

# The Inverse Spectral Problem for Transmission Eigenvalues

Samuel Cogar\*, David Colton<sup>†</sup>, and Yuk-J Leung

Department of Mathematical Sciences, University of Delaware,  
Newark, Delaware 19716, USA

Email: [cogar@udel.edu](mailto:cogar@udel.edu), [colton@udel.edu](mailto:colton@udel.edu) and [yleung@udel.edu](mailto:yleung@udel.edu)

Keywords: inverse scattering, modified far field operator, modified transmission eigenvalues

## Abstract

In this paper, we consider the inverse medium problem of determining the spherically stratified index of refraction  $n(r)$  from given spectral data. We begin by introducing a modified transmission eigenvalue problem depending on a parameter  $\eta$  and an associated modified far field operator. We prove that this operator is injective with dense range provided that  $k$  is not a modified transmission eigenvalue, and we show that  $n(r)$  is uniquely determined by the modified transmission eigenvalues corresponding to  $\eta$  whenever  $0 < n(r) < \eta^2$  for  $0 \leq r \leq 1$ .

## 1. Introduction

In this paper, we revisit the inverse spectral problem for transmission eigenvalues that was previously considered in [1], [2], [8], and [17]. The transmission eigenvalue problem originally arose in inverse scattering theory and has been the subject matter of numerous investigations in recent years. For a survey of recent developments in this area, we refer the reader to the monograph [5]. The transmission eigenvalue problem is a non-selfadjoint boundary value problem for a pair of fields  $w$  and  $v$  in a bounded, simply connected domain  $D$  in  $\mathbb{R}^3$

---

\*supported by the Army Research Office through the National Defense Science and Engineering Graduate Fellowship

<sup>†</sup>supported in part by a grant from the Air Force Office of Scientific Research  
2010 *Mathematics Subject Classification*. Primary: 35P05, 35P25, 35R30.

with sufficiently smooth boundary  $\partial D$  such that

$$\begin{aligned} \Delta w + k^2 n(x)w &= 0 \text{ in } D \\ \Delta v + k^2 v &= 0 \text{ in } D \\ v = w, \frac{\partial v}{\partial \nu} &= \frac{\partial w}{\partial \nu} \text{ on } \partial D \end{aligned} \tag{1.1}$$

where  $k > 0$  is the wave number,  $\nu$  is the unit outward normal to  $\partial D$ , and  $n(x)$ ,  $x \in \mathbb{R}^3$ , is the index of refraction which is continuous in the closure  $\overline{D}$  of  $D$ , satisfies  $n(x) > 0$  for  $x \in \overline{D}$ , and is such that  $n(x) - 1$  has compact support  $\overline{D}$ .

The transmission eigenvalue problem (1.1) arises in a study of the scattering problem

$$\begin{aligned} \Delta u + k^2 n(x)u &= 0 \text{ in } \mathbb{R}^3 \\ u &= e^{ikx \cdot d} + u^s \\ \lim_{r \rightarrow \infty} r \left( \frac{\partial u^s}{\partial r} - ik u^s \right) &= 0 \end{aligned} \tag{1.2}$$

for  $u \in H_{loc}^1(\mathbb{R}^3)$ , where  $r = |x|$ ,  $d$  is a unit vector, and  $u^s$  denotes the scattered field. From (1.2) it is easy to deduce that, for fixed  $k$ ,  $u^s$  has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} \left\{ u_\infty(\hat{x}, d) + O\left(\frac{1}{r}\right) \right\} \tag{1.3}$$

as  $r \rightarrow \infty$  where  $\hat{x} = \frac{x}{|x|}$  [6]. The function  $u_\infty(\hat{x}, d)$  is called the *far field pattern* corresponding to  $u^s$ . From (1.3) we can now define the *far field operator*  $F : L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)$ , where  $\mathbb{S}^2$  is the unit sphere in  $\mathbb{R}^3$ , by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d)g(d)ds(d). \tag{1.4}$$

It is then possible to show [6] that  $F$  is injective with dense range provided  $k$  is not a *transmission eigenvalue*, i.e. a value of  $k$  such that there exists a non-trivial solution to (1.1). Since the far field operator plays a central role in much of the recent developments in inverse scattering theory, the spectral theory of the transmission eigenvalue problem has become a problem of particular interest.

Due to the fact that the transmission eigenvalue problem is non-selfadjoint, a natural question to ask (if we drop the condition that  $k > 0$ ) is do there exist complex eigenvalues? This question is also of importance in attempts to develop the linear sampling method for solving the inverse scattering problem in the time domain [13]. Although the answer to this question remains open in general, for the case when the medium is spherically stratified and the eigenfunctions are also spherically stratified there exists a considerable amount of results establishing conditions under which there exist complex eigenvalues (cf. [7], [8], [9], [16], and Chapter 5 of [5]). Such results are possible since in the

case of spherically stratified media with spherically stratified eigenfunctions the transmission eigenvalue problem (1.1) can be reduced to a problem in ordinary differential equations. In particular, if  $n(x) = n(r)$  is a function only of  $r = |x|$  and  $v$  and  $w$  are spherically symmetric, we can set

$$\begin{aligned} w(x) &= a_0 \frac{y(r)}{r} \\ v(x) &= b_0 \frac{\sin kr}{kr} \end{aligned}$$

where  $a_0, b_0$  are constants and from (1.1) we have that

$$\begin{aligned} y'' + k^2 n(r)y &= 0 \\ y(0) = 0, y'(0) &= 1 \end{aligned} \tag{1.5}$$

where the second initial condition is a normalization condition. Assuming that  $D$  is a ball of radius 1, we can now conclude that  $k$  is a transmission eigenvalue if and only if

$$d(k) := \det \begin{pmatrix} y(1) & \frac{\sin k}{k} \\ y'(1) & \cos k \end{pmatrix} = 0$$

