Modified Transmission Eigenvalues in Inverse Scattering Theory

S. Cogar*, D. Colton†, S. Meng‡, and P. Monk†

Abstract

We consider the scattering of an acoustic plane wave by an inhomogeneous medium of compact support. Our aim is to introduce a new class of target signatures that can be used to detect changes in the material properties of the scattering object from a knowledge of the far field pattern of the scattered field. To this end we introduce a modified far field operator depending on a parameter η and show that this operator is injective with dense range provided η is not an eigenvalue of a new problem called the modified transmission eigenvalue problem. It is explained why this class of target signatures is preferable in some ways to previously studied target signatures that are based on scattering resonances, transmission eigenvalues, or Stekloff eigenvalues.

Key words. inverse scattering, nondestructive testing, modified transmission eigenvalues, Herglotz wave functions, non-selfadjoint eigenvalue problems

AMS subject classifications. 35J25, 35P05, 35P25, 35R30

1. Introduction

The spectral properties of the far field operator play a central role in mathematical scattering theory. In particular, the theory of scattering resonances is a rich and beautiful part of scattering theory and for a comprehensive survey of this area we refer the reader to [11]. More recently, attention has been given to the development of the theory of transmission eigenvalues and for a discussion of this topic in scattering theory see [5]. These two areas of research are in a certain sense complementary, as the following example illustrates. Consider the

*Corresponding author. Department of Mathematical Sciences, University of Delaware, Newark, DE 19716 (cogar@udel.edu). This material is based upon work supported by the Army Research Office through the National Defense Science and Engineering Graduate (NDSEG) Fellowship, 32 CFR 168a.

†Department of Mathematical Sciences, University of Delaware, Newark, DE 19716 (colton@udel.edu, monk@udel.edu). This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-17-1-0147.

‡Institute for Mathematics and its Applications, University of Minnesota, Twin Cities, Minneapolis, MN 55455 (shixumeng@ima.umn.edu).

scattering of an incident field u^i by a homogeneous ball in $\mathbb{R}^3$ of radius one with index of refraction two, i.e. if u^s is the scattered field and $u = u^i + u^s$ is the total field then

$$\Delta u + 4k^2 u = 0, \quad |x| < 1 \tag{1.1a}$$

$$\Delta u + k^2 u = 0, \quad |x| > 1 \tag{1.1b}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{1.1c}$$

where $r = |x|$, k is the wave number and u is continuously differentiable across $|x| = 1$. Then there is a scattering resonance k, $\mathrm{Im}(k) < 0$, such that there exists a solution of (1.1a)–(1.1c) where $u^i = 0$ and $u^s = h_0^{(1)}(kr)$ for $r > 1$ where $h_0^{(1)}$ denotes a spherical Hankel function. On the other hand, there is a transmission eigenvalue k, $\mathrm{Im}(k) = 0$, such that there exists a solution of (1.1a)–(1.1c) where $u^s = 0$ and $u^i = j_0(kr)$ for $r > 1$ where j_0 denotes a spherical Bessel function. Due to the fact that, in principle, both scattering resonances and transmission eigenvalues can be determined from the measured scattering data, efforts have been made over the years to use these eigenvalues as a "target signature" in various areas of application, most notably in the use of scattering resonances in the "singularity expansion method" [3] and the use of transmission eigenvalues in nondestructive testing [14]. However, the success of these efforts has been limited by a number of problems. In particular, for the case of scattering resonances, these eigenvalues all lie in the lower half-plane and hence determining them from measured scattering data is problematic. On the other hand, real transmission eigenvalues exist only if the index of refraction is real and hence this excludes their use in the case when absorption is present. Furthermore, in both cases, the eigenvalues are determined by the material properties of the scatterer, i.e. it is not possible to choose the interrogation frequency a priori.

In an effort to overcome the above drawbacks in the use of scattering resonances and transmission eigenvalues as target signatures, a new approach to this problem has recently been proposed [6], [7]. In this approach, a modified far field operator is introduced which allows the wave number to be fixed and real while introducing a new parameter which now serves the role of a target signature. In particular, for the modification used in [6] and [7], this new parameter turned out to be the well-known Stekloff eigenvalue and examples were given showing how such eigenvalues are effective in detecting changes in the index of refraction due to either flaws or changes in the material properties of the medium. Such an approach is particularly promising in their potential use for the nondestructive testing of anisotropic materials since in this case traditional imaging techniques are problematic due to the fact that the solution of the inverse scattering problem for anisotropic media is no longer uniquely determined [12].

A problem with using Stekloff eigenvalues as a target signature is that difficulties arise in the detection of voids which lie along a nodal line of the Stekloff eigenfunction. More generally, the use of Stekloff eigenvalues as a target signature does not provide any mechanism to improve the sensitivity in detecting eigenvalue changes due to changes in the index of refraction. The purpose of this paper is to introduce a new modified far field operator and corresponding eigenvalue problem which depends on a second parameter in addition to the eigenvalue parameter. This second parameter can then be adjusted to remove the above problem associated with the sensitivity of the eigenvalues to changes in the index of refraction as well as to avoid nodal lines of the eigenfunctions.

The plan of our paper is as follows. In the next section of our paper we introduce the modified far field operator discussed in the above paragraph. We then discuss the modified transmission eigenvalue problem associated with the operator and show that the injectivity of this modified far field operator can be characterized in terms of the eigenvalues of the modified transmission eigenvalue problem. This result then allows us to determine the eigenvalues from the measured far field data. A special section is devoted to the case of a complex index of refraction, basing our analysis on Agmon's theory of non-selfadjoint elliptic equations [1]. Our paper is concluded by a detailed numerical investigation of the applicability of our new class of eigenvalue problems to their potential use in nondestructive testing.

2. A modified far field operator

Fix $k > 0$ and let $d = 2, 3$. For each unit vector $\hat{d} \in \mathbb{S}^{d-1}$ (the unit sphere in $\mathbb{R}^d$), let $u_\infty(\cdot, \hat{d})$ be the far-field pattern corresponding to the scattering problem

$$\Delta u + k^2 n(x) u = 0 \text{ in } \mathbb{R}^d, \tag{2.1a}$$

$$u(x) = e^{ikx\cdot\hat{d}} + u^s(x), \tag{2.1b}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \tag{2.1c}$$

where $r = |x|$. We define $D \subset \mathbb{R}^d$ to be a bounded set with connected complement containing the support of $1 - n$. We assume that D contains the origin and has Lipschitz boundary ∂D, and we assume that $n \in L^\infty(D)$ has positive real part and nonnegative imaginary part. Let B be either a ball centered at the origin containing D in its interior or $B = D$, and let $h_\infty(\cdot, \hat{d})$ be the far-field pattern corresponding to the transmission problem

$$\Delta h_1 + k^2 h_1 = 0 \text{ in } \mathbb{R}^d \setminus \overline{B}, \tag{2.2a}$$

$$\frac{1}{\gamma}\Delta h_2 + k^2 \eta h_2 = 0 \text{ in } B, \tag{2.2b}$$

$$h_1(x) = e^{ikx\cdot\hat{d}} + h_1^s(x), \tag{2.2c}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial h_1^s}{\partial r} - ikh_1^s \right) = 0, \tag{2.2d}$$

$$h_1 = h_2 \text{ on } \partial B, \tag{2.2e}$$

$$\frac{\partial h_1}{\partial \nu} = \frac{1}{\gamma}\frac{\partial h_2}{\partial \nu} \text{ on } \partial B, \tag{2.2f}$$

where $\gamma > 0$ is a fixed constant not equal to one and η is a (possibly complex) constant. The Sommerfeld radiation conditions (2.1c) and (2.2d) are assumed to hold uniformly in all directions. Note that there exists a unique solution to both (2.1a)–(2.1c) and (2.2a)–(2.2f) [5, Chapter 1].

We define the *modified far field operator* $\mathcal{F} : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ by

$$(\mathcal{F}g)(\hat{x}) := \int_{\mathbb{S}^{d-1}} \left[u_\infty(\hat{x}, \hat{d}) - h_\infty(\hat{x}, \hat{d}) \right] g(\hat{d}) ds(\hat{d}), \ \hat{x} \in \mathbb{S}^{d-1}. \tag{2.3}$$

A slight modification of the proof of Theorem 2.1 in [8] yields the following result.

Theorem 2.1. *The modified far field operator $\mathcal{F}$ is injective with dense range if and only if there does not exist a nontrivial solution w, v to the modified interior transmission problem*

$$\Delta w + k^2 n(x) w = 0 \ \textit{in } B, \tag{2.4a}$$

$$\frac{1}{\gamma}\Delta v + k^2 \eta v = 0 \ \textit{in } B, \tag{2.4b}$$

$$w = v \ \textit{on } \partial B, \tag{2.4c}$$

$$\frac{\partial w}{\partial \nu} = \frac{1}{\gamma}\frac{\partial v}{\partial \nu} \ \textit{on } \partial B, \tag{2.4d}$$

where v is a generalized Herglotz wave function, i.e. of the form

$$v(x) = \int_{\mathbb{S}^{d-1}} h_2(x, \hat{d}) g(\hat{d}) ds(\hat{d}), \ x \in B,$$

for some $g \in L^2(\mathbb{S}^{d-1})$, where h_1, h_2 satisfy (2.2a)–(2.2f).

We call (2.4a)–(2.4d) the *modified transmission eigenvalue problem.* This problem with $\gamma = 1$ was studied in [8] in the context of an inverse spectral problem, where the eigenparameter considered was the wave number k. In this paper, we consider η as the eigenparameter and fix $k > 0$. Whenever (2.4a)–(2.4d) has a nontrivial solution $(w, v) \in H^1(B) \times H^1(B)$, we call η a *modified transmission eigenvalue.*

3. Solvability of the modified interior transmission problem

In this section, we show that the *modified interior transmission problem*

$$\Delta w + k^2 n(x) w = f \text{ in } B, \tag{3.1a}$$

$$\frac{1}{\gamma}\Delta v + k^2 \eta v = g \text{ in } B, \tag{3.1b}$$

$$w - v = \ell_1 \text{ on } \partial B, \tag{3.1c}$$

$$\frac{\partial w}{\partial \nu} - \frac{1}{\gamma}\frac{\partial v}{\partial \nu} = \ell_2 \text{ on } \partial B, \tag{3.1d}$$

satisfies the Fredholm alternative, where $f, g \in L^2(B)$, $\ell_1 \in H^{1/2}(\partial B)$, and $\ell_2 \in H^{-1/2}(\partial B)$. We note that setting the right-hand sides of (3.1a)–(3.1d) to zero yields the modified transmission eigenvalue problem (2.4a)–(2.4d). We introduce a variational formulation of this problem using the space

$$\mathcal{H}(B) := \{(u, v) \in H^1(B) \times H^1(B) \mid u - v \in H_0^1(B)\}.$$

