline{S} g_1^\alpha \right\|_{\mathcal{L}}^2 < \alpha^\xi. \tag{4.20}$$

Then we see that for $g^\alpha = (g_1^\alpha, g_2^\alpha)$ we have

$$\begin{aligned}\alpha\mathcal{B}(g^\alpha) &= \alpha\left|(N_0 g_2^\alpha, g_1^\alpha)_{\partial C}\right| + \alpha^{1-\xi}\left\|Sg_2^\alpha - \overline{S}g_1^\alpha\right\|_{\mathcal{L}}^2\\ &\leq \alpha\left|(N_0 g_2^\alpha, g_1^\alpha)_{\partial C}\right| + \alpha.\end{aligned}$$

Finally, we use the fact that T is bounded to obtain

$$\begin{aligned}\left|(N_0 g_2^\alpha, g_1^\alpha)_{\partial C}\right| &= \left|(TSg_2^\alpha, \overline{S}g_1^\alpha)_{\mathcal{L}}\right|\\ &\leq \left|(TSg_2^\alpha, Sg_2^\alpha)_{\mathcal{L}}\right| + \left|(TSg_2^\alpha, \overline{S}g_1^\alpha - Sg_2^\alpha)_{\mathcal{L}}\right|\\ &\leq \|T\|\,\|Sg_2^\alpha\|_{\mathcal{L}}^2 + \|T\|\,\|Sg_2^\alpha\|_{\mathcal{L}}\left\|Sg_2^\alpha - \overline{S}g_1^\alpha\right\|_{\mathcal{L}}\\ &< \|T\|\,\|Sg_2^\alpha\|_{\mathcal{L}}\left(\|Sg_2^\alpha\|_{\mathcal{L}} + \alpha^{\xi/2}\right),\end{aligned}$$

which is bounded above by some constant independent of α as $\alpha \to 0$. Thus, we conclude that the sequence $\{\mathcal{B}(g^\alpha)\}$ is bounded as $\alpha \to 0$. □

This lemma may be combined with a simple modification of the proof of Theorem 1 in the appendix of [2] to prove the main theorem of this section, which relates the range of G to the GLSM functional $J_\alpha(\phi;\cdot)$.

Theorem 4.9. *In addition to the assumptions of Lemma 4.8, assume that $\mathcal{N}$ has dense range. Given some function p such that $\frac{p(\alpha)}{\alpha} = O(1)$, consider a minimizing sequence $\{g^\alpha\}$ satisfying*

$$J_\alpha(\phi; g^\alpha) \leq j_\alpha(\phi) + p(\alpha)$$

for each $\alpha > 0$. Then $\phi \in R(G)$ if and only if the sequence $\{\mathcal{B}(g^\alpha)\}$ is bounded as $\alpha \to 0$.

We may combine this theorem (for $\phi = \Phi(\cdot, z)$) with Theorems 4.2 and 4.3 in order to compute modified exterior transmission eigenvalues from measured scattering data with the generalized linear sampling method. We simply sample a region in the complex plane and compute the indicator function $\mathcal{B}(g^\alpha)$ for some small value of $\alpha > 0$ for each choice of η. The eigenvalues correspond to locations in the complex plane for which $\mathcal{B}(g^\alpha)$ is large. We will see in the next section that in practice we minimize a regularized version of the GLSM cost functional $J_\alpha(\phi;\cdot)$, and we remark that the above results may be appropriately modified to this case following along the lines of [4].

5. Numerical examples

In this section we perform detailed numerical testing of computing modified exterior transmission eigenvalues from measured scattering data, and we investigate their potential use as a target signature in nondestructive testing. In order to generate simulated scattering data to test our methods, we use the finite element software `FreeFem++` [14]. The scattering problem (2.1a)–(2.1e) and the auxiliary scattering problem (2.2a)–(2.2e) are solved by truncating the unbounded domain and imposing the exact boundary conditions in terms of a Dirichlet-to-Neumann operator on a circular artificial boundary (see [6, p. 25] for details). For simplicity we restrict our attention to $\mathbb{R}^2$, but the same process may be carried out in $\mathbb{R}^3$.