and it can be shown ([7], [8], [9], [16]) that in general  $d(k)$  has complex zeros, i.e. there exist complex transmission eigenvalues in the case of spherically stratified media (we will always restrict our attention to the case when the eigenfunctions are also spherically stratified). In particular, if we know all of the transmission eigenvalues in this case, both real and complex including their multiplicities, does this information uniquely determine  $n(r)$ ? The first results for this problem were obtained by McLaughlin and Polyakov more than twenty years ago [17], where it was shown that uniqueness is obtained for the inverse spectral problem provided  $0 < n(r) < \frac{1}{9}$  under the assumption that  $n(1) = 1$  and  $n'(1) = 0$ . This bound on  $n(r)$  was improved to  $0 < n(r) < 1$  by Aktosun, Gintides, and Papanicolaou in [2] again under the assumption that  $n(1) = 1$  and  $n'(1) = 0$ . Different proofs of this result were given in [1] and [8] (see Chapter 5 of [5] where the condition in [8] that  $n(0)$  is known a priori is removed). The purpose of this paper is to establish uniqueness for the inverse spectral problem not only for  $0 < n(r) < 1$  but for all  $n(r) > 0$  and without assuming that  $n(1)$  and  $n'(1)$  are known. This will be accomplished by introducing a new set of spectral data that is arrived at by considering a modified far field operator instead of (1.4) (see also [12]). In particular, we will consider the *modified far field operator*  $\mathcal{F} : L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)$  defined by

$$(\mathcal{F}g)(\hat{x}) := \int_{\mathbb{S}^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)]g(d)ds(d)$$

where  $h_\infty$  is the far field pattern of the solution to (1.2) with  $n(x)$  replaced by  $\eta^2$ , where  $\eta > 0$  is a constant, and we will show that  $\mathcal{F}$  is injective with dense

range provided  $k$  is not an eigenvalue of the modified transmission problem

$$\begin{aligned} \Delta w + k^2 n(x)w &= 0 \text{ in } D \\ \Delta v + k^2 \eta^2 v &= 0 \text{ in } D \\ v = w, \frac{\partial v}{\partial \nu} &= \frac{\partial w}{\partial \nu} \text{ on } \partial D. \end{aligned} \tag{1.6}$$

Returning now to the special case of a spherically stratified media, it can easily be seen that  $k$  is a modified transmission eigenvalue if and only if

$$\tilde{d}(k) := \det \begin{pmatrix} y(1) & \frac{\sin k\eta}{k\eta} \\ y'(1) & \cos k\eta \end{pmatrix} = 0 \tag{1.7}$$

where  $y(r)$  is as previously defined in (1.5). Note that  $\eta = 1$  corresponds to the standard transmission eigenvalue problem whereas  $\eta \neq 1$  yields a new set of spectral data. It is amusing to note that if we set  $\eta = \frac{\alpha}{k}$  where  $\alpha > 0$  then from (1.7) we have that  $y(r)$  satisfies

$$\begin{aligned} y'' + k^2 n(r)y &= 0 \\ y(0) = 0, y(1) - \frac{\tan \alpha}{\alpha} y'(1) &= 0 \end{aligned}$$

i.e. in this case we have an inverse Sturm-Liouville problem! Hence, by choosing two different values of  $\alpha$ , we can uniquely determine  $n(r)$  (cf. [15, Chapter 4]). However, in this paper we choose  $\eta$  to be independent of  $k$  and show that in this case the eigenvalues corresponding to the modified transmission problem uniquely determine  $n(r)$  provided that  $\eta^2 > n(r)$ .

The plan of our paper is as follows. In the next section of this paper we will introduce the modified far field operator described above and establish conditions for when it is injective with dense range. We will then consider the inverse spectral problem for a spherically stratified medium with spherically stratified eigenfunctions and establish our desired uniqueness theorem for the inverse spectral problem.

## 2. The modified far field operator

In this section we introduce the modified far field operator and the modified interior transmission problem. We note that this operator was previously introduced in [12] where it was used for different purposes than for what is considered here. For  $k > 0$  and a unit vector  $d$  recall the scattering problem

$$\Delta u + k^2 n(x)u = 0 \text{ in } \mathbb{R}^3 \tag{2.1a}$$

$$u = e^{ikx \cdot d} + u^s \tag{2.1b}$$

$$\lim_{r \rightarrow \infty} r \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0 \tag{2.1c}$$

where the refractive index  $n(x)$  has the properties given previously and recall that  $D = \{x \in \mathbb{R}^3 | n(x) \neq 1\}$ . Throughout this section we assume that  $D$  is simply connected with a connected  $C^2$  boundary  $\partial D$  and that  $D$  contains the origin.

We now consider the transmission problem

$$\Delta h_1 + k^2 h_1 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D} \quad (2.2a)$$

$$\Delta h_2 + k^2 \eta^2 h_2 = 0 \text{ in } D \quad (2.2b)$$

$$h_1 = h_2, \quad \frac{\partial h_1}{\partial \nu} = \frac{\partial h_2}{\partial \nu} \text{ on } \partial D \quad (2.2c)$$

$$h_1 = e^{ikx \cdot d} + h_1^s \quad (2.2d)$$

$$\lim_{r \rightarrow \infty} r \left( \frac{\partial h_1^s}{\partial r} - ik h_1^s \right) = 0 \quad (2.2e)$$

where  $\eta > 0$  is a constant. The Sommerfeld radiation conditions (2.1c) and (2.2e) are assumed to hold uniformly in all directions. Note that there exists a unique solution to both (2.1) and (2.2) [6, Theorem 8.7].

**Definition 2.1.** We define the modified far field operator  $\mathcal{F} : L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)$  by

$$(\mathcal{F}g)(\hat{x}) := \int_{\mathbb{S}^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)]g(d)ds(d), \quad \hat{x} \in \mathbb{S}^2,$$

where  $u_\infty$  is the far field pattern corresponding to the scattering problem (2.1) and  $h_\infty$  is the far field pattern corresponding to the transmission problem (2.2).