Defining a lifting function $\varphi \in H^1(B)$ such that $\varphi|_{\partial B} = \ell_1$ and writing $u = w - \varphi$, we have that $(u, v) \in H^1(B) \times H^1(B)$ satisfies

$$\Delta u + k^2 n u = f - \Delta\varphi - k^2 n\varphi \text{ in } B, \tag{3.2a}$$

$$\frac{1}{\gamma}\Delta v + k^2 \eta v = g \text{ in } B, \tag{3.2b}$$

$$u - v = 0 \text{ on } \partial B, \tag{3.2c}$$

$$\frac{\partial u}{\partial \nu} - \frac{1}{\gamma}\frac{\partial v}{\partial \nu} = \ell_2 - \frac{\partial \varphi}{\partial \nu} \text{ on } \partial B, \tag{3.2d}$$

which is equivalent to the variational problem of finding $(u, v) \in \mathcal{H}(B)$ satisfying

$$a_\eta((u, v), (u', v')) = \ell(u', v') \quad \forall (u', v') \in \mathcal{H}(B) \tag{3.3}$$

where the sesquilinear form $a_\eta(\cdot, \cdot)$ is given by

$$a_\eta((u, v), (u', v')) := (\nabla u, \nabla u') - \frac{1}{\gamma}(\nabla v, \nabla v') - k^2(nu, u') + k^2\eta(v, v')$$

and the antilinear functional ℓ is given by

$$\ell(u', v') := -(f, u') + (g, v') + \langle \ell_2, v' \rangle - (\nabla\varphi, \nabla u') + k^2(n\varphi, u').$$

In writing this formulation, we have used the definitions

$$(f, g) := \int_B f\overline{g}\, dA \text{ and } \langle f, g \rangle := \int_{\partial B} f\overline{g}\, ds,$$

where the latter integral must be understood in the sense of the duality pairing $H^{-1/2}(\partial B) \times H^{1/2}(\partial B)$. In order to show that the variational problem (3.3) satisfies the Fredholm property, we begin by defining the sesquilinear forms $\hat{a}(\cdot,\cdot)$ and $b_\eta(\cdot,\cdot)$ as

$$\hat{a}((u,v),(u',v')) := (\nabla u, \nabla u') - \frac{1}{\gamma}(\nabla v, \nabla v') + k^2(u,u') - k^2\alpha(v,v')$$
$$b_\eta((u,v),(u',v')) := ((n+1)u,u') - (\eta+\alpha)(v,v')$$

for a constant $\alpha > 0$ such that $1-\alpha$ has the same sign as $1-\frac{1}{\gamma}$, from which we see that

$$a_\eta((u,v),(u',v')) = \hat{a}((u,v),(u',v')) - k^2 b_\eta((u,v),(u',v')) \quad \forall (u,v),(u',v') \in \mathcal{H}(B).$$

By means of the Riesz representation theorem, we define the bounded linear operators $A_\eta, \hat{A}, B_\eta : \mathcal{H}(B) \to \mathcal{H}(B)$ by

$$(A_\eta(u,v),(u',v'))_{\mathcal{H}(B)} = a_\eta((u,v),(u',v')),$$
$$(\hat{A}(u,v),(u',v'))_{\mathcal{H}(B)} = \hat{a}((u,v),(u',v')),$$
$$(B_\eta(u,v),(u',v'))_{\mathcal{H}(B)} = b_\eta((u,v),(u',v'))$$

for all $(u,v),(u',v') \in \mathcal{H}(B)$, and we see that $A_\eta = \hat{A} - k^2 B_\eta$. Thus, in order to show that A_η is a Fredholm operator of index zero, we need only show that $\hat{A}$ is invertible and that B_η is compact. Compactness of B_η follows from the compact embedding of $\mathcal{H}(B)$ into $L^2(B) \times L^2(B)$. Since $\hat{a}(\cdot,\cdot)$ is not coercive due to the opposite signs in the gradient terms, we will appeal to T-coercivity [4] in order to show invertibility of $\hat{A}$. For a given isomorphism $T : \mathcal{H}(B) \to \mathcal{H}(B)$, we define the sesquilinear form

$$\hat{a}^T((u,v),(u',v')) := \hat{a}((u,v),T(u',v'))$$

for $(u,v),(u',v') \in \mathcal{H}(B)$. Choosing $T(u,v) = (u, -v+2u)$ when $\gamma > 1$ and $T(u,v) = (u-2v,-v)$ when $\gamma < 1$ yields that $\hat{a}^T(\cdot,\cdot)$ is coercive and hence that $\hat{a}(\cdot,\cdot)$ is T-coercive.

For the convenience of the reader, we show T-coercivity of $\hat{a}(\cdot,\cdot)$ when $\gamma > 1$. In this case we choose $\alpha < 1$ and we have that

$$\begin{aligned}\hat{a}^T((u,v),(u',v')) &= \hat{a}((u,v),(u',-v'+2u')) \\ &= (\nabla u, \nabla u') + \frac{1}{\gamma}(\nabla v, \nabla v') - \frac{2}{\gamma}(\nabla v, \nabla u') + k^2(u,u') \\ &\quad + k^2\alpha(v,v') - 2k^2\alpha(v,u').\end{aligned}$$

By the reverse triangle inequality, we obtain

$$\begin{aligned}\left|\hat{a}^T((u,v),(u,v))\right| \geq{}& (\nabla u, \nabla u) + \frac{1}{\gamma}(\nabla v, \nabla v) + k^2(u,u) + k^2\alpha(v,v) \\ &- \frac{2}{\gamma}\left|(\nabla v, \nabla u)\right| - 2k^2\alpha\left|(v,u)\right|.\end{aligned}$$

By Young's inequality, for all $\epsilon_1, \epsilon_2 > 0$ we have that

$$2\left|(\nabla v, \nabla u)\right| \leq \epsilon_1(\nabla v, \nabla v) + \epsilon_1^{-1}(\nabla u, \nabla u)$$

and

$$2\left|(v, u)\right| \leq \epsilon_2(v, v) + \epsilon_2^{-1}(u, u).$$

Thus, we arrive at the inequality

$$\begin{aligned}\left|\hat{a}^T((u,v),(u,v))\right| \geq \left(1 - \frac{1}{\gamma}\epsilon_1^{-1}\right)(\nabla u, \nabla u) + \frac{1}{\gamma}(1-\epsilon_1)(\nabla v, \nabla v) \\ + k^2(1-\alpha\epsilon_2^{-1})(u,u) + k^2\alpha(1-\epsilon_2)(v,v),\end{aligned}$$

and hence choosing $\frac{1}{\gamma} < \epsilon_1 < 1$ and $\alpha < \epsilon_2 < 1$ yields coercivity of $\hat{a}^T(\cdot,\cdot)$. Similar arguments establish the result when $0 < \gamma < 1$.

Applying the Lax-Milgram lemma and the fact that T is an isomorphism (note that $T^2 = I$), we have that $\hat{A}$ is invertible. Therefore, we conclude that A_η is a Fredholm operator of index zero, implying that the modified interior transmission problem satisfies the Fredholm alternative. In particular, if η is not a modified transmission eigenvalue, then the modified interior transmission problem (3.1a)–(3.1d) is well-posed. We will use this result extensively in later sections.

4. Properties of the eigenvalues

In this section, we return to the modified transmission eigenvalue problem (2.4a)–(2.4d) and study the properties of the modified transmission eigenvalues. Recall from the previous section that $A_\eta = \hat{A} - k^2 B_\eta$, where $\hat{A}$ is invertible and B_η is compact for each $\eta \in \mathbb{C}$. We observe that the mapping $\eta \mapsto B_\eta$ is analytic, and from the analytic Fredholm theory [9, Theorem 8.26] we conclude that the existence of at least one η for which A_η is invertible implies that A_η is invertible for all but a discrete set of η. In other words, the set of modified transmission eigenvalues is discrete provided that (2.4a)–(2.4d) is well-posed for some η. Indeed, if n is real-valued, then we may choose $\eta = i\tau$ for some $\tau > 0$. We observe that if $A_\eta(u,v) = 0$ for some $(u,v) \in \mathcal{H}(B)$, then taking the imaginary part of the equation

$$a_\eta((u,v),(u,v)) = 0$$

implies that $\tau \left\|v\right\|^2 = 0$, from which it follows that $v = 0$ and $w = 0$. Thus, we obtain injectivity and hence invertibility of $A_{i\tau}$ by the Fredholm property.

Now, in order to show that (2.4a)–(2.4d) is self-adjoint whenever n is real-valued, we appeal to a different technique, and we require some real η_0 which is not a modified transmission eigenvalue, the existence of which is guaranteed by the discreteness result we derived above from analytic Fredholm theory. Moreover, in order to emphasize the connection between the modified transmission eigenvalue problem and the standard transmission eigenvalue problem, we assume that $\eta = 1$ is not a modified transmission eigenvalue. Note that $\eta = 1$ is a modified transmission eigenvalue if and only if $k > 0$ is a transmission eigenvalue in the sense that the problem

$$\Delta w + k^2 n(x) w = 0 \text{ in } B, \tag{4.1a}$$

$$\frac{1}{\gamma}\Delta v + k^2 v = 0 \text{ in } B, \tag{4.1b}$$

$$w = v \text{ on } \partial B, \tag{4.1c}$$

$$\frac{\partial w}{\partial \nu} = \frac{1}{\gamma}\frac{\partial v}{\partial \nu} \text{ on } \partial B, \tag{4.1d}$$

admits a nontrivial solution $(w, v) \in H^1(B) \times H^1(B)$. We see that (4.1a)–(4.1d) differs from the usual transmission eigenvalue problem for anisotropic media, as the term $\frac{1}{\gamma}$ occurs in the principal part of the second equation. However, as γ is constant, the transformation

$$\tilde{w} = \frac{1}{\gamma} w, \ \tilde{v} = \frac{1}{\gamma} v, \ \tilde{k} = \sqrt{\gamma} k$$

recovers the standard transmission eigenvalue problem. We remark that under certain restrictions on the index of refraction n, the transmission eigenvalues are discrete without finite accumulation points [5]. Moreover, if $\mathrm{Im}(n) > 0$ a.e. in an open subset of B, $k > 0$ cannot be a transmission eigenvalue [9], and hence $\eta = 1$ cannot be a modified transmission eigenvalue. In fact, in this case no real modified transmission eigenvalues exist, as we will show in Section 6.