In order to construct the approximation to the near field operator, we use N_{inc} point sources y_j, $j = 1, \ldots, N_{inc}$, distributed uniformly on the measurement manifold ∂C, which we assume to be a circle for simplicity, and we compute an $N_{inc} \times N_{inc}$ matrix $\mathbf{U}$ with $\mathbf{U}_{\ell,m} \approx u^s(y_\ell, y_m)$. By the same process we may approximate the auxiliary near field operator N_0 as an $N_{inc} \times N_{inc}$ matrix $\mathbf{U}_0$ with $(\mathbf{U}_0)_{\ell,m} \approx u_0^s(y_\ell, y_m)$, but in the case when B, B_1, and C are chosen to be balls centered at the origin we compute $\mathbf{U}_0$ analytically using separation of variables. In order to add noise to the data, we choose $\delta > 0$ and set

$$\mathbf{U}^\delta_{\ell,m} = \mathbf{U}_{\ell,m}\left(1 + \delta\frac{\zeta_{\ell,m} + \mathrm{i}\mu_{\ell,m}}{\sqrt{2}}\right),\ \ell, m = 1, \ldots, N_{inc},$$

where $\zeta_{\ell,m}$ and $\mu_{\ell,m}$ are uniformly distributed random numbers in $[-1, 1]$ computed using the `rand` command in `MATLAB`. Once the simulated data has been computed with suitable noise added, we compute the data vector $\boldsymbol{\phi}_z$ with ℓth entry given by $(\boldsymbol{\phi}_z)_\ell = \Phi_\infty(y_\ell, z)$, $\ell = 1, \ldots, N_{inc}$, for some $z \in B_1 \setminus \overline{B}$. We now describe our implementation of the generalized linear sampling method to compute modified exterior transmission eigenvalues given the noisy near field matrix $\mathbf{U}^\delta$. We begin by using the trapezoidal rule on ∂C in order to approximate the single layer potential and the conjugate single layer potential as matrices $\mathbf{S}_2$ and $\mathbf{S}_1$, respectively, with each subscript matching the component of g to which the operator is applied in the GLSM cost functional (4.18). We then choose a region in the complex plane in which to sample values of the eigenparameter η in a Cartesian grid, and for each sampled value of η we compute the matrix $\mathbf{U}_0$ and construct the approximation of the noisy modified near field operator $\mathcal{N}^\delta$ as $\boldsymbol{\mathcal{U}}^\delta = \mathbf{U}^\delta - \mathbf{U}_0$.

The GLSM cost functional J given by (4.18) does not have a minimizer in general, and as a result we instead use the regularized cost functional

$$\begin{aligned} J_\alpha^\delta(\phi; g) := \alpha\left|(N_0 g_2, g_1)_{\partial C}\right| + \alpha\delta\left\|g_1\right\|^2_{\partial C} + \alpha\delta\left\|g_2\right\|^2_{\partial C} \\ + \alpha_1\left\|S g_2 - \overline{S} g_1\right\|^2_{B_1\setminus\overline{B}} + \left\|\mathcal{N}^\delta g_2 - \phi\right\|^2_{\partial C}, \end{aligned} \tag{5.1}$$

where the parameter $\delta > 0$ is an estimate of the noise in the data, i.e. if N^δ is the noisy near field operator then δ satisfies $\left\|N^\delta - N\right\| \leq \delta$. Note that we have replaced the term $\alpha^{1-\xi}$ with α_1 for simplicity. In order to construct the discrete regularized cost functional $\mathbf{J}^\delta_\alpha(\cdot)$, we define the matrices

$$\mathbf{N}_0 := \begin{pmatrix} 0 & \mathbf{U}_0 \\ 0 & 0 \end{pmatrix},\ \boldsymbol{\mathcal{N}}^\delta := \begin{pmatrix} 0 & \boldsymbol{\mathcal{U}}^\delta \\ 0 & 0 \end{pmatrix},\ \mathbf{S} := \begin{pmatrix} \mathbf{S}_1 & -\mathbf{S}_2 \end{pmatrix},$$