**Definition 2.2.** Given a solution  $h_1, h_2$  to (2.2), the function defined by

$$H_2(x) := \int_{\mathbb{S}^2} h_2(x, d)g(d)ds(d)$$

for some  $g \in L^2(\mathbb{S}^2)$  is called a generalized Herglotz wave function.

In the case  $\eta = 1$ , we have that the unique solution of (2.2) is  $h_1 = h_2 = e^{ikx \cdot d}$ ,  $h_1^s = 0$ , so  $H_2$  is the standard Herglotz wave function. Furthermore,  $h_1^s = 0$  implies that  $h_\infty = 0$  so  $\mathcal{F}$  is the standard far field operator. The following theorem is an analogue of Theorem 8.9 in [6].

**Theorem 2.3.** The modified far field operator  $\mathcal{F}$  is injective with dense range if and only if there does not exist a nontrivial solution  $w, H_2$  to the modified interior transmission problem

$$\Delta w + k^2 n(x)w = 0 \quad (2.3a)$$

$$\Delta H_2 + k^2 \eta^2 H_2 = 0 \text{ in } D \quad (2.3b)$$

$$w = H_2, \quad \frac{\partial w}{\partial \nu} = \frac{\partial H_2}{\partial \nu} \text{ on } \partial D \quad (2.3c)$$

where  $H_2$  is a generalized Herglotz wave function.

**Proof.** Let  $g$  be in the nullspace of  $\mathcal{F}$ , i.e.

$$\int_{\mathbb{S}^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)]g(d)ds(d) = 0$$

for all  $\hat{x} \in \mathbb{S}^2$ . Define

$$h_g(x) := \int_{\mathbb{S}^2} e^{ikx \cdot d} g(d)ds(d) \text{ for } x \in \mathbb{R}^3.$$

Then

$$w_\infty(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d)g(d)ds(d)$$

is the far field pattern corresponding to the scattering problem

$$\Delta w + k^2 n(x)w = 0 \text{ in } \mathbb{R}^3$$

$$w = h_g + w^s$$

$$\lim_{r \rightarrow \infty} r \left( \frac{\partial w^s}{\partial r} - ikw^s \right) = 0$$

and

$$(H_1)_\infty(\hat{x}) := \int_{\mathbb{S}^2} h_\infty(\hat{x}, d)g(d)ds(d)$$

is the far field pattern corresponding to the transmission problem

$$\Delta H_1 + k^2 H_1 = 0 \text{ in } \mathbb{R}^3 \setminus \bar{D}$$

$$\Delta H_2 + k^2 \eta^2 H_2 = 0 \text{ in } D$$

$$H_1 = H_2, \quad \frac{\partial H_1}{\partial \nu} = \frac{\partial H_2}{\partial \nu} \text{ on } \partial D$$

$$H_1 = h_g + H_1^s$$

$$\lim_{r \rightarrow \infty} r \left( \frac{\partial H_1^s}{\partial r} - ikH_1^s \right) = 0$$

where

$$H_2(x) := \int_{\mathbb{S}^2} h_2(x, d)g(d)ds(d)$$

is a generalized Herglotz wave function. Since

$$w_\infty(\hat{x}) - (H_1)_\infty(\hat{x}) = \int_{\mathbb{S}^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)]g(d)ds(d) = 0$$

for all  $\hat{x} \in \mathbb{S}^2$  we have that  $w_\infty = (H_1)_\infty$  and hence  $w^s = H_1^s$  in  $\mathbb{R}^3 \setminus D$  by Rellich's lemma. In particular, since  $w$  and  $H_1$  arise from the same incident field,  $w = H_1$  in  $\mathbb{R}^3 \setminus D$  so the Cauchy data for  $w$  and  $H_1$  coincide. Therefore,

the Cauchy data for  $w$  and  $H_2$  coincide so  $w$  and  $H_2$  satisfy the modified interior transmission problem

$$\begin{aligned}\Delta w + k^2 n(x)w &= 0 \text{ in } D \\ \Delta H_2 + k^2 \eta^2 H_2 &= 0 \text{ in } D \\ w = H_2, \frac{\partial w}{\partial \nu} &= \frac{\partial H_2}{\partial \nu} \text{ on } \partial D.\end{aligned}$$

If  $g$  is nonzero then the solution  $w, H_2$  is nontrivial so we have shown that if  $\mathcal{F}$  is not injective then there exists a nontrivial solution  $w, H_2$  to the modified interior transmission problem with  $H_2$  a generalized Herglotz wave function.

Conversely, suppose that  $w, H_2$  satisfies (2.3a)–(2.3c) where

$$H_2(x) = \int_{\mathbb{S}^2} h_2(x, d)g(d)ds(d)$$

is a generalized Herglotz wave function for some nonzero  $g \in L^2(\mathbb{S}^2)$ . Define

$$h_g(x) := \int_{\mathbb{S}^2} e^{ikx \cdot d} g(d)ds(d), \quad u_g^s(x) := \int_{\mathbb{S}^2} u^s(x, d)g(d)ds(d) \text{ for } x \in \mathbb{R}^3 \setminus \overline{D},$$

and let  $w := h_g + u_g^s$  in  $\mathbb{R}^3 \setminus \overline{D}$ . We first show that this extension of  $w$  provides a solution of  $\Delta w + k^2 n(x)w = 0$  such that  $w$  is in the Sobolev space  $H_{loc}^2(\mathbb{R}^3)$ . Choose a ball  $B \subset \mathbb{R}^3$  such that  $\overline{D} \subseteq B$ . Since  $w$  satisfies the Helmholtz equation in  $\mathbb{R}^3 \setminus \overline{D}$ , we may apply Green's formula to  $w$  in the region  $B \setminus \overline{D}$  to obtain