Theorem 4.1. *If $\eta = 1$ is not a modified transmission eigenvalue, then the set of modified transmission eigenvalues is discrete without finite accumulation points. In addition, if n is real-valued, then eigenvalues exist and are real.*

Proof. Given $g \in L^2(B)$, we consider the auxiliary source problem of finding $(w_g, v_g) \in H^1(B) \times H^1(B)$ satisfying

$$\Delta w_g + k^2 n w_g = 0 \text{ in } B, \tag{4.2a}$$

$$\frac{1}{\gamma}\Delta v_g + k^2 v_g = k^2 g \text{ in } B, \tag{4.2b}$$

$$w_g = v_g \text{ on } \partial B, \tag{4.2c}$$

$$\frac{\partial w_g}{\partial \nu} = \frac{1}{\gamma}\frac{\partial v_g}{\partial \nu} \text{ on } \partial B. \tag{4.2d}$$

From the discussion in Section 3, we have that (4.2a)–(4.2d) satisfies the Fredholm property. Under the assumption that $\eta = 1$ is not a modified transmission eigenvalue, the Fredholm alternative implies that (4.2a)–(4.2d) has a unique solution $(w_g, v_g) \in H^1(B) \times H^1(B)$ that satisfies the estimate

$$\|w_g\|_{H^1(B)} + \|v_g\|_{H^1(B)} \leq c \|g\|_{L^2(B)}. \tag{4.3}$$

Thus, we may define the linear operator $T_1 : L^2(B) \to L^2(B)$ by $T_1 g := v_g$. We see from (4.3) that T_1 is bounded from $L^2(B)$ into $H^1(B)$, and hence the compact embedding of $H^1(B)$ into $L^2(B)$ implies that $T_1 : L^2(B) \to L^2(B)$ is compact. From (2.4a)–(2.4d) and (4.2a)–(4.2d) we see that η is a modified transmission eigenvalue if and only if

$$(1-\eta)T_1 g = g \tag{4.4}$$

for some nonzero $g \in L^2(B)$, and hence compactness of T_1 implies that the set of eigenvalues is discrete without finite accumulation points. We now consider the case when n is real-valued, and we introduce a variational formulation of (4.2a)–(4.2d), which is to find $(w_g, v_g) \in \mathcal{H}(B)$ satisfying

$$(\nabla w_g, \nabla w') - \frac{1}{\gamma}(\nabla v_g, \nabla v') - k^2(n w_g, w') + k^2(v_g, v') = k^2(g, v') \tag{4.5}$$

for all $(w', v') \in \mathcal{H}(B)$. For $g, h \in L^2(B)$, we have that

$$\begin{aligned}
k^2(T_1 g, h) &= k^2(v_g, h) \\
&= k^2\overline{(h, v_g)} \\
&= \overline{(\nabla w_h, \nabla w_g)} - \frac{1}{\gamma}\overline{(\nabla v_h, \nabla v_g)} - k^2\overline{(n w_h, w_g)} + k^2\overline{(v_h, v_g)} \\
&= (\nabla w_g, \nabla w_h) - \frac{1}{\gamma}(\nabla v_g, \nabla v_h) - k^2(n w_g, w_h) + k^2(v_g, v_h) \\
&= k^2(g, v_h) \\
&= k^2(g, T_1 h),
\end{aligned}$$

implying that T_1 is self-adjoint. Therefore, if n is real-valued, then the relation (4.4) implies that modified transmission eigenvalues exist and are real in addition to the set of eigenvalues being discrete and having no finite accumulation points. □

Remark 4.2. The choice of $\gamma \neq 1$ is necessary in the following sense. If n is taken to be a constant with $B = D$ and we choose $\gamma = 1$, then the modified transmission eigenvalue problem (2.4a)–(2.4d) becomes

$$\Delta w + k^2 n w = 0 \text{ in } B, \tag{4.6a}$$

$$\Delta v + k^2 \eta v = 0 \text{ in } B, \tag{4.6b}$$

$$w = v \text{ on } \partial B, \tag{4.6c}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \partial B. \tag{4.6d}$$

We note that for any solution $u \in H^1(B)$ to $\Delta u + k^2 n u = 0$ in B, it follows that $(u, u) \in H^1(B) \times H^1(B)$ solves (4.6a)–(4.6d) when $\eta = n$. Thus, the eigenspace corresponding to $\eta = n$ has infinite dimension, and hence the spectrum cannot be represented by a compact operator as in the proof of Theorem 4.1.

5. Determination of eigenvalues from far field data

In this section, we show that modified transmission eigenvalues may be determined from far field data. We begin by defining

$$A_0(x) := \begin{cases} \frac{1}{\gamma} I, & x \in B \\ I, & x \notin B \end{cases} \quad \text{and } n_0(x) := \begin{cases} \eta, & x \in B \\ 1, & x \notin B \end{cases}$$

for $x \in \mathbb{R}^d$, and we recall that for $g \in L^2(\mathbb{S}^{d-1})$ the function

$$v_g(x) := \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\hat{d}} g(\hat{d}) ds(\hat{d}),\ x \in \mathbb{R}^d$$

is called the Herglotz wave function with kernel g [9]. We denote by $\Phi_\infty(\cdot,\cdot)$ the far field pattern of the radiating fundamental solution $\Phi(\cdot,\cdot)$ of the Helmholtz equation in $\mathbb{R}^d$ given by

$$\Phi(x,z) := \begin{cases} \frac{e^{ik|x-z|}}{4\pi|x-z|} & \text{in } \mathbb{R}^3, \\ \frac{i}{4} H_0^{(1)}(k\,|x-z|) & \text{in } \mathbb{R}^2, \end{cases}$$

where $H_0^{(1)}$ is the zeroth order Hankel function of the first kind. If $z \in D$ and g_z solves the modified far field equation

$$\mathcal{F} g_z = \Phi_\infty(\cdot, z), \tag{5.1}$$

then we have that

$$w_\infty - v_\infty = \Phi_\infty(\cdot, z),$$

where w_∞ is the far field pattern corresponding to the scattering problem

$$\begin{aligned} \Delta w_z + k^2 n w_z = 0 \text{ in } \mathbb{R}^d, \\ w_z = v_{g_z} + w_z^s, \\ \lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial w_z^s}{\partial r} - ikw_z^s \right) = 0 \end{aligned}$$

and v_∞ is the far field pattern corresponding to the transmission problem

$$\begin{aligned} \nabla \cdot A_0 \nabla v_z + k^2 n_0 v_z = 0 \text{ in } \mathbb{R}^d, \\ v_z = v_{g_z} + v_z^s, \\ \lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial v_z^s}{\partial r} - ikv_z^s \right) = 0. \end{aligned}$$

By Rellich's lemma [9], we have that

$$w_z^s - v_z^s = \Phi(\cdot, z) \text{ in } \mathbb{R}^d \setminus \overline{B},$$

and since w_z^s and v_z^s arise from the same incident field v_{g_z}, we obtain

$$w_z - v_z = \Phi(\cdot, z) \text{ in } \mathbb{R}^d \setminus \overline{B}.$$

Thus, we see that $(w_z, v_z) \in H^1(B) \times H^1(B)$ satisfies

$$\Delta w_z + k^2 n w_z = 0 \text{ in } B, \tag{5.2a}$$

$$\frac{1}{\gamma}\Delta v_z + k^2 \eta v_z = 0 \text{ in } B, \tag{5.2b}$$

$$w_z - v_z = \Phi(\cdot, z) \text{ on } \partial B, \tag{5.2c}$$

$$\frac{\partial w_z}{\partial \nu} - \frac{1}{\gamma}\frac{\partial v_z}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \text{ on } \partial B, \tag{5.2d}$$

with w_z and v_z having the decompositions

$$w_z = v_{g_z} + w_z^s, \quad v_z = v_{g_z} + v_z^s. \tag{5.3}$$

In general, the solution to (5.2a)–(5.2d) (which exists and is unique provided η is not a modified transmission eigenvalue) will not have the decomposition (5.3). However, the fields w_z, v_z may each be decomposed as the sum of an incident field and a radiating field by Green's formula, and by the following lemma we see that the two incident fields coincide.

Lemma 5.1. *Assume that η is not a modified transmission eigenvalue. Then the problem (5.2a)–(5.2d) has a unique solution $(w_z, v_z) \in H^1(B) \times H^1(B)$, and the fields w_z, v_z may be decomposed as $w_z = v_z^i + w_z^s$ and $v_z = v_z^i + v_z^s$, where $v_z^i \in H^1(B)$ satisfies the Helmholtz equation in B and $w_z^s, v_z^s \in H^1_{loc}(\mathbb{R}^d)$ each satisfies the Sommerfeld radiation condition.*

Proof. Assuming that η is not a modified transmission eigenvalue, the Fredholm alternative implies the existence of a unique solution $(w_z, v_z) \in H^1(B) \times H^1(B)$ to (2.4a)–(2.4d). From Green's formula , we have the representations

$$\begin{aligned} w_z(x) = &\int_{\partial B} \left[\Phi(x,y)\frac{\partial w_z(y)}{\partial \nu} - w_z(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y) \\ &- k^2 \int_B [1 - n(y)]\Phi(x,y) w(y) dy, \ x \in B, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} v_z(x) = &\int_{\partial B} \left[\Phi(x,y)\frac{\partial v_z(y)}{\partial \nu} - v_z(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y) \\ &- \int_B \left[\left(1 - \frac{1}{\gamma}\right)\Delta v_z + k^2(1-\eta) v_z\right] \Phi(x,y) dy, \ x \in B. \end{aligned} \tag{5.5}$$

In order to obtain the correct decomposition of v_z, we may rewrite (5.5) as

$$\begin{aligned} v_z(x) = &\int_{\partial B} \left[\Phi(x,y)\frac{1}{\gamma}\frac{\partial v_z(y)}{\partial \nu} - v_z(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y) \\ &+ \left(1 - \frac{1}{\gamma}\right)\int_{\partial B} \Phi(x,y)\frac{\partial v_z(y)}{\partial \nu} ds(y) \\ &- \int_B \left[\left(1 - \frac{1}{\gamma}\right)\Delta v_z + k^2(1-\eta) v_z\right] \Phi(x,y) dy, \ x \in B. \end{aligned} \tag{5.6}$$

With the decompositions (5.4) and (5.6), we define

$$w_z^i(x) := \int_{\partial B} \left[\Phi(x,y) \frac{\partial w_z(y)}{\partial \nu} - w_z(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right] ds(y), \; x \in B,$$

and

$$v_z^i(x) := \int_{\partial B} \left[\Phi(x,y) \frac{1}{\gamma} \frac{\partial v_z(y)}{\partial \nu} - v_z(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right] ds(y), \; x \in B,$$

and we note that both w_z^i and v_z^i satisfy the Helmholtz equation in B. Moreover, we define $w_z^s := w_z - w_z^i$ and $v_z^s := v_z - v_z^i$, and we observe that each satisfies the Sommerfeld radiation condition when extended to $\mathbb{R}^d \setminus \overline{B}$.