and the vectors

$$\mathbf{g} := \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix},\ \text{ and } \mathbf{b}_z := \begin{pmatrix} \boldsymbol{\phi}_z \\ 0 \end{pmatrix},$$

where each of $\mathbf{g}_1$ and $\mathbf{g}_2$ is an $N_{inc} \times 1$ vector representing the discrete version of the arguments g_1 and g_2, respectively. We may now write the discrete regularized cost functional as

$$\mathbf{J}_\alpha^\delta(\mathbf{g}) := \alpha\,|\mathbf{g}^*(\mathbf{N}_0\mathbf{g})| + \delta\alpha\mathbf{g}^*\mathbf{g} + \alpha_1(\mathbf{S}\mathbf{g})^*(\mathbf{S}\mathbf{g}) + (\boldsymbol{\mathcal{N}}^\delta\mathbf{g} - \boldsymbol{b}_z)^*(\boldsymbol{\mathcal{N}}^\delta\mathbf{g} - \boldsymbol{b}_z), \quad (5.2)$$

where * refers to the Hermitian transpose. This cost functional is difficult to minimize in $X := \mathbb{C}^{2N_{inc}}$ as it is neither differentiable nor convex [3], and we follow similar procedures to those found in [4]. We first choose the two components $\mathbf{g}_{0,1}, \mathbf{g}_{0,2} \in \tilde{X} := \mathbb{C}^{N_{inc}}$ of the starting point $\mathbf{g}_0$ as

$$\mathbf{g}_{0,2} = \arg\min_{\mathbf{g}\in\tilde{X}}\Big(\beta_2\,\|\mathbf{g}\|^2 + \left\|\boldsymbol{\mathcal{N}}^\delta\mathbf{g} - \mathbf{b}_z\right\|^2\Big),$$
$$\mathbf{g}_{0,1} = \arg\min_{\mathbf{g}\in\tilde{X}}\Big(\beta_1\,\|\mathbf{g}\|^2 + \|\mathbf{S}_1\mathbf{g} - \mathbf{S}_2\mathbf{g}_{0,2}\|^2\Big),$$

where we choose β_2 and β_1 such that $\delta\,\|\mathbf{g}_{0,2}\| = \left\|\boldsymbol{\mathcal{N}}^\delta\mathbf{g}_{0,2} - \mathbf{b}_z\right\|$ and $\|\mathbf{g}_{0,1}\| = \|\mathbf{g}_{0,2}\|$, respectively. The computation of $\mathbf{g}_{0,2}$ is accomplished using the well-known Morozov discrepancy principle and corresponds to the standard linear sampling method; indeed, the linear sampling method would have us plot $\|\mathbf{g}_{0,2}\|$ for various values of η in order to detect eigenvalues. The choice of β_1 is not entirely necessary, but it is designed to avoid choosing $\mathbf{g}_{0,1}$ with large norm relative to the other terms. Once the initial point $\mathbf{g}_0$ is chosen along with regularization parameters β_1 and β_2, we must choose the value of α and α_1 in addition to a suitable optimization algorithm in order to minimize $\mathbf{J}_\alpha^\delta(\cdot)$. We adopt a heuristic from [4] in which we set $\alpha_3 = 1$ and $\alpha = \max(\beta_1, \beta_2)/\,\|\mathbf{N}_0\|$, and we use limited memory BFGS from the Complex Optimization Toolbox [21] described in [20] (with the conjugate cogradient approximated by a finite difference scheme) in order to compute the minimizer $\mathbf{g}_\eta^{\mathrm{glsm}}$. We then evaluate the indicator function

$$\mathbf{I}(\mathbf{g}) := |\mathbf{g}^*(\mathbf{N}_0\mathbf{g})| + \delta\mathbf{g}^*\mathbf{g} + \frac{\alpha_1}{\alpha}(\mathbf{S}\mathbf{g})^*(\mathbf{S}\mathbf{g}) \quad (5.3)$$

at $\mathbf{g} = \mathbf{g}_\eta^{\mathrm{glsm}}$. For each sampled value of η, we repeat this process for 5 random choices of z in $B_1 \setminus \overline{B}$. By plotting the values of the indicator function evaluated at each minimizer $\mathbf{g}_\eta^{\mathrm{glsm}}$ (averaged over the randomly chosen z) against η in the sampled region of the complex plane, we obtain a contour map whose peaks should correspond to the modified exterior transmission eigenvalues. Exact eigenvalues may be computed for the case when C, B, B_1, D, and D_1 are all balls centered at the origin using separation of variables and a suitable root-finding scheme, and we use this procedure in order to test our implementation of the generalized sampling method.