$$\begin{aligned}w(x) &= \int_{\partial(B \setminus \overline{D})} \left[ w(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y) \Phi(x, y) \right] ds(y) \\ &= \int_{\partial B} \left[ w(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y) \Phi(x, y) \right] ds(y) \\ &\quad - \int_{\partial D} \left[ w(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y) \Phi(x, y) \right] ds(y)\end{aligned}\tag{2.4}$$

for all  $x \in B \setminus \overline{D}$ . Applying Green's second identity to the second integral of (2.4) we have that

$$\begin{aligned}\int_{\partial D} \left[ w(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y) \Phi(x, y) \right] ds(y) \\ &= \int_D \left[ w(y) \Delta_y \Phi(x, y) - \Phi(x, y) \Delta w(y) \right] dy \\ &= \int_D \left[ -k^2 w(y) \Phi(x, y) + k^2 n(y) w(y) \Phi(x, y) \right] dy \\ &= - \int_D k^2 m(y) w(y) \Phi(x, y) dy,\end{aligned}$$

where  $m := 1 - n$ . Thus

$$w(x) = \int_D k^2 m(y) w(y) \Phi(x, y) dy + \int_{\partial B} \left[ w(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y) \Phi(x, y) \right] ds(y) \quad (2.5)$$

for all  $x \in B \setminus \overline{D}$ . Since the first term in (2.5) is a volume potential and  $mw \in L^2(D)$  it is in  $H_{loc}^2(\mathbb{R}^3)$  [6, Theorem 8.2]. For the second term in (2.5),  $x \in B$  and  $y \in \partial B$  so this term is infinitely differentiable and hence  $w$  is in  $H^2(B)$ . Since  $B$  is a ball of arbitrarily large radius,  $w \in H_{loc}^2(\mathbb{R}^3)$  as desired. Now, defining

$$H_1(x) := \int_{\mathbb{S}^2} h_1(x, d) g(d) ds(d), \quad H_1^s(x) := \int_{\mathbb{S}^2} h_1^s(x, d) g(d) ds(d) \text{ for } x \in \mathbb{R}^3 \setminus \overline{D},$$

we observe from (2.2) that  $H_1$  satisfies the transmission problem

$$\begin{aligned} \Delta H_1 + k^2 H_1 &= 0 \text{ in } \mathbb{R}^3 \setminus \overline{D} \\ \Delta H_2 + k^2 \eta^2 H_2 &= 0 \text{ in } D \\ H_1 &= H_2, \quad \frac{\partial H_1}{\partial \nu} = \frac{\partial H_2}{\partial \nu} \text{ on } \partial D \\ H_1 &= h_g + H_1^s \\ \lim_{r \rightarrow \infty} r \left( \frac{\partial H_1^s}{\partial r} - ik H_1^s \right) &= 0. \end{aligned}$$

Thus the Cauchy data of  $w$  and  $H_1$  coincide. Since  $w$  and  $H_1$  arise from the same incident field  $h_g$ , the Cauchy data of their scattered fields  $u_g^s$  and  $H_1^s$  coincide. Applying Green's formula yields  $u_g^s = H_1^s$  in  $\mathbb{R}^3 \setminus \overline{D}$ . Therefore, the far field patterns of  $H_1^s$  and  $u_g^s$  coincide so

$$\int_{\mathbb{S}^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)] g(d) ds(d) = (u_g)_\infty(\hat{x}) - (H_1)_\infty(\hat{x}) = 0$$

for all  $\hat{x} \in \mathbb{S}^2$  and  $\mathcal{F}$  is not injective since  $g$  was assumed to be nonzero. The fact that  $\mathcal{F}$  has dense range follows from the fact that  $\mathcal{F}$  is injective exactly as in the proof of Corollary 1.16 in [5] since both  $u_\infty(\hat{x}, d)$  and  $h_\infty(\hat{x}, d)$  satisfy the reciprocity principle.  $\square$

**Definition 2.4.** *We say that  $k$  is a modified transmission eigenvalue (corresponding to  $\eta$ ) if there exists a nontrivial solution  $w, H_2$  to the modified interior transmission problem (2.3a)–(2.3c).*



With this definition we may restate Theorem 2.3 to say that the modified far field operator  $\mathcal{F}$  is injective with dense range if and only if  $k$  is not a modified transmission eigenvalue corresponding to  $\eta$  with  $H_2$  a generalized Herglotz wave function. (In this regard, see also Theorem 3.1 of [12].) In the case  $\eta = 1$  the modified transmission eigenvalue problem reduces to the standard transmission eigenvalue problem so the modified transmission eigenvalues are precisely the transmission eigenvalues.

### 3. The inverse spectral problem

In this section, we establish an inverse spectral theorem generalizing that of Aktosun and Papanicolaou in [1] and Colton and Leung in [8]. Throughout this section we always assume that  $n(x) = n(r)$  is spherically symmetric with  $n \in C^3[0, 1]$ .

In [1] and [8] the authors considered the (normalized) transmission eigenvalue problem for an isotropic spherically stratified medium in  $\mathbb{R}^3$  of finding nontrivial  $w, v$  satisfying

$$\Delta w + k^2 n(r)w = 0 \text{ in } B \quad (3.1a)$$

$$\Delta v + k^2 v = 0 \text{ in } B \quad (3.1b)$$

$$w = v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \partial B \quad (3.1c)$$

where  $B$  is the open unit ball in  $\mathbb{R}^3$ . They looked for spherically symmetric eigenfunctions

$$w(x) = a_0 \frac{y(r)}{r}, \quad v(x) = b_0 \frac{\sin kr}{kr},$$

where  $a_0, b_0$  are constants, in which case  $y(r)$  satisfies

$$y'' + k^2 n(r)y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad (3.2)$$

and  $k$  is a transmission eigenvalue if and only if the determinant

$$d(k) := \det \begin{pmatrix} y(1) & \frac{\sin k}{k} \\ y'(1) & \cos k \end{pmatrix} \quad (3.3)$$

is zero where now  $k$  is allowed to be complex. The following theorem was then established.