Then for all $x \in B$ the boundary conditions (5.2c)–(5.2d) imply that

$$(w_z^i - v_z^i)(x) = \int_{\partial B} \left[\Phi(x,y) \frac{\partial \Phi(y,z)}{\partial \nu(y)} - \Phi(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right] ds(y).$$

Let $B_r := \{y \in \mathbb{R}^d \setminus \overline{B} : |y| < r\}$ and $S_r := \{y \in \mathbb{R}^d \setminus \overline{B} : |y| = r\}$ with $r > 0$ large enough that B is contained in the ball of radius r, and observe that $\partial B_r = S_r \cup \partial B$. With the normal vector ν to ∂B_r directed into the exterior of B_r, we may apply Green's second identity to obtain

$$\begin{aligned}(w_z^i - v_z^i)(x) = &\int_{S_r} \left[\Phi(x,y) \frac{\partial \Phi(y,z)}{\partial \nu} - \Phi(y,z) \frac{\partial \Phi(x,y)}{\partial \nu} \right] ds(y) \\ &- \int_{B_r} \left[\Phi(x,y) \Delta_y \Phi(y,z) - \Phi(y,z) \Delta_y \Phi(x,y) \right] dy.\end{aligned}$$

Since $x, z \in B$, we have that $\Phi(x,\cdot)$ and $\Phi(\cdot,z)$ each satisfies the Helmholtz equation in B_r, and hence the integral over B_r vanishes identically. Since $\Phi(x,\cdot)$ and $\Phi(\cdot,z)$ each satisfies the Sommerfeld radiation condition, we have that the integral over S_r vanishes in the limit $r \to \infty$. Thus, it follows that $w_z^i = v_z^i$ in B, and we arrive at the desired decomposition of w_z and v_z. □

We construct a factorization of $\mathcal{F}$ in the following way. Define the space of generalized incident fields

$$H_{inc}(B) := \{v^i \in H^1(B) \mid \Delta v^i + k^2 v^i = 0 \text{ in } B\}.$$

Define $P_1 : H_{inc}(B) \to L^2(B)$ as

$$P_1 v^i = k^2(1-n)v^i.$$

Define $G_1 : L^2(B) \to L^2(\mathbb{S}^{d-1})$ as $G_1 f = w_\infty$, where w_∞ is the far field pattern corresponding to the radiating solution w^s of

$$\Delta w^s + k^2 n w^s = f \text{ in } \mathbb{R}^d.$$

Define $P_2 : H_{inc}(B) \to L^2(B)$ as

$$P_2 v^i = \nabla \cdot (I - A_0) \nabla v^i + k^2(1-n_0)v^i.$$

Define $G_2 : L^2(B) \to L^2(\mathbb{S}^{d-1})$ as $G_2 f = v_\infty$, where v_∞ is the far field pattern corresponding to the radiating solution v^s of

$$\nabla \cdot A_0 \nabla v^s + k^2 n_0 v^s = f \text{ in } \mathbb{R}^d.$$

For a plane wave $v^i(x) = e^{ikx\cdot\hat{d}}$, we note that $G_1 P_1 v^i$ and $G_2 P_2 v^i$ are the far field patterns corresponding to the scattering problems (2.1a)–(2.1c) and (2.2a)–(2.2f), respectively. Finally, defining $\mathcal{B} : H_{inc}(B) \to L^2(\mathbb{S}^{d-1})$ as $\mathcal{B} := G_1 P_1 - G_2 P_2$, we have the factorization $\mathcal{F} = \mathcal{B} H$, where $H : L^2(\mathbb{S}^{d-1}) \to H_{inc}(B)$ is the Herglotz operator defined as $Hg = v_g$, and we note that $\mathcal{B}$ is compact by compactness of G_1 and G_2 and boundedness of P_1 and P_2. For use in the proof of the next theorem, we note that the operator H has dense range [5, Lemma 2.1]. The following two theorems constitute the main results of this section.

Theorem 5.2. *Assume that η is not a modified transmission eigenvalue and let $z \in D$. Then for every $\epsilon > 0$ there exists $g_z^\epsilon \in L^2(\mathbb{S}^{d-1})$ that satisfies*

$$\lim_{\epsilon \to 0} \|\mathcal{F} g_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^{d-1})} = 0 \tag{5.7}$$

such that $\{v_{g_z^\epsilon}\}$ converges and hence $\left\|v_{g_z^\epsilon}\right\|_{H^1(B)}$ is bounded as $\epsilon \to 0$.

Proof. From our assumption that η is not a modified transmission eigenvalue, the Fredholm alternative implies that there exists a unique solution $(w_z, v_z) \in H^1(B) \times H^1(B)$ to (5.2a)–(5.2d), and from Lemma 5.1 it follows that the fields w_z, v_z may be decomposed as $w_z = v_z^i + w_z^s$ and $v_z = v_z^i + v_z^s$, where $v_z^i \in H_{inc}(B)$ and w_z^s, v_z^s each satisfies the Sommerfeld radiation condition. By construction of the operator $\mathcal{B}$, we see that

$$\mathcal{B} v_z^i = \Phi_\infty(\cdot, z).$$

By density of the range of the Herglotz operator H in $H_{inc}(B)$, for each $\epsilon > 0$ there exists $g_z^\epsilon \in L^2(\mathbb{S}^{d-1})$ such that

$$\left\|v_{g_z^\epsilon} - v_z^i\right\|_{H^1(B)} < \frac{\epsilon}{M},$$

where we choose $M > \|\mathcal{B}\|$. From boundedness of $\mathcal{B}$ we obtain

$$\|\mathcal{F} g_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^{d-1})} = \left\|\mathcal{B} v_{g_z^\epsilon} - \mathcal{B} v_z^i\right\|_{L^2(\mathbb{S}^{d-1})} \le \|\mathcal{B}\| \left\|v_{g_z^\epsilon} - v_z^i\right\|_{H^1(B)} < \epsilon$$

for all $\epsilon > 0$, implying that (5.7) is satisfied. Moreover, we see that $\{v_{g_z^\epsilon}\}_{\epsilon > 0}$ converges to v_z^i in $H^1(B)$, and hence $\left\|v_{g_z^\epsilon}\right\|_{H^1(B)}$ is bounded as $\epsilon \to 0$. □

Theorem 5.3. *Assume that η is a modified transmission eigenvalue and $g_z^\epsilon \in L^2(\mathbb{S}^{d-1})$ satisfies (5.7). Then $\left\|v_{g_z^\epsilon}\right\|_{H^1(B)}$ cannot be bounded as $\epsilon \to 0$ for almost every $z \in B_\rho$, where B_ρ is an arbitrary ball of radius ρ in D.*

Proof. Suppose to the contrary that for z in a small ball $B_\rho \subset D$ the sequence $\{v_{g_z^\epsilon}\}_{\epsilon>0}$ is bounded in $H^1(B)$ as $\epsilon \to 0$. Then up to a subsequence $\{v_{g_z^\epsilon}\}_{\epsilon>0}$ converges weakly to some $v_z^i \in H_{inc}(B)$. By compactness of $\mathcal{B}$, we conclude that $\mathcal{B}v_{g_z^\epsilon} \to \mathcal{B}v_z^i$ in $L^2(\mathbb{S}^{d-1})$ as $\epsilon \to 0$, from which we observe that

$$\lim_{\epsilon\to 0} \left\| \mathcal{F}g_z^\epsilon - \mathcal{B}v_z^i \right\|_{L^2(\mathbb{S}^{d-1})} = 0.$$

Thus, we see that $\mathcal{B}v_z^i = \Phi_\infty(\cdot, z)$, and as before we have that $(w_z, v_z) \in H^1(B)\times H^1(B)$ satisfies (5.2a)–(5.2d) with w_z and v_z arising from the incident field v_z^i.

We consider the equivalent problem (3.2a)–(3.2d) introduced in Section 3 of finding $(u_z, v_z) \in H^1(B) \times H^1(B)$ satisfying

$$\Delta u_z + k^2 n u_z = -\Delta\varphi_z - k^2 n \varphi_z \text{ in } B, \tag{5.8a}$$

$$\frac{1}{\gamma}\Delta v_z + k^2 \eta v_z = 0 \text{ in } B, \tag{5.8b}$$

$$u_z - v_z = 0 \text{ on } \partial B, \tag{5.8c}$$

$$\frac{\partial u_z}{\partial \nu} - \frac{1}{\gamma}\frac{\partial v_z}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} - \frac{\partial \varphi_z}{\partial \nu} \text{ on } \partial B, \tag{5.8d}$$

where we have chosen the lifting function $\varphi_z \in H^1(B)$ such that $\varphi_z|_{\partial B} = \Phi(\cdot, z)$ and we have written $u_z = w_z - \varphi_z$. As given in the same section, an equivalent variational formulation of this problem is to find $(u_z, v_z) \in H^1(B) \times H^1(B)$ satisfying

$$a_\eta((u_z, v_z), (u', v')) = \ell(u', v') \quad \forall (u', v') \in \mathcal{H}(B) \tag{5.9}$$

where the sesquilinear form $a_\eta(\cdot, \cdot)$ is given by

$$a_\eta((u, v), (u', v')) := (\nabla u, \nabla u') - \frac{1}{\gamma}(\nabla v, \nabla v') - k^2(nu, u') + k^2\eta(v, v')$$

and the antilinear function ℓ is given by

$$\ell(u', v') := \left\langle \frac{\partial \Phi(\cdot, z)}{\partial \nu}, v' \right\rangle - (\nabla \varphi_z, \nabla u') + k^2(n\varphi_z, u').$$

As the variational problem (5.9) satisfies the Fredholm property, we have that solvability is equivalent to $\ell(u_\eta, v_\eta) = 0$ for all solutions $(u_\eta, v_\eta) \in \mathcal{H}(B)$ of the homogeneous adjoint problem

$$(\nabla u_\eta, \nabla u') - \frac{1}{\gamma}(\nabla v_\eta, \nabla v') - k^2(\overline{n}u_\eta, u') + k^2\overline{\eta}(v_\eta, v') = 0 \quad \forall (u', v') \in \mathcal{H}(B),$$

i.e. for all solutions $(u_\eta, v_\eta) \in H^1(B) \times H^1(B)$ to

$$\Delta u_\eta + k^2 \overline{n} u_\eta = 0 \text{ in } B,$$

$$\frac{1}{\gamma}\Delta v_\eta + k^2 \overline{\eta} v_\eta = 0 \text{ in } B,$$

$$u_\eta - v_\eta = 0 \text{ on } \partial B,$$

$$\frac{\partial u_\eta}{\partial \nu} - \frac{1}{\gamma}\frac{\partial v_\eta}{\partial \nu} = 0 \text{ on } \partial B.$$

We see that

$$\ell(u_\eta, v_\eta) = \left\langle \frac{\partial \Phi(\cdot, z)}{\partial \nu}, v_\eta \right\rangle - (\nabla \varphi_z, \nabla u_\eta) + k^2(n\varphi_z, u_\eta). \tag{5.10}$$

After integrating by parts in the second term and using the fact that $\varphi_z|_{\partial B} = \Phi(\cdot, z)$ and that $\Delta u_\eta + k^2 \overline{n} u_\eta = 0$ in B, we may write (5.10) as

$$\ell(u_\eta, v_\eta) = \int_{\partial B} \left[u_\eta(y) \frac{\partial \Phi(y, z)}{\partial \nu} - \Phi(y, z) \frac{\partial u_\eta}{\partial \nu}(y) \right] ds(y).$$

Thus, the solvability condition for each $z \in B_\rho$ becomes

$$\int_{\partial B} \left[u_\eta(y) \frac{\partial \Phi(y, z)}{\partial \nu} - \Phi(y, z) \frac{\partial u_\eta}{\partial \nu}(y) \right] ds(y) = 0.$$

Green's representation theorem implies that $u_\eta(z) = 0$ for all z in B_ρ, from which the unique continuation principle [9] implies that $u_\eta = 0$ in B and hence that $v_\eta = 0$ in B. However, this result contradicts the fact that since η is an eigenvalue, some nonzero pair (u_η, v_η) must exist. Therefore, we conclude that the sequence $\{v_{g_z^\epsilon}\}_{\epsilon > 0}$ cannot be bounded in $H^1(B)$ as $\epsilon \to 0$. □

We observe that (5.7) is satisfied whenever $\mathcal{F}$ has dense range, which by Theorem 2.1 holds whenever η is not an eigenvalue with v_z^i a Herglotz wave function. These two results imply that we may in principle detect modified transmission eigenvalues by using regularization methods to solve the modified far field equation (5.1) for $g_z \in L^2(\mathbb{S}^{d-1})$ (with $z \in D$ given) and then computing the norm of v_{g_z} in $H^1(B)$. The modified transmission eigenvalues are precisely those η for which this norm is large. For a justification of this approach see [2]. In practice, we use the norm of the regularized solution g_z as a proxy for $\|v_{g_z}\|_{H^1(B)}$ in order to reduce computational expense. We plot the norm of g_z against the sampled values of η and look for the eigenvalues as sharp peaks in the graph. Of course, this approach must be carried out in a discrete setting, which we will describe and numerically investigate in Section 7.