We remark that although the theory indicates that we should evaluate the GLSM indicator function $\mathbf{I}$ at the minimizer $\mathbf{g}_\eta^{\mathrm{glsm}}$, it has been noted that when applying generalized linear sampling to shape reconstruction it often suffices to evaluate $\mathbf{I}(\mathbf{g}_0)$ and hence avoid minimizing the GLSM cost functional [6]. Our

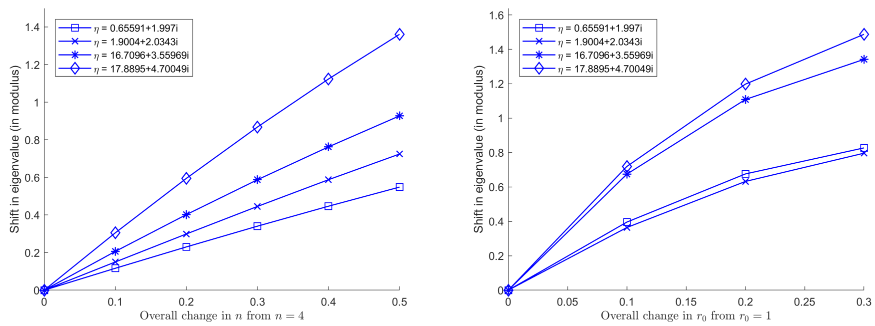
Order	Smallest eigenvalue	Second-smallest eigenvalue
0	0.65591+1.9970i	16.710+3.5597i
1	1.9004+2.0343i	17.890+4.7005i
2	5.3742+0.66951i	26.525+5.7074i
3	8.5088-0.21271i	87.874-0.20658i
4	34.043-0.061097i	14.329-0.042867i
5	41.878-0.0044474i	22.270-0.0035143i
6	52.205-0.00016804i	109.59-0.00026141i
7	65.243-0.0000039840i	121.07-0.0000075302i

Table 1: The two smallest eigenvalues (in magnitude) corresponding to the first few orders of Bessel functions for the annulus $D_1 \setminus \overline{D}$ with $A = I$ and $n = 4$, computed using separation of variables.

findings support this approach when computing eigenvalues as well (c.f. Figure 3), and as for shape reconstruction some optimization can improve the results (c.f. Figure 4). However, we have observed that demanding a high degree of accuracy in the optimization scheme can harm the results, and as a consequence we severely limit the number of iterations performed in the optimization scheme.

We begin by investigating the sensitivity of modified exterior transmission eigenvalues to changes in the material properties of an inhomogeneous medium. In order to allow us to compute exact eigenvalues using separation of variables (in which each eigenvalue is the root of a transcendental function involving Bessel functions of nonnegative integer order), we consider an isotropic medium of constant index of refraction such that D and D_1 are disks of radius 1 and 2, respectively, and we choose $B = D$ and $B_1 = D_1$. We also choose the measurement manifold C to be a disk of radius $r_c = 0.5$ centered at the origin. We consider $A = I$ and $n = 4$, and we choose $\gamma = 2$. The choice of γ greatly affects the distribution and sensitivity of the eigenvalues, and although the precise effect of γ on these properties is not known, we have observed that $\gamma = 2$ appears to be a good choice for this particular scatterer. In Table 1 we first show a few of the smallest eigenvalues computed using separation of variables.