**Theorem 3.1.** *Assume that  $n \in C^3[0, 1]$ ,  $n(1) = 1$ , and  $n'(1) = 0$ . Then, if  $0 < n(r) < 1$  for  $0 \leq r < 1$ , the transmission eigenvalues (including multiplicity) uniquely determine  $n(r)$ .*

Our goal is to prove an inverse spectral theorem that is valid for all  $n(r) > 0$  and does not require the assumptions that  $n(1) = 1$  and  $n'(1) = 0$ . From this point forward we make no assumption about the value of  $n(1)$  or  $n'(1)$ . We

will show that for  $\eta$  sufficiently large the modified transmission eigenvalues are sufficient to determine  $n(1)$ ,  $n'(1)$ , and  $n(r)$ .

For a given constant  $\eta > 0$  we consider the (normalized) modified transmission eigenvalue problem for an isotropic spherically stratified medium in  $\mathbb{R}^3$  of finding nontrivial  $\tilde{w}, \tilde{v}$  satisfying

$$\Delta \tilde{w} + k^2 n(r) \tilde{w} = 0 \text{ in } B \quad (3.4a)$$

$$\Delta \tilde{v} + k^2 \eta^2 \tilde{v} = 0 \text{ in } B \quad (3.4b)$$

$$\tilde{w} = \tilde{v}, \quad \frac{\partial \tilde{w}}{\partial \nu} = \frac{\partial \tilde{v}}{\partial \nu} \text{ on } \partial B \quad (3.4c)$$

where  $B$  is the open unit ball in  $\mathbb{R}^3$ . We look for spherically symmetric eigenfunctions

$$\tilde{w}(x) = \tilde{a}_0 \frac{y(r)}{r}, \quad \tilde{v}(x) = \tilde{b}_0 \frac{\sin k\eta r}{k\eta r},$$

where  $\tilde{a}_0, \tilde{b}_0$  are constants, noting that since (3.1a) and (3.4a) are identical,  $y(r)$  is the solution to (3.2) as before. Then  $k$  is a modified transmission eigenvalue if and only if the determinant

$$\tilde{d}(k) := \det \begin{pmatrix} y(1) & \frac{\sin k\eta}{k\eta} \\ y'(1) & \cos k\eta \end{pmatrix} \quad (3.5)$$

is zero. We assume that  $n(r) < \eta^2$  for  $0 \leq r \leq 1$ .

By an asymptotic analysis similar to that in [10] we have that

$$\tilde{d}(k) = \frac{1}{k[n(0)n(1)]^{1/4}} \left[ \sin k\delta \cos k\eta - \frac{\sqrt{n(1)}}{\eta} \cos k\delta \sin k\eta + O\left(\frac{1}{k}\right) \right] \quad (3.6)$$

as  $k \rightarrow \infty$  along the real line, where

$$\delta := \int_0^1 \sqrt{n(\rho)} d\rho. \quad (3.7)$$

Note that the leading term of the expression in brackets is almost-periodic as defined in [14] and takes both positive and negative values, so if  $\delta \neq \eta$  then there exist infinitely many positive zeros of  $\tilde{d}(k)$  and hence infinitely many positive modified transmission eigenvalues corresponding to  $\eta$ . From (3.6) we can determine  $\delta$  from the modified transmission eigenvalues  $\{\tilde{k}_j\}$  in the following way. Let  $\tilde{D}(k) = [n(0)n(1)]^{1/4} k \tilde{d}(k)$  and rewrite (3.6) as

$$\tilde{D}(k) = A \sin k(\delta + \eta) + B \sin k(\delta - \eta) + O\left(\frac{1}{k}\right) \quad (3.8)$$

where

$$A = \frac{1}{2} \left( 1 - \frac{\sqrt{n(1)}}{\eta} \right), \quad B = \frac{1}{2} \left( 1 + \frac{\sqrt{n(1)}}{\eta} \right). \quad (3.9)$$

From the representation (3.8) the zeros  $\{\tilde{k}_j\}$  of  $\tilde{d}(k)$  have density  $(\delta + \eta)/\pi$  if  $n(1) \neq \eta^2$  and  $\delta \neq \eta$  (cf. Theorem 2.5 and its corollaries in [9]) so  $\delta$  is determined by the modified transmission eigenvalues if  $n(r) < \eta^2$  for  $0 \leq r \leq 1$ . We now wish to apply Hadamard's factorization theorem to the determinant  $\tilde{d}(k)$ . In order to take advantage of known results for the standard determinant  $d(k)$ , we will use the function

$$f(k) := \tilde{d}\left(\frac{k}{\eta}\right) = \det \begin{pmatrix} \tilde{y}(1) & \frac{\sin k}{k} \\ \tilde{y}'(1) & \cos k \end{pmatrix},$$

where  $\tilde{y}$  satisfies

$$\tilde{y}'' + k^2 \tilde{n}(r) \tilde{y} = 0, \quad \tilde{y}(0) = 0, \quad \tilde{y}'(0) = 1,$$

with  $\tilde{n}(r) = \frac{n(r)}{\eta^2}$ . Note that  $f(k)$  is the determinant for the standard interior transmission problem with index of refraction  $\tilde{n}(r)$ .

We observe that  $f(k)$  is an even entire function of order one and if  $0 < \tilde{n}(r) < 1$  then  $f(k)$  has a zero of order two at the origin [4]. From these observations, we have the following lemma.