With the ability to detect modified transmission eigenvalues from far field data established by Theorems 5.2 and 5.3, we now investigate the effect of a flaw in the scatterer on the eigenvalues. In particular, we suppose that the real-valued refractive index n is perturbed by δn which results in perturbations of the eigenfunction pair $(w, v) \in H^1(B) \times H^1(B)$ by $(\delta w, \delta v)$ and the eigenvalue η by $\delta\eta$. From the variational formulation (3.3) of the modified transmission eigenvalue problem, we see that the perturbed eigenfunctions satisfy

$$\begin{aligned} \frac{1}{\gamma}(\nabla(v + \delta v), \nabla v') - (\nabla(w + \delta w), \nabla w') + k^2((n + \delta n)(w + \delta w), w') \\ - k^2(\eta + \delta\eta)(v + \delta v, v') = 0 \quad \forall (w', v') \in \mathcal{H}(B). \end{aligned}$$

Since $(w, v) \in \mathcal{H}(B)$ is an eigenfunction pair corresponding to η, the above expression becomes

$$\left[\frac{1}{\gamma}(\nabla \delta v, \nabla v') - (\nabla \delta w, \nabla w') + k^2(n\delta w, w') - k^2\eta(\delta v, v')\right] \\ + k^2(\delta n(w + \delta w), w') - k^2\delta\eta(v + \delta v, v') = 0 \quad \forall (w', v') \in \mathcal{H}(B).$$

Choosing $(w', v') = (w, v)$ and recalling that n was assumed to be real-valued, we observe that the expression in brackets vanishes and we have that

$$k^2(\delta n(w + \delta w), w) - k^2\delta\eta(v + \delta v, v) = 0.$$

Finally, by assuming only small changes (neglecting quadratic terms) and simplifying, we arrive at the perturbation estimate

$$\delta\eta = \frac{(\delta n w, w)}{(v, v)}. \tag{5.11}$$

It is not clear what conclusion to draw from this estimate if η is of multiplicity greater than one, as the choice of (w, v) is not unique. We will return to eigenvalues with multiplicity in our numerical investigations in Section 7.2. However, if η has multiplicity one and w is negligible in a neighborhood of the flaw, then the numerator in (5.11) will be negligible as well, and we would not expect a noticeable shift in the eigenvalue. In contrast to Stekloff eigenvalues [6], this perturbation estimate does not depend explicitly on the wave number k. In addition, we note that the estimate (5.11) depends on γ only through the eigenfunction pair (w, v). Though the effect of γ on η and (w, v) is not clear at this point, the following remark lends some insight into their relationship.

Remark 5.4. As our only restriction on γ is that it is positive and distinct from unity, the question naturally arises of whether an optimal value of γ exists and how one may best choose it. While further investigation is required in order to understand the relationship between γ and the modified transmission eigenvalues, the following example of spherically stratified media lends some insight. Consider the unit ball B in $\mathbb{R}^3$ with constant index of refraction. If we seek radially symmetric eigenfunctions to the modified transmission eigenvalue problem (2.4a)–(2.4d), then we find that η is a modified transmission eigenvalue if and only if it is a zero of the determinant

$$d(\eta) := \sqrt{\frac{\eta}{\gamma}} j_0\left(k\sqrt{n}\right) j_0'\left(k\sqrt{\gamma\eta}\right) - \sqrt{n} j_0'\left(k\sqrt{n}\right) j_0\left(k\sqrt{\gamma\eta}\right),$$

where j_0 is the zeroth order spherical Bessel function of the first kind. We observe that as γ increases, the spherical Bessel functions become more oscillatory, which results in an increase in the density of the zeros. A similar determinant function appears for higher orders of Bessel function, and the same result holds. Thus, we might expect that γ affects the density of the eigenvalues. In the case of noisy data, it may be advantageous to choose γ small in order to reduce the density of the eigenvalues and hence better identify the sharp peaks corresponding to modified transmission eigenvalues.

6. Modified transmission eigenvalues for complex n

In this section, we investigate further the case when the refractive index n is complex-valued. In particular, we assume that $n(x) = n_1(x) + i\frac{n_2(x)}{k}$ with $n_1(x) > 0$ and $n_2(x) \geq 0$ for $x \in B$. In the case of absorbing media, i.e. whenever n_2 is not identically zero, the modified transmission eigenvalue problem is non-selfadjoint, and hence the existence of modified transmission eigenvalues is not clear at this point. However, from the discussion at the beginning of Section 4 (see also the proof of Theorem 4.1), we have discreteness of the set of eigenvalues provided that some η exists which is not a modified transmission eigenvalue. To this end, let $(w, v) \in H^1(B) \times H^1(B)$ be a nontrivial solution to (2.4a)–(2.4d) for some η. Then setting $(w', v') = (w, v)$ and all right-hand sides to be zero in the variational formulation (3.3) yields

$$\int_B \left(\frac{1}{\gamma} |\nabla v|^2 - |\nabla w|^2 + k^2 n |w|^2 - k^2 \eta |v|^2 \right) dx = 0. \tag{6.1}$$

Taking the imaginary part of (6.1) and simplifying the resulting expression, we observe that η must satisfy

$$\mathrm{Im}(\eta) \int_B |v|^2 \, dx = \int_B \mathrm{Im}(n) \, |w|^2 \, dx. \tag{6.2}$$

Under the assumption that $\mathrm{Im}(n)$ is nonzero on an open subset of B, we observe that each integral in (6.2) must be positive, and we conclude that $\mathrm{Im}(\eta) > 0$, which implies that no real modified transmission eigenvalues exist in this case. Thus, we see that (2.4a)–(2.4d) is well-posed for any real η, and we obtain discreteness of modified transmission eigenvalues.

We now consider the existence of modified transmission eigenvalues for general refractive index of the form given above. Assume that the following problem has only a trivial solution $\psi \in H^1(B)$ where

$$\Delta \psi + k^2 n \psi = 0 \text{ in } B, \tag{6.3a}$$

$$\psi = 0 \text{ on } \partial B. \tag{6.3b}$$

If $n_2(x)$ is not zero, then this assumption is satisfied. If $n_2(x)$ is zero, this means that we choose the wave number k such that k^2 is not a (generalized) Dirichlet eigenvalue. We assume in this section that $n \in C^\infty(\overline{B})$ and B is a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary.

To facilitate the analysis, we first introduce the following source problem. For any given $\eta \in \mathbb{C}$ and $g \in L^2(B)$, find a nontrivial solution $(w, v) \in H^1(B) \times H^1(B)$ such that

$$\Delta w + k^2 n w = 0 \text{ in } B, \tag{6.4a}$$

$$\frac{1}{\gamma} \Delta v + k^2 \eta v = k^2 g \text{ in } B, \tag{6.4b}$$

$$w = v \text{ on } \partial B, \tag{6.4c}$$

$$\frac{\partial w}{\partial \nu} = \frac{1}{\gamma} \frac{\partial v}{\partial \nu} \text{ on } \partial B. \tag{6.4d}$$

From the discussion in Section 3, the problem (6.4a)–(6.4d) satisfies the Fredholm property. Then the existence of a solution to (6.4a)–(6.4d) is equivalent to the uniqueness of the solutions. We study the uniqueness of solutions to (6.4a)–(6.4d) for η such that $|\eta|$ is sufficiently large, $\arg \eta$ is fixed and $\eta \notin [0, \infty)$. The uniqueness is based on the following a priori estimate. We postpone the proof to Section 6.1.

Theorem 6.1. *Assume that* $(w, v) \in H^1(B) \times H^1(B)$ *and* $g \in L^2(B)$ *satisfy equations* (6.4a)–(6.4d). *Let* η *be such that* $|\eta|$ *is sufficiently large,* $\arg \eta$ *is fixed and* $\eta \notin [0, \infty)$. *Then* $v \in H^2(B)$ *and*

$$\|v\|_{L^2(B)} \leq c \frac{1}{|\eta|} \|g\|_{L^2(B)}, \tag{6.5}$$

where c *is a constant independent of* g.

Let $z \in \mathbb{C}$ be fixed such that $|z|$ is sufficiently large, $\arg z$ is fixed and $z \notin [0, \infty)$. We consider the problem (6.4a)–(6.4d) with $\eta = z$. Note that the assumptions in Theorem 6.1 are satisfied, and from Theorem 6.1 we have that the problem (6.4a)–(6.4d) has at most one solution. As the problem (6.4a)–(6.4d) satisfies the Fredholm property, there exists a unique solution (w, v) to (6.4a)–(6.4d). We define the operator $T_z : L^2(B) \to L^2(B)$ by

$$T_z g = v, \tag{6.6}$$

where (w, v) is the unique solution to (6.4a)–(6.4d) with $\eta = z$. One observes that if η is an eigenvalue of (6.4a)–(6.4d) (where $g = 0$), then $(z - \eta)^{-1}$ is an eigenvalue of T_z since if $\frac{1}{\gamma}\Delta v + k^2 \eta v = 0$ then $\frac{1}{\gamma}\Delta v + k^2 z v = k^2 (z - \eta) v$. The analysis of the eigenvalues η to (6.4a)–(6.4d) (where $g = 0$) then reduces to the analysis of the operator T_z.

Our approach to the existence of eigenvalues is based on the spectral theory of Hilbert-Schmidt operators. The following lemma follows from Theorem 16.4 in [1] or Corollary 31 (page 1115) in [10] Volume II.

Lemma 6.2. *Let* H *be a Hilbert space and* S *be a bounded linear operator from* H *to* H. *If* λ^{-1} *is in the resolvent of* S, *define*

$$(S)_\lambda = S(I - \lambda S)^{-1}.$$

Assume $S : H \to H$ *is a Hilbert-Schmidt operator. For the operator* S, *assume there exist* N *rays with bounded growth where the angle between any two adjacent rays is less than* $\frac{\pi}{2}$: *more precisely assume there exist* $0 \leq \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi$ *such that* $\theta_k - \theta_{k-1} < \frac{\pi}{2}$ *for* $k = 2, \cdots, N$ *and* $2\pi - \theta_N + \theta_1 < \frac{\pi}{2}$ *satisfying the condition that there exists* $r_0 > 0$, $c > 0$ *such that* $\|(S)_{re^{i\theta_k}}\| = O(\frac{1}{r})$ *for* $k = 1, \cdots, N$ *and* $r \geq r_0$. *Then the space spanned by the nonzero generalized eigenfunctions of* S *is dense in the closure of the range of* S.

We shall apply Lemma 6.2 with respect to the operator T_z. To begin with we shall prove that $T_z : L^2(B) \to L^2(B)$ is a Hilbert-Schmidt operator. Note that as B belongs to either $\mathbb{R}^2$ or $\mathbb{R}^3$, the following lemma is a particular case of Theorem 13.5 in [1].