In Figure 2 we show the shift in a few of the smallest eigenvalues (in magnitude) due to both an overall change in the refractive index n and a change in the radius r_0 of the circular cavity. Here we have only shown eigenvalues corresponding to the two lowest orders of Bessel functions when computed by separation of variables (from the first two rows of Table 1), as experience has shown that these are the only eigenvalues likely to be detected in the presence of noise. For a more complicated medium, this association does not exist, and the eigenvalues which are robust to noise must be determined by experimentation. We see that at least one of the eigenvalues exhibits a noticeable shift in each case, and the shift appears to be monotonic for the eigenvalues considered. We note that in Figure 2b we have kept B fixed as the unit disk despite the radius of D increasing. If we were to consider the radius of D decreasing instead, we



(a) Overall change in n (b) Change in the radius r_0 of D

Figure 2: Plots of the shifts in the eigenvalues due to overall changes in n (left) and changes in the radius r_0 of the cavity D (right). We observe that at least one of the eigenvalues exhibits a noticeable shift in each case, and the shift appears to be monotonic for the eigenvalues considered.

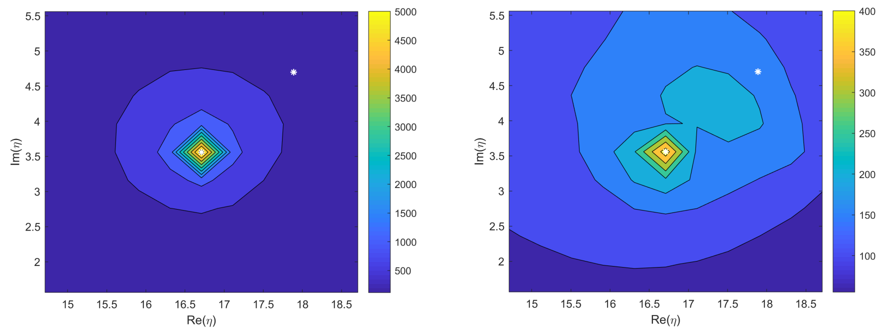
would be required to choose the radius of B sufficiently small to remain in D.

In order to test our method for detecting modified transmission eigenvalues using generalized linear sampling, we focus on the two eigenvalues in Figure 2 that display the greatest change in magnitude in the presence of a flaw. In Figure 3 we show the plot of the indicators $\mathbf{I}(\mathbf{g}_0)$ and $\mathbf{I}(\mathbf{g}_\eta^{\text{glsm}})$ for comparison, where we have added 1.6% percent noise to the data. Here we use a uniform 21×21 grid. We observe that the eigenvalue $\eta = 16.7096 + 3.55969i$ is detected using either of the indicators, and the full GLSM indicator $\mathbf{I}(\mathbf{g}_\eta^{\text{glsm}})$ shows some influence from an additional eigenvalue. In some cases this eigenvalue is roughly detected by the full GLSM indicator, but this detection does not occur often enough to be reliable. We see that the peak in Figure 3a is of greater magnitude, but experience shows that it is not always as localized as the peak in Figure 3b.

We now investigate the effect of a localized flaw on the eigenvalues by considering the same scatterer as in the previous two examples except that the refractive index n is now given by

$$n(x) = \begin{cases} 4, & x \in (D_1 \setminus \overline{D}) \setminus \overline{D_f}, \\ 1, & \text{otherwise}, \end{cases}$$

where D_f is a disk of radius $r_f > 0$ centered at $(x_f, y_f) \in D_1 \setminus \overline{D}$. For this example we choose $r_f = 0.2$ and $(x_f, y_f) = (1.5\cos(\pi/3), 1.5\sin(\pi/3))$, and we use a 21×21 grid on a smaller search region that only contains the eigenvalue of interest. In Figure 4a we see that the location of the shifted eigenvalue is difficult to discern from the plot of the indicator $\mathbf{I}(\mathbf{g}_0)$. The occurrence of this "double peak" has been observed in other similar eigenvalue problems (see Figures 5 and 6 in [8] for example). However, in Figure 4b we clearly see the detection of the eigenvalue corresponding to the flawed medium, and it is shifted relative to the unflawed medium (the eigenvalues of which are represented as white stars in each