**Lemma 3.2.** *The function  $\tilde{d}(k)$  has the following properties:*

1. *The determinant  $\tilde{d}(k)$  is an even entire function of order one.*
2. *If  $n(r) < \eta^2$ , then  $\tilde{d}(k)$  has a zero of order two at the origin.*

**Proof.** Since  $\tilde{d}(k) = f(k\eta)$  and  $f(k)$  is even and entire,  $\tilde{d}(k)$  must also be even and entire. Furthermore, scaling the argument of an entire function with a positive real number does not affect the order, so  $\tilde{d}(k)$  is of order one.

Now if  $n(r) < \eta^2$  for  $0 \leq r < 1$ , then  $0 < \tilde{n}(r) < 1$  for  $0 \leq r < 1$ , so  $f(k)$  has a zero of order two at the origin [4]. Thus, by Hadamard's factorization theorem, there exists an entire function  $g$  such that  $f(k) = k^2 g(k)$  and  $g(0) \neq 0$ . For  $\tilde{g}(k) := \eta^2 g(k\eta)$  we have that

$$\tilde{d}(k) = f(k\eta) = (k\eta)^2 g(k\eta) = k^2 \tilde{g}(k).$$

Since  $\tilde{g}(0) = \eta g(0) \neq 0$ ,  $\tilde{d}(k)$  has a zero of order two at the origin.  $\square$

Under the assumptions of Lemma 3.2,  $\tilde{d}(\sqrt{k})$  is an entire function of order  $1/2$ . Thus if the zeros  $\{\tilde{k}_j\}$  of  $\tilde{d}(k)$  are known (including multiplicity) then by Hadamard's factorization theorem

$$\tilde{d}(k) = \tilde{c} k^2 \prod_{j=1}^{\infty} \left(1 - \frac{k^2}{\tilde{k}_j^2}\right) \quad (3.10)$$

for some nonzero constant  $\tilde{c}$ .

We now use (3.8) and (3.10) to show that  $n(1)$  is uniquely determined by the modified transmission eigenvalues corresponding to  $\eta$  when  $n(r) < \eta^2$  for  $0 \leq r \leq 1$ . To this end we first recall the following result concerning almost-periodic functions, which we call the Bohr integral lemma (cf. Lemma 1 of [11] or p. 16 of [3] for a proof).

**Lemma 3.3.** *If  $\varphi(k)$  is an entire function which is almost-periodic and bounded on the real line then each of the limits*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T \varphi(k) \sin(\alpha k) dk \quad (3.11a)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T \varphi(k) \cos(\alpha k) dk \quad (3.11b)$$

*exists for any real  $\alpha$  and a fixed constant  $a$ .*

**Lemma 3.4.** *If  $n(r) < \eta^2$  for  $0 \leq r \leq 1$  then the modified transmission eigenvalues corresponding to  $\eta$  uniquely determine  $n(1)$ .*

**Proof.** First note that  $n(1) \neq \eta^2$  so from the preceding discussion  $\delta$  is known. Define

$$\psi(k) := k^3 \prod_{j=1}^{\infty} \left( 1 - \frac{k^2}{k_j^2} \right). \quad (3.12)$$

With

$$C := \frac{1}{\tilde{c}[n(0)n(1)]^{1/4}},$$

we have  $\psi(k) = C\tilde{D}(k)$  from (3.10) and the definition of  $\tilde{D}(k)$ . From (3.8) we have that

$$\psi(k) = C \left[ A \sin k(\delta + \eta) + B \sin k(\delta - \eta) \right] + O\left(\frac{1}{k}\right). \quad (3.13)$$

Applying Bohr's integral lemma with  $a > 0$  sufficiently large, we see that the limits

$$M_1 := \lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T \psi(k) \sin k(\delta + \eta) dk,$$

$$M_2 := \lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T \psi(k) \sin k(\delta - \eta) dk,$$

exist and are known. Computing these limits by (3.13), we have

$$M_1 = \frac{CA}{2}, \quad M_2 = \frac{CB}{2}, \quad (3.14)$$

so by definition of  $A$  and  $B$  in (3.9), we have that

$$\frac{M_1}{M_2} = \frac{1 - \sqrt{n(1)}/\eta}{1 + \sqrt{n(1)}/\eta},$$

and hence  $n(1)$  is known.  $\square$

Under the assumptions of Lemma 3.4, we have that  $n(1)$  is determined and hence the constant  $\tilde{c}[n(0)]^{1/4}$  is determined by either of the equations in (3.14). We conclude that  $[n(0)]^{1/4}\tilde{d}(k)$  is uniquely determined by the modified transmission eigenvalues  $\{\tilde{k}_j\}$  corresponding to  $\eta$ . In order to prove the desired uniqueness theorem, we will need representations for  $y(1)$  and  $y'(1)$ . From Chapter 5 of [5], we have

$$y(1) = \frac{1}{[n(0)n(1)]^{1/4}} \left[ \frac{\sin k\delta}{k} + \int_0^\delta K(\delta, t) \frac{\sin kt}{k} dt \right] \quad (3.15a)$$

$$\begin{aligned} y'(1) &= \left[ \frac{n(1)}{n(0)} \right]^{1/4} \left[ \cos k\delta + \frac{\sin k\delta}{2k} \int_0^\delta p(s) ds + \int_0^\delta K_\xi(\delta, t) \frac{\sin kt}{k} dt \right] \\ &\quad - \frac{n'(1)}{4[n(0)]^{1/4}[n(1)]^{5/4}} \left[ \frac{\sin k\delta}{k} + \int_0^\delta K(\delta, t) \frac{\sin kt}{k} dt \right], \end{aligned} \quad (3.15b)$$

where  $K(\xi, t)$  is the solution to the Goursat problem

$$K_{\xi\xi} - K_{tt} - p(\xi)K = 0, \quad 0 < t < \xi < \delta \quad (3.16a)$$

$$K(\xi, 0) = 0, \quad 0 \leq \xi \leq \delta \quad (3.16b)$$

$$K(\xi, \xi) = \frac{1}{2} \int_0^\xi p(s) ds \quad (3.16c)$$

and  $\xi, p(\xi)$  are given by

$$\xi := \int_0^r \sqrt{n(\rho)} d\rho \quad (3.17)$$

$$p(\xi) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16} \frac{[n'(r)]^2}{[n(r)]^3}. \quad (3.18)$$

For future use, we recall the following result by Rundell and Sacks [18] (see also p. 162 of [15]).