Lemma 6.3. *Assume a bounded operator $T_z : L^2(B) \to L^2(B)$ satisfies that $T_z : L^2(B) \to H^2(B)$ is bounded. Then $T_z : L^2(B) \to L^2(B)$ is a Hilbert-Schmidt operator.*

To apply Lemma 6.2, we study the the resolvent of T_z and the range of T_z, respectively.

Lemma 6.4. *Assume that λ satisfies $z - \lambda \notin \mathbb{R}$, $\arg \lambda$ is fixed and $|\lambda|$ is sufficiently large. Then $(T_z)_\lambda = T_z(1 - \lambda T_z)^{-1}$ is bounded from $L^2(B)$ to $L^2(B)$ and*

$$\|(T_z)_\lambda\| \leq c\frac{1}{|\lambda|}.$$

Proof. (a) First we show that $T_z(1 - \lambda T_z)^{-1}$ is bounded from $L^2(B)$ to $L^2(B)$. Note that $T_z : L^2(B) \to L^2(B)$ is Hilbert-Schmidt and hereon compact. In order to show that $I - \lambda T_z$ has a bounded inverse, from Fredholm theory it is sufficient to show that

$$\big(I - \lambda T_z\big)v = 0$$

has only a trivial solution. Indeed assume that $\big(I - \lambda T_z\big)v = 0$, i.e. $T_z(\lambda v) = v$. From the definition of T_z, there exists $w \in H^1(B)$ such that

$$\begin{aligned} \Delta w + k^2 n w &= 0 \text{ in } B, \\ \frac{1}{\gamma}\Delta v + k^2 z v &= k^2 \lambda v \text{ in } B, \\ w &= v \text{ on } \partial B, \\ \frac{\partial w}{\partial \nu} &= \frac{1}{\gamma}\frac{\partial v}{\partial \nu} \text{ on } \partial B. \end{aligned}$$

Then (w, v) satisfies equations (6.4a)–(6.4d) with $\eta = z - \lambda$ and $g = 0$. From the assumptions, $z - \lambda \notin \mathbb{R}$ and $|\lambda|$ is sufficiently large, and hence the assumptions in Theorem 6.1 are satisfied for $\eta = z - \lambda$. Consequently we have that $v = 0$. This proves that $I - \lambda T_z$ has a bounded inverse.

(b) Second we show that

$$\|T_z(1 - \lambda T_z)^{-1}\| \leq c\frac{1}{|\lambda|}.$$

Assume that $T_z(1 - \lambda T_z)^{-1} g = v$ where g and v belong to $L^2(B)$. It is sufficient to show that

$$\|v\|_{L^2(B)} \leq c\frac{1}{|\lambda|}\|g\|_{L^2(B)}$$

for any $g \in L^2(B)$. Let $u = (1 - \lambda T_z)^{-1} g$. Then we obtain

$$\lambda v = \lambda T_z(1 - \lambda T_z)^{-1} g = (1 - \lambda T_z)^{-1} g - (I - \lambda T_z)(1 - \lambda T_z)^{-1} g = u - g.$$

This yields that

$$\|v\|_{L^2(B)} \le \frac{1}{|\lambda|}\big(\|u\|_{L^2(B)} + \|g\|_{L^2(B)}\big). \tag{6.7}$$

Now we estimate $\|u-g\|_{L^2(B)}$. As $u = (1-\lambda T_z)^{-1}g$, we have that $g = (1-\lambda T_z)u$, i.e. $T_z(\lambda u) = u - g$. From the definition of T_z, $u - g \in H^2(B)$ and there exists $w \in H^1(B)$ such that

$$\begin{aligned} \Delta w + k^2 n w &= 0 \text{ in } B, \\ \frac{1}{\gamma}\Delta(u-g) + k^2 z(u-g) &= k^2 \lambda u \text{ in } B, \\ w &= u - g \text{ on } \partial B, \\ \frac{\partial w}{\partial \nu} &= \frac{1}{\gamma}\frac{\partial(u-g)}{\partial \nu} \text{ on } \partial B. \end{aligned}$$

The above equations can be written as

$$\Delta w + k^2 n w = 0 \text{ in } B, \tag{6.8a}$$

$$\frac{1}{\gamma}\Delta(u-g) + k^2(z-\lambda)(u-g) = k^2\lambda g \text{ in } B, \tag{6.8b}$$

$$w = u - g \text{ on } \partial B, \tag{6.8c}$$

$$\frac{\partial w}{\partial \nu} = \frac{1}{\gamma}\frac{\partial(u-g)}{\partial \nu} \text{ on } \partial B. \tag{6.8d}$$

As z is fixed and $|\lambda|$ is sufficiently large, from (6.8a)–(6.8d) and Theorem 6.1 we have

$$\|u-g\|_{L^2(B)} \le c\frac{1}{|\lambda|}\|\lambda g\|_{L^2(B)} \le c\|g\|_{L^2(B)}.$$

Then from estimate (6.7) we conclude that

$$\|v\|_{L^2(B)} \le c\frac{1}{|\lambda|}\|g\|_{L^2(B)}.$$

This proves the lemma. □

Lemma 6.5. *T_z has dense range in $L^2(B)$.*

Proof. Let $f \in L^2(B)$ be such that

$$(T_z g, f) = 0 \quad \text{for any} \quad g \in L^2(B), \tag{6.9}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(B)$. To prove the lemma, it is sufficient to show that $f = 0$. Indeed let $f_n \in C_0^\infty(B)$ be such that f_n converges to f in $L^2(B)$-norm. Now let us define

$$g_n = \frac{1}{k^2}\left(\frac{1}{\gamma}\Delta f_n + k^2 z f_n\right).$$

Then $w = 0$, $v = f_n$, $g = g_n$ and $\eta = z$ satisfy equations (6.4a)–(6.4d). By the definition of T_z, one immediately derives $T_z g_n = f_n$. Then equation (6.9) yields

$$(f_n, f) = 0 \quad \text{for any} \quad n.$$

Note that as f_n converges to f in $L^2(B)$-norm, we conclude that $f = 0$ in $L^2(B)$. This proves the lemma. □

Theorem 6.6. *Assume that* $n(x) = n_1(x) + i\frac{n_2(x)}{k}$ *and* $n_2(x) \geq 0$. *Assume that* k *is such that equations* (6.3a)–(6.3b) *have only a trivial solution. Then there exist infinitely many eigenvalues* η, *the eigenvalues form a discrete set, and the space spanned by the nonzero generalized eigenfunctions of* T_z *is dense in* $L^2(B)$. *Moreover, there are only a finite number of eigenvalues outside* $\{\eta : 0 \leq \arg \eta < \epsilon\}$ *for any positive* ϵ.

Proof. We shall apply Lemma 6.2 with respect to the operator T_z. From Lemma 6.3, T_z is a Hilbert-Schmidt operator. Let us choose $0 \leq \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi$ such that $\theta_k - \theta_{k-1} < \frac{\pi}{2}$ for $k = 2, \cdots, N$, $2\pi - \theta_N + \theta_1 < \frac{\pi}{2}$ and $z - re^{i\theta_k} \notin R$. From Lemma 6.4, there exists $r_0 > 0$ such that $\|(S)_{re^{i\theta_k}}\| = O(\frac{1}{r})$ for $k = 1, \cdots, N$ and $r \geq r_0$. Then all of the assumptions in Lemma 6.2 are satisfied. This yields that the space spanned by the nonzero generalized eigenfunctions of T_z is dense in the closure of the range of T_z. From Lemma 6.5, T_z has dense range in $L^2(B)$, and hence there exist infinitely many eigenvalues of T_z and the space spanned by the nonzero generalized eigenfunctions of T_z is dense in $L^2(B)$. The discreteness of the eigenvalues follows from compactness of $T_z : L^2(B) \to L^2(B)$. Furthermore, since any η outside the set $\{\eta : 0 < \arg \eta < \epsilon \text{ or } 2\pi - \epsilon < \arg \eta < 2\pi\}$ with sufficiently large $|\eta|$ satisfies the assumptions in Theorem 6.1, we have that (w, v) is zero if (w, v) satisfies equation (6.4a)–(6.4d) with $g = 0$. This shows that such η is not an eigenvalue. Since the set of eigenvalues is discrete and the imaginary part of every eigenvalue is nonnegative from the discussion at the beginning of this section, this proves that there are only a finite number of eigenvalues outside $\{\eta : 0 \leq \arg \eta < \epsilon\}$ for any positive ϵ, and we have proven the theorem. □

6.1. Proof of Theorem 6.1

In this section, we shall prove the a priori estimate in Theorem 6.1. The analysis relies on the analysis of PDEs with a small (or large) parameter. In particular, equations (6.4a)–(6.4b) give rise to two Dirichlet-to-Neumann maps with a small (or large) parameter, and equations (6.4c)–(6.4d) relate the two Dirichlet-to-Neumann maps. In particular as shall be shown later, the choice of $\gamma \neq 1$ yields the ellipticity of the problem. Here we adapt the semiclassical analysis (small parameter) approach. We refer to [18] for more details.

To begin with, we introduce the following notations. Let h be a small parameter. We define $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$ and $D^h_{x_j} = \frac{h}{i}\frac{\partial}{\partial x_j}$ where the superscript h

represents a scaling by the small parameter h. For an open bounded manifold B in $\mathbb{R}^d$ we introduce the semiclassical Sobolev spaces $H^s_{sc}(B)$ equipped with the norm $\|\cdot\|_{H^s_{sc}(B)}$, where $\|u\|_{H^s_{sc}(B)} := \inf\{\|\tilde{u}\|_{H^s_{sc}(\mathbb{R}^d)}, \tilde{u}|_B = u\}$ and $\|u\|^2_{H^s_{sc}(\mathbb{R}^d)} := \int_{\mathbb{R}^d}(1+h^2|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi$.

Definition 6.7. *Let $a(x,\xi)$ be in $C^\infty(\mathbb{R}^{2(d-1)})$. We say a is a symbol of order m, denoted as $a \in S^m$, if*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}$$

for all α and β where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$. For $a \in S^m$ we define the semiclassical pseudo-differential operator $op_h(a)$ by

$$op_h(a)u = \frac{1}{(2\pi)^d}\int e^{ix\xi}a(x,h\xi)\hat{u}(\xi)d\xi.$$

We denote $\mathrm{op}(a)$ as the usual pseudo-differential operator where one simply takes $h=1$ in the above definitions.

We introduce a set of local coordinates near $\Gamma = \partial B$. Let $x = (x', x_d)$ and $\xi = (\xi', \xi_d)$ where (x,ξ) is the local coordinate in the cotangent bundle $T^*(\Gamma \times (0,\epsilon))$ and (x',ξ') is the local coordinate in the cotangent bundle $T^*\Gamma$.