(a) $\mathbf{I}(\mathbf{g}_0)$ (b) $\mathbf{I}(\mathbf{g}_\eta^{\text{glsm}})$

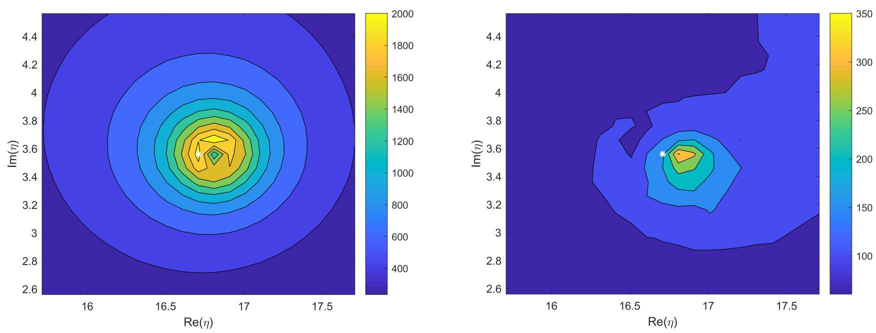
Figure 3: Plots of the indicators $\mathbf{I}(\mathbf{g}_0)$ (left) and $\mathbf{I}(\mathbf{g}_\eta^{\text{glsm}})$ (right). The exact eigenvalues are represented by the white star symbol. We observe that only one eigenvalue is detected in each case.

figure). This example shows that using the full GLSM indicator $\mathbf{I}(\mathbf{g}_\eta^{\text{glsm}})$ has the potential to correct for shortcomings in the initial guess for the optimization scheme and as a result improve the detection of eigenvalues.

As a final test, we investigate the detection of eigenvalues for an anisotropic, absorbing medium in which the cavity is not circular. In particular, we let

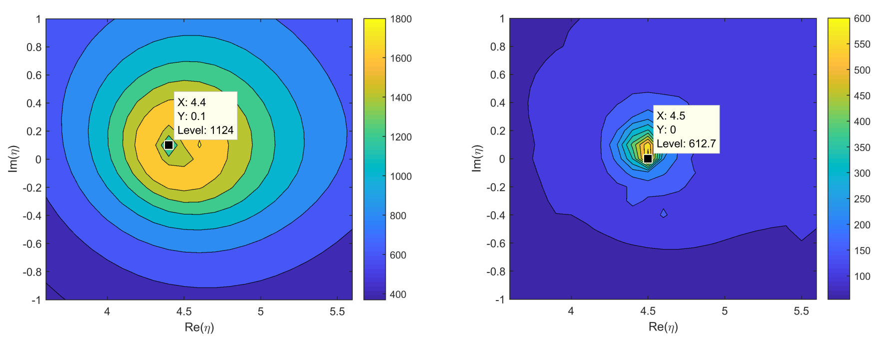
$$A = \begin{pmatrix} 3.5 & -0.25 \\ -0.5 & 3.5 \end{pmatrix}$$

and $n = 4+4i$, and we let D be the square centered at the origin with side length $\sqrt{2}$. We maintain the same choice of D_1 as the disk of radius 2, and we let B and B_1 be the disks of radius 0.7 and 2, respectively. The eigenvalues of A are approximately 3.1464 and 3.8536, and as a result we may still choose $\gamma = 2$. We add approximately 1.6% noise to the data. Since we do not know *a priori* where the eigenvalues are located for this medium, we searched different rectangular regions near the origin until one was found, and Figure 5 shows the detection of this eigenvalue using a 21×21 grid on the region shown for both GLSM indicators. We see that the indicator $\mathbf{I}(\mathbf{g}_\eta^{\text{glsm}})$ displays a more localized peak. In Figure 6 we search over the same grid for an eigenvalue when a circular flaw of radius $r_f = 0.2$ centered at $(x_f, y_f) = (1.5\cos(\pi/3), 1.5\sin(\pi/3))$ has been introduced as we did in the previous example. In both cases we have marked the peak in the contour plot, and we see that there is a shift in the eigenvalue for both indicators. We remark that we used the same noise in both examples in order to avoid its effect on the eigenvalue. In practice one would use an extremely fine grid to determine the eigenvalue for the unflawed medium to a high degree of accuracy despite the noise, allowing for more reliable information from an observed shift in an eigenvalue. However, even our crude example provides some information for this complicated medium.