**Theorem 3.5.** *Let  $K(\xi, t)$  satisfy (3.16a)–(3.16c). Then  $p \in C^1[0, \delta]$  is uniquely determined by the Cauchy data  $K(\delta, t), K_\xi(\delta, t)$ .*

Now that we have determined  $\delta$  and  $n(1)$ , we may use Lemma 3.3 and Theorem 3.5 to prove the following theorem. As part of the proof, we show that  $n'(1)$  is uniquely determined under the assumptions of the theorem.

**Theorem 3.6.** *Assume that  $n \in C^3[0, 1]$ . If  $\eta > 0$  and  $0 < n(r) < \eta^2$  for  $0 \leq r \leq 1$ , then the modified transmission eigenvalues corresponding to  $\eta$  (including multiplicity) uniquely determine  $n(r)$ .*

**Proof.** We consider the modified interior transmission problem corresponding to  $\eta$  and define  $\tilde{d}(k)$  by (3.5). By the previous discussion  $\delta$  is determined by the modified transmission eigenvalues corresponding to  $\eta$ . By Lemma 3.4,  $n(1)$  is uniquely determined by the modified transmission eigenvalues corresponding to  $\eta$ , so  $[n(0)]^{1/4}\tilde{d}(k)$  is uniquely determined. Substituting the representations of  $y(1)$  and  $y'(1)$  given by (3.15a)–(3.15b) into the determinant  $\tilde{d}(k)$ , we have

$$\begin{aligned}\tilde{d}(k) &= \frac{1}{[n(0)n(1)]^{1/4}} \left[ \frac{\sin k\delta}{k} + \int_0^\delta K(\delta, t) \frac{\sin kt}{k} dt \right] \cos k\eta \\ &\quad - \left[ \frac{n(1)}{n(0)} \right]^{1/4} \left[ \cos k\delta + \frac{\sin k\delta}{2k} \int_0^\delta p(s) ds + \int_0^\delta K_\xi(\delta, t) \frac{\sin kt}{k} dt \right] \frac{\sin k\eta}{k\eta} \\ &\quad - \frac{n'(1)}{4[n(0)]^{1/4}[n(1)]^{5/4}} \left[ \frac{\sin k\delta}{k} + \int_0^\delta K(\delta, t) \frac{\sin kt}{k} dt \right] \frac{\sin k\eta}{k\eta}.\end{aligned}\quad (3.19)$$

From (3.19) we have that

$$\frac{\ell\pi}{\eta} [n(0)]^{1/4} \tilde{d}\left(\frac{\ell\pi}{\eta}\right) = \frac{(-1)^\ell}{[n(1)]^{1/4}} \left[ \sin\left(\frac{\ell\pi\delta}{\eta}\right) + \int_0^\delta K(\delta, t) \sin\left(\frac{\ell\pi t}{\eta}\right) dt \right] \quad (3.20)$$

for  $\ell \in \mathbb{N}$ . Using the change of variables  $s = \frac{t}{\eta}$  in the integral in (3.20) we have

$$\int_0^\delta K(\delta, t) \sin\left(\frac{\ell\pi t}{\eta}\right) dt = \eta \int_0^{\delta/\eta} K(\delta, \eta s) \sin(\ell\pi s) ds.$$

Note that  $\{\sin(\ell\pi s)\}_{\ell \in \mathbb{N}}$  is complete in  $L^2[0, 1]$ , so it is complete in  $L^2[0, \frac{\delta}{\eta}]$  if  $\frac{\delta}{\eta} < 1$  [19]. From the assumption that  $n(r) < \eta^2$  we have that  $\delta < \eta$  so  $K(\delta, t)$  is known from (3.20). We now proceed to determine  $K_\xi(\delta, t)$ . We first use the knowledge of  $K(\delta, t)$  to determine  $n'(1)$ .

Applying integration by parts in (3.15b) we see that  $y'(1)$  has the asymptotic behavior

$$\begin{aligned}y'(1) &= \left[ \frac{n(1)}{n(0)} \right]^{1/4} \left[ \cos k\delta + \frac{\sin k\delta}{2k} \int_0^\delta p(s) ds \right] \\ &\quad - \frac{n'(1)}{4k[n(0)]^{1/4}[n(1)]^{5/4}} \sin k\delta + O\left(\frac{1}{k^2}\right).\end{aligned}\quad (3.21)$$

Substituting (3.21) into the determinant  $\tilde{d}(k)$  we see that

$$\begin{aligned}[n(0)]^{1/4} \tilde{d}(k) &= [n(0)]^{1/4} y(1) \cos k\eta \\ &\quad - [n(1)]^{1/4} \left[ \cos k\delta + \frac{\sin k\delta}{2k} \int_0^\delta p(s) ds \right] \frac{\sin k\eta}{k\eta} \\ &\quad + \frac{n'(1)}{4k^2\eta[n(1)]^{5/4}} \sin k\delta \sin k\eta + O\left(\frac{1}{k^3}\right).\end{aligned}\quad (3.22)$$