Let $\mu(x) \in C^\infty(\overline{B})$. Assume that $-h^2\Delta - \mu$ is elliptic and $|\xi|^2 - \mu \neq 0$ for any ξ and $x \in \overline{B}$. We have that in the tubular neighborhood of Γ the semiclassical symbol of $-h^2\Delta - \mu$ is

$$\xi_d^2 + h\alpha\xi_d + R(x,\xi') - \mu$$

where $\alpha(x)$ is a smooth function depending on x. The symbol of $R(x,\xi')$ is $\mathfrak{g}^{k\ell}\xi'_k\xi'_\ell$ (here we use the Einstein summation), where $(\mathfrak{g}_{k\ell})$ is the Riemannian metric on Γ and $(\mathfrak{g}^{k\ell})$ is the inverse of $(\mathfrak{g}_{k\ell})$. Moreover

$$\xi_n^2 + R(x,\xi') - \mu = (\xi_d - \rho_1(x,\xi'))(\xi_d - \rho_2(x,\xi'))$$

where $\rho_2 = -i\sqrt{R(x,\xi') - \mu}$ and $\rho_1 = -\rho_2$ are symbols of order 1 with $\mathrm{Im}(\rho_2) < 0$ and $\mathrm{Im}(\rho_1) > 0$.

Now let us recall the following lemma. For a detailed proof one can refer to [17, p. 10–11].

Lemma 6.8. *Assume that $\mu(x) \in C^\infty(\overline{B})$ satisfies $|\xi|^2 - \mu \neq 0$ for any ξ and $x \in \overline{B}$. Assume that $v \in H^1(B)$ and $g \in L^2(B)$ satisfy*

$$h^2\Delta v + \mu v = h^2k^2\gamma g \qquad in \qquad B,$$

where h is sufficiently small. Let $\gamma_1 = v|_{x_d=0}$ and $\gamma_0 = D^h_{x_d}v|_{x_d=0}$. Then

$$\gamma_0 = -op_h(\rho_2)\gamma_1 + g_1, \tag{6.10}$$

where g_1 satisfies the estimate

$$\|g_1\|_{H^{\frac{1}{2}}_{sc}(\Gamma)} \le c\left(h^{\frac{1}{2}}\|v\|_{H^1_{sc}(B)} + h^{\frac{3}{2}}\|g\|_{L^2(B)} + h\|\gamma_1\|_{H^{\frac{1}{2}}_{sc}(\Gamma)}\right). \tag{6.11}$$

Moreover, if $\gamma_1 \in H^{\frac{3}{2}}(\Gamma)$*, then*

$$\|v\|_{H^2_{sc}(B)} \le c\left(h^2\|g\|_{L^2(B)} + h^{\frac{1}{2}}\|\gamma_1\|_{H^{\frac{3}{2}}(\Gamma)}\right), \tag{6.12}$$

where c *is a constant independent of* g*.*

Let us also recall the following lemma. For a detailed proof one can refer to [16, p. 1102–1104].

Lemma 6.9. *Assume that* k *is such that equations* (6.3a)–(6.3b) *have only a trivial solution. Let* $w \in H^1(B)$ *satisfy*

$$\Delta w + k^2 n w = 0 \qquad in \qquad B.$$

Then in the local coordinates $x = (x', x_d)$ *and* $\xi = (\xi', \xi_d)$ *we have that*

$$D_{x_3} w = \big(op(a_1) + op(a_0)\big) w \quad on \quad \Gamma \tag{6.13}$$

where $a_1(x', \xi') = i\sqrt{\mathfrak{g}^{k\ell}\xi'_k\xi'_\ell}$ *and* a_0 *is a symbol of order zero.*

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1: Recall that $|\eta|$ is sufficiently large, and let h be a small parameter defined by $h = |k^2\gamma\eta|^{-1/2}$. Now multiplying equation (6.4b) by γh^2 yields

$$h^2\Delta v + \mu v = h^2 k^2 \gamma g \qquad \text{in} \qquad B,$$

where $\mu = h^2k^2\gamma\eta$. Since $\eta \notin [0, \infty)$, we have that $|\xi|^2 - \mu \neq 0$ for any ξ and $x \in \overline{B}$. Then from Lemma 6.8

$$\gamma_0 = -\text{op}_h(\rho_2)\gamma_1 + g_1, \tag{6.14}$$

where g_1 satisfies the estimate (6.11). Since k is such that equations (6.3a)–(6.3b) have only a trivial solution, we may apply Lemma 6.9. In particular multiplying equation (6.13) by h yields

$$D^h_{x_3} w|_{x_3=0} = h\big(\text{op}(a_1) + \text{op}(a_0)\big) w|_{x_3}, \tag{6.15}$$

where $a_1(x', \xi') = i\sqrt{\mathfrak{g}^{k\ell}\xi'_k\xi'_\ell}$ and a_0 is a symbol of order zero. From equations (6.4c)–(6.4d),

$$w|_{x_d=0} = w|_{x_d=0} = \gamma_1 \quad \text{and} \quad D^h_{x_d} w|_{x_d=0} = \frac{1}{\gamma} D^h_{x_d} v|_{x_d=0} = \frac{1}{\gamma}\gamma_0.$$

Then we can write equation (6.15) as

$$\frac{1}{\gamma}\gamma_0 = h\big(\text{op}(a_1) + \text{op}(a_0)\big)\gamma_1. \tag{6.16}$$

From equations (6.14) and (6.16) we can derive an equation for γ_1

$$-\mathrm{op}_h(\rho_2)\gamma_1 + g_1 = \gamma\big(h\big(\mathrm{op}(a_1) + \mathrm{op}(a_0)\big)\gamma_1\big).$$

Note that since a_1 is homogeneous in ξ', we have that $h\mathrm{op}(a_1) = \mathrm{op}_h(a_1)$. This yields that

$$\mathrm{op}_h(\rho_2 + \gamma a_1)\gamma_1 = g_1 - h\gamma \mathrm{op}(a_0)\gamma_1. \tag{6.17}$$

Note that $\rho_2 = -i\sqrt{\mathfrak{g}^{k\ell}\xi'_k\xi'_\ell - \mu}$, $a_1 = i\sqrt{\mathfrak{g}^{k\ell}\xi'_k\xi'_\ell}$ and $\gamma \neq 1$, and hence $\rho_2+\gamma a_1$ is an elliptic symbol of order one and we may apply its parametrix Q (of order -1) to equation (6.17) to obtain

$$\gamma_1 = Q(g_1 - h\gamma\mathrm{op}(a_0)\gamma_1) + h\mathrm{op}_h(a_{-1})\gamma_1,$$

where a_{-1} is a symbol of order -1. This yields the estimate

$$\|\gamma_1\|_{H^{\frac{3}{2}}_{sc}(\Gamma)} \le c\left(\|g_1\|_{H^{\frac{1}{2}}_{sc}(\Gamma)} + h\|\mathrm{op}(a_0)\gamma_1\|_{H^{\frac{1}{2}}_{sc}(\Gamma)} + h\|\gamma_1\|_{H^{\frac{1}{2}}_{sc}(\Gamma)}\right), \tag{6.18}$$

where c is a constant independent of g_1 and γ_1.

From the definition of the semiclassical norm $\|\cdot\|_{H^{\frac{1}{2}}_{sc}(\Gamma)}$, we have that

$$\|\mathrm{op}(a_0)\gamma_1\|_{H^{\frac{1}{2}}_{sc}(\Gamma)} \le ch^{-\frac{1}{2}}\|\gamma_1\|_{H^{\frac{1}{2}}_{sc}(\Gamma)}. \tag{6.19}$$

Now from estimates (6.13), (6.18), and (6.19) we obtain

$$\|\gamma_1\|_{H^{\frac{3}{2}}_{sc}(\Gamma)} \le c\left(h^{\frac{1}{2}}\|v\|_{H^1_{sc}(B)} + h^{\frac{3}{2}}\|g\|_{L^2(B)} + h^{\frac{1}{2}}\|\gamma_1\|_{H^{\frac{1}{2}}_{sc}(\Gamma)}\right),$$

and for sufficiently small h we have

$$\|\gamma_1\|_{H^{\frac{3}{2}}_{sc}(\Gamma)} \le c\big(h^{\frac{1}{2}}\|v\|_{H^1_{sc}(B)} + h^{\frac{3}{2}}\|g\|_{L^2(B)}\big). \tag{6.20}$$

From estimates (6.12) and (6.20) we have that for sufficiently small h

$$\|v\|_{H^2_{sc}(B)} \quad \le \quad ch^2\|g\|_{L^2(B)}. \tag{6.21}$$

From the definition of semiclassical norm $\|\cdot\|_{H^2_{sc}(B)}$ and $h = |k^2\gamma\eta|^{-1/2}$ we can derive

$$\|v\|_{L^2(B)} \le c\frac{1}{|\eta|}\|g\|_{L^2(B)},$$

where c is a constant independent of g. This proves the theorem. $\square$

7. Numerical Examples

We now provide a series of numerical examples in support of the theoretical results we have obtained. In order to generate simulated scattering data to test our methods and to construct a finite element eigensolver, we use the finite element software `FreeFem++` [15]. The scattering problem is solved by truncating the unbounded domain and imposing the exact boundary conditions in terms of a Dirichlet-to-Neumann operator on a circular artificial boundary (see [5, p. 25] for details). In each case we use 51 incoming waves with directions d_j, $j = 1, \ldots, 51$, distributed uniformly on the unit circle, and we compute a 51×51 matrix U with $U_{\ell,m} \approx u_\infty(d_j, d_\ell)$. By the same process we obtain a matrix H approximating h_∞. In examples where we add noise to the data, we choose $\epsilon > 0$ and set

$$U^\epsilon_{\ell,m} = U_{\ell,m}\left(1 + \epsilon\frac{\zeta_{\ell,m} + i\mu_{\ell,m}}{\sqrt{2}}\right),\ \ell, m = 1, \ldots, 51,$$

where $\zeta_{\ell,m}$ and $\mu_{\ell,m}$ are uniformly distributed random numbers in $[-1, 1]$ computed using the `rand` command in `MATLAB`. Once the simulated scattering data has been computed, we compute the data vector b_z with the ℓth entry $b_{z,\ell} = \Phi_\infty(d_\ell, z)$, $\ell = 1, \ldots, 51$, for some $z \in D$. Finally, we set $F^\epsilon = U^\epsilon - H$ and approximate the Herglotz kernel g using Tikhonov regularization as

$$g^\epsilon_z = ((F^\epsilon)^* F^\epsilon + \alpha I)^{-1}(F^\epsilon)^* b_z.$$

This computation is performed using the `regtools` package [13]. We repeat this process for 32 randomly placed $z \in D$ and the norm of g^ϵ_z is averaged. We use the ℓ^2-norm of g^ϵ_z as a proxy for $\|v_g\|_{H^1(B)}$. The choice of 51 directions and 32 sampling points is arbitrary, but sufficiently many incoming waves are required in order to accurately approximate the modified far field operator $\mathcal{F}$.