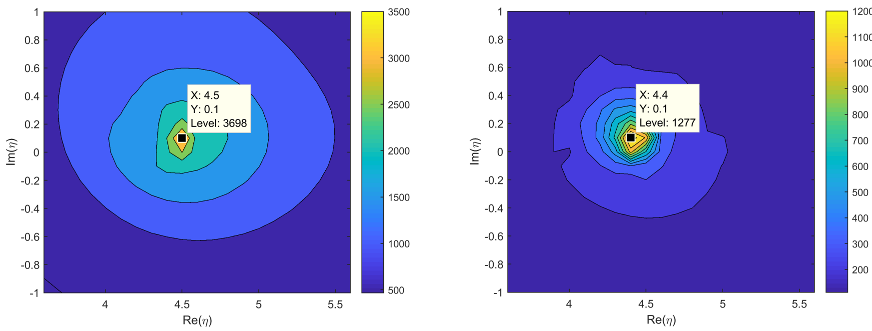
(a) $\mathbf{I}(\mathbf{g}_0)$ (b) $\mathbf{I}(\mathbf{g}_\eta^{\mathrm{glsm}})$

Figure 4: Plots of the indicators $\mathbf{I}(\mathbf{g}_0)$ (left) and $\mathbf{I}(\mathbf{g}_\eta^{\mathrm{glsm}})$ (right). The exact eigenvalues for the unflawed medium (i.e. $D_f = \varnothing$) are shown by the white star symbol. The detection of the shifted eigenvalue is not entirely clear using the indicator $\mathbf{I}(\mathbf{g}_0)$, whereas we clearly see the shifted eigenvalue using the full GLSM indicator $\mathbf{I}(\mathbf{g}_\eta^{\mathrm{glsm}})$.



(a) $\mathbf{I}(\mathbf{g}_0)$ (b) $\mathbf{I}(\mathbf{g}_\eta^{\mathrm{glsm}})$

Figure 5: Plots of the indicators $\mathbf{I}(\mathbf{g}_0)$ (left) and $\mathbf{I}(\mathbf{g}_\eta^{\mathrm{glsm}})$ (right) for the unflawed anisotropic, absorbing media.



(a) $\mathbf{I}(\mathbf{g}_0)$ (b) $\mathbf{I}(\mathbf{g}_\eta^{\mathrm{glsm}})$

Figure 6: Plots of the indicators $\mathbf{I}(\mathbf{g}_0)$ (left) and $\mathbf{I}(\mathbf{g}_\eta^{\mathrm{glsm}})$ (right) for the flawed anisotropic, absorbing media.

6. Conclusion

We have demonstrated that the introduction of a modified near field operator allows for the detection of changes in the material properties of an anisotropic medium through the study of an eigenvalue problem associated with the operator. Our study of this modified exterior transmission problem exhibits some peculiarities when compared to similar methods arising from far field operators. In particular, the problem is not self-adjoint (even for nonabsorbing media), and the auxiliary near field operator has a nonsymmetric factorization. The first issue implies that eigenvalues may be complex, which are detectable due to our use of an artificial eigenparameter η appearing only in the auxiliary problem, and the second issue presents difficulties in applying the current formulation of the generalized linear sampling method for computing eigenvalues, which required us to modify a recently developed nonsymmetric version of generalized linear sampling for this purpose.

While the most important unanswered question concerns the existence of modified exterior transmission eigenvalues, some quantification of the shift in an eigenvalue due to a change in the material properties of the medium (as derived for similar problems in [8] and [11]) would be of immense value, in addition to results on the distribution of the eigenvalues in the complex plane. The latter result would be particularly important for this problem since the eigenvalues are not confined to the upper half-plane as is the case for problems involving modified far field operators. A central question for practical purposes is the effect of the choice of γ on both the sensitivity and distribution of the eigenvalues, beyond our numerical investigations which indicate that some values are better than others. Despite (or perhaps because of) these open questions, this class of eigenvalues is certainly deserving of further study from both a theoretical and practical perspective.

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