Define

$$H(k) := 4k^2\eta[n(1)]^{5/4} \left( [n(0)]^{1/4}\tilde{d}(k) - [n(0)]^{1/4}y(1)\cos k\eta \right. \\ \left. + [n(1)]^{1/4} \left[ \cos k\delta + \frac{\sin k\delta}{2k} \int_0^\delta p(s)ds \right] \frac{\sin k\eta}{k\eta} \right),$$

so from (3.22) we have

$$H(k) = n'(1)\sin k\delta\sin k\eta + O\left(\frac{1}{k}\right). \quad (3.23)$$

Since  $K(\delta, t)$  is known for  $0 \leq t \leq \delta$ , we have that  $[n(0)]^{1/4}y(1)$  is known from (3.15a). Furthermore,

$$K(\delta, \delta) = \frac{1}{2} \int_0^\delta p(s)ds$$

by (3.16c), so  $H(k)$  is known. From (3.23) we may rewrite  $H(k)$  as

$$H(k) = \frac{n'(1)}{2} \left[ \cos k(\delta - \eta) - \cos k(\delta + \eta) \right] + O\left(\frac{1}{k}\right). \quad (3.24)$$

Since the leading term of  $H(k)$  is almost-periodic, we may choose  $a > 0$  large enough in (3.11b) and apply Bohr's integral lemma with  $\alpha = \delta - \eta$  to  $H(k)$  in order to conclude that the limit

$$M := \lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T H(k) \cos k(\delta - \eta) dk \quad (3.25)$$

exists and is known. Computing this limit by (3.24), we have that

$$M = \frac{n'(1)}{4}, \quad (3.26)$$

so  $n'(1)$  is uniquely determined. From (3.19) we now have that

$$\frac{\ell\pi}{\delta} [n(0)]^{1/4} \tilde{d}\left(\frac{\ell\pi}{\delta}\right) \\ = [n(0)]^{1/4} y(1) \frac{\ell\pi}{\delta} \cos\left(\frac{\ell\pi\eta}{\delta}\right) \\ - \frac{1}{\eta} \sin\left(\frac{\ell\pi\eta}{\delta}\right) [n(1)]^{1/4} \left[ (-1)^\ell + \frac{\delta}{\ell\pi} \int_0^\delta K_\xi(\delta, t) \sin\left(\frac{\ell\pi t}{\delta}\right) dt \right] \\ - \frac{n'(1)}{4\eta[n(1)]^{5/4}} \sin\left(\frac{\ell\pi\eta}{\delta}\right) \int_0^\delta K(\delta, t) \sin\left(\frac{\ell\pi t}{\delta}\right) dt \quad (3.27)$$

for  $\ell \in \mathbb{N}$ . Since  $n'(1)$  is known we see that the first and third terms in the right-hand side of (3.27) are also known. Hence by completeness of  $\{\sin(\frac{\ell\pi t}{\delta})\}_{\ell \in \mathbb{N}}$

in  $L^2[0, \delta]$ ,  $K_\xi(\delta, t)$  is known from (3.27). By Theorem 3.5,  $p(\xi)$  is uniquely determined by our knowledge of  $K(\delta, t)$  and  $K_\xi(\delta, t)$ , but this implies that  $n(r)$  is known (cf. Chapter 5 of [5]).  $\square$

## References

- [1] T. Aktosun and V. G. Papanicolaou, Reconstruction of the wave speed from transmission eigenvalues for the spherically symmetric variable-speed wave equation, *Inverse Problems* 29 (2013) 065007.
- [2] T. Aktosun, D. Gintides, and V. G. Papanicolaou, The uniqueness in the inverse problem for the spherically symmetric variable-speed wave equation, *Inverse Problems*, 27 (2011) 115004.
- [3] A. S. Besicovitch, *Almost Periodic Functions*, Dover Publications, Inc., New York, 1955.
- [4] F. Cakoni, D. Colton, and D. Gintides, The interior transmission eigenvalue problem, *SIAM J. Math. Anal.* 42 (2010) 2912–2921.
- [5] F. Cakoni, D. Colton, and H. Haddar, *Inverse Scattering Theory and Transmission Eigenvalues*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 88, SIAM, Philadelphia, 2016.
- [6] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd ed., Springer, New York, 2013.
- [7] D. Colton and Y.J. Leung, The existence of complex transmission eigenvalues for spherically stratified media, *Applicable Analysis*, to appear.
- [8] D. Colton and Y.J. Leung, Complex eigenvalues and the inverse spectral problem for transmission eigenvalues, *Inverse Problems* 29 (2013) 104008.
- [9] D. Colton, Y.J. Leung, and S. Meng, Distribution of complex transmission eigenvalues for spherically stratified media, *Inverse Problems* 31 (2015) 035006.
- [10] D. Colton, L. Päivärinta, and J. Sylvester, The interior transmission problem, *Inverse Problems and Imaging* 1 (2007) 13–28.
- [11] R. Cooke, Almost-periodic functions, *The American Mathematical Monthly* 88 (1981) 515–526.
- [12] G. Giorgi and H. Haddar, Computing estimates of material properties from transmission eigenvalues, *Inverse Problems* 28 (2012) 055009.
- [13] Y. Guo, P. Monk, and D. Colton, Toward a time domain approach to the linear sampling method, *Inverse Problems* 29 (2013) 095016.



- [14] Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd ed., Cambridge University Press, Cambridge, 2004.
- [15] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, 2nd ed., Springer, New York, 2011.
- [16] Y.J. Leung and D. Colton, Complex transmission eigenvalues for spherically stratified media, *Inverse Problems* 28 (2012) 075005.
- [17] J. R. McLaughlin and P. L. Polyakov, On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, *Journal of Differential Equations* 107 (1994) 351–382.
- [18] W. Rundell and P. Sacks, Reconstruction techniques for classical inverse Sturm-Liouville problems, *Mathematics of Computation* 58 (1992) 161–183.
- [19] R. Young, *An Introduction to Nonharmonic Fourier Series*, 1st ed., Academic Press, Inc., San Diego, CA, 2001.