In our finite element eigensolver, we avoid imposing the transmission conditions (2.4c)–(2.4d) by using the following equivalent formulation of the modified transmission eigenvalue problem (2.4a)–(2.4d). Writing $u_1 = w - v$ and $u_2 = w - \frac{1}{\gamma}v$, we have the equivalent formulation of finding $(u_1, u_2) \in \tilde{\mathcal{H}}(B) := H^1_0(B) \times H^1(B)$ satisfying

$$\Delta u_1 + \frac{k^2}{1-\gamma}n(u_1 - \gamma u_2) = \frac{k^2\gamma^2}{1-\gamma}\eta(u_1 - u_2) \text{ in } B \tag{7.1a}$$

$$\frac{1}{\gamma}\Delta u_2 + \frac{k^2}{\gamma(1-\gamma)}n(u_1 - \gamma u_2) = \frac{k^2}{1-\gamma}\eta(u_1 - u_2) \text{ in } B \tag{7.1b}$$

$$\frac{\partial u_2}{\partial \nu} = 0 \text{ on } \partial B \tag{7.1c}$$

An equivalent variational formulation of (7.1a)–(7.1c) is to find $(u_1, u_2) \in \tilde{\mathcal{H}}(B)$

satisfying

$$(\nabla u_1, \nabla u_1') + \frac{1}{\gamma}(\nabla u_2, \nabla u_2') - \frac{k^2}{1-\gamma}\,(n(u_1 - \gamma u_2), u_1') - \frac{k^2}{\gamma(1-\gamma)}\,(n(u_1 - \gamma u_2), u_2')$$
$$= \eta\left[-\frac{k^2\gamma^2}{1-\gamma}(u_1 - u_2, u_1') - \frac{k^2}{1-\gamma}(u_1 - u_2, u_2')\right] \quad \forall (u_1', u_2') \in \tilde{\mathcal{H}}(B). \quad (7.2)$$

We use `FreeFem++` to assemble the finite element matrices $\mathbf{A}$ (left-hand side) and $\mathbf{B}$ (bracketed expression on the right-hand side) of (7.2) with $\mathbb{P}_1$ elements, and we read these matrices into `MATLAB` and use the `eigs` command to solve the generalized eigenvalue problem $\mathbf{Au} = \eta\mathbf{Bu}$. The original fields w, v may be recovered as

$$w = \frac{1}{1-\gamma}(u_1 - \gamma u_2), \quad v = \frac{\gamma}{1-\gamma}(u_1 - u_2).$$

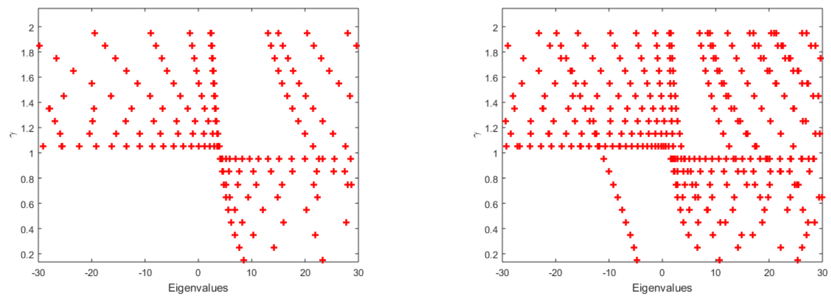
We performed tests with the unit disk and an L-shaped domain formed by removing the square $[0.1, 1.1]\times[-1.1, 0.1]$ from the square $[-0.9, 1.1]\times[-1.1, 0.9]$ (see Figure 4). Both of these regions were tested in [6] and hence provide a direct comparison to results for Stekloff eigenvalues. We also performed tests for a square and an annulus, but the results are similar to the disk and L-shaped domain and we chose not to include them here. In the case of $B \neq D$, we choose B to be a disk of radius 1.5 centered at the origin. Except for our example for absorbing media, we choose the constant refractive index $n = 4$ in D. In the case $B \neq D$, we extend n by one in $B \setminus D$. We choose $k = 1$, and we investigate different values of γ.

Before we continue, we present a result on the effect of γ on the eigenvalues. Since γ cannot be equal to one, we might expect that some qualitative change occurs in the eigenvalues at this value, and Figure 1 indicates that this expectation is correct. For the disk with $B = D$ (Figure 1a), no eigenvalues below $\eta = n = 4$ have been observed whenever $0 < \gamma < 1$, whereas one occurs in the case $B \neq D$ (Figure 1b). We will see in a later section that this eigenvalue is highly sensitive to the presence of flaws. It should be noted that a higher index of refraction may result in a negative eigenvalue even in the case $B = D$.

7.1. Changes in modified transmission eigenvalues due to flaws

In this section, we numerically investigate the effect of a flaw in the material on the modified transmission eigenvalues. The perturbation estimate (5.11) suggests that the change in an eigenvalue should depend in part on the size of the flaw and the magnitude of the eigenfunction w in a neighborhood of the flaw, at least for an eigenvalue of multiplicity one. From this observation, we might expect that each eigenvalue will exhibit different sensitivity to the location and size of a flaw, and we may infer this sensitivity from a plot of the corresponding eigenfunction (see Figure 4). For simplicity, we consider a circular flaw D_0 of radius r_c centered at (x_c, y_c), and we define the refractive index n as

$$n(x) = \begin{cases} 1, & x \in D_0 \\ 4, & x \in D \setminus D_0. \end{cases}$$



(a) disk, $B = D$ (b) disk, $B \neq D$

Figure 1: Behavior of the eigenvalues for a disk as γ varies in the range $0.1 \leq \gamma \leq 2$. The crosses on each horizontal line represent the eigenvalues for that choice of γ. The same general pattern appears for the L-shaped domain.

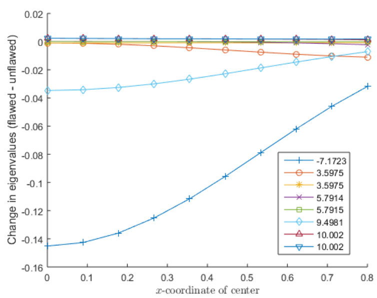
We investigate the sensitivity of the eigenvalues to the size and position of a flaw. We have observed that the choice $B \neq D$ greatly improves the ability to compute eigenvalues from far field data, and we limit our sensitivity investigation to this case.

First, we fix the radius of the flaw at $r_c = 0.05$ and vary the position of the center (x_c, y_c). For the disk (Figures 2a and 2b), we choose $y_c = 0$ and $0 \leq x_c \leq 0.8$, and for the L-shaped domain (Figures 2c and 2d), we choose $y_c = 0$ and $-0.4 \leq x_c \leq 0.8$.

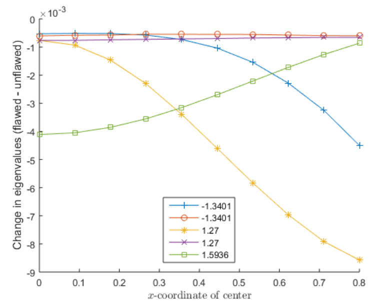
Second, we vary the radius of the flaw in the range $0.01 \leq r_c \leq 0.2$ with a fixed center (x_c, y_c) for each region. For the disk (Figures 3a and 3b), we choose $(x_c, y_c) = (0, 0.3)$, and for the L-shaped domain (Figures 3c and 3d), we choose $(x_c, y_c) = (0.1, 0.4)$.

Testing has shown that for $\gamma = 2$, the choice $B \neq D$ reduces the sensitivity of the eigenvalues to flaws compared to $B = D$, as was observed for Stekloff eigenvalues in [6]. However, for $\gamma = 0.5$, the choice $B \neq D$ improves sensitivity, likely because some eigenfunctions are larger in the center of the domain than for $\gamma = 2$. These eigenfunctions correspond to the two most sensitive eigenvalues for each region, and even though these eigenvalues have varying sensitivity to position of the flaw, one of them maintains a noticeable shift that is an order of magnitude greater than for $\gamma = 2$. Thus, we see that the ability to choose γ freely provides a great advantage in finding an effective target signature. Moreover, since the choice of γ only affects the computation of h_∞ and not the measured scattering data u_∞, one may in practice use eigenvalues corresponding to different values of γ as target signatures.

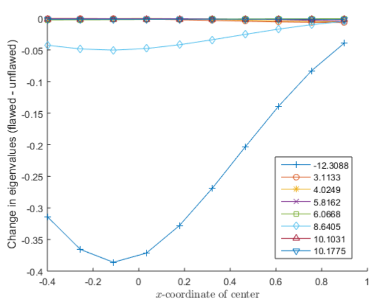
As we stated earlier, the shift in each eigenvalue is related to the corresponding eigenfunction w in the neighborhood of the flaw. In Figure 4, we plot the eigenfunction w corresponding to two eigenvalues for each region with $\gamma = 0.5$. When an eigenvalue has multiplicity one, the perturbation estimate (5.11) consistently predicts whether an eigenvalue will shift based on the magnitude of



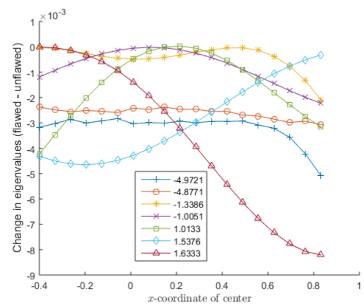
(a) disk, $\gamma = 0.5$



(b) disk, $\gamma = 2$

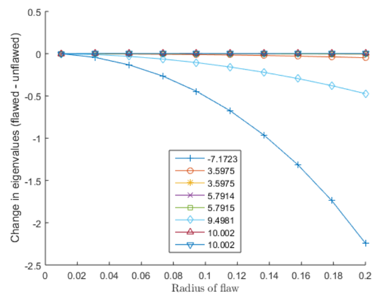


(c) L-shaped domain, $\gamma = 0.5$

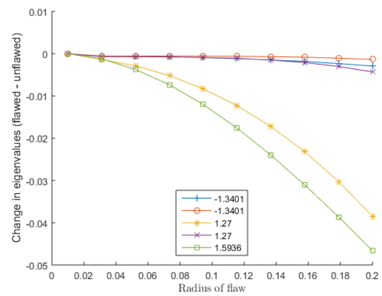


(d) L-shaped domain, $\gamma = 2$

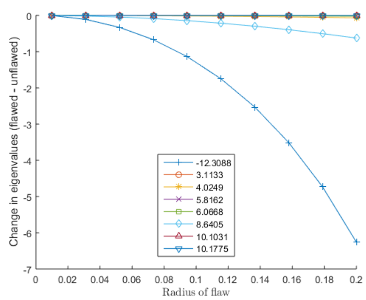
Figure 2: Sensitivity of modified transmission eigenvalues to changes in the position of a flaw for the case $B \neq D$ and both $\gamma = 0.5$ (left column) and $\gamma = 2$ (right column). We plot the change in the first few eigenvalues for each region due to the presence of the flaw. The numbers in each legend refer to the computed eigenvalue of the unflawed region. Note that the sensitivity of the first few eigenvalues to changes in the index of refraction due to a flaw is significantly increased if $\gamma = 0.5$ instead of $\gamma = 2$.



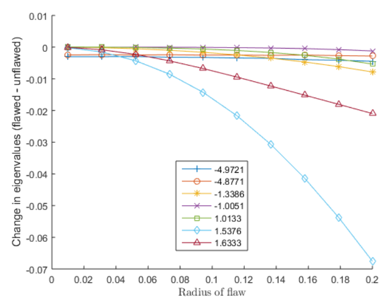
(a) disk, $\gamma = 0.5$



(b) disk, $\gamma = 2$

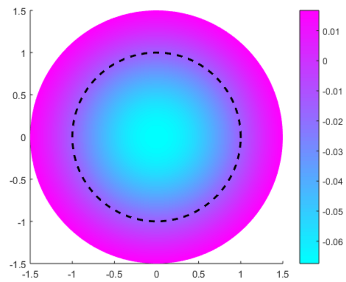


(c) L-shaped domain, $\gamma = 0.5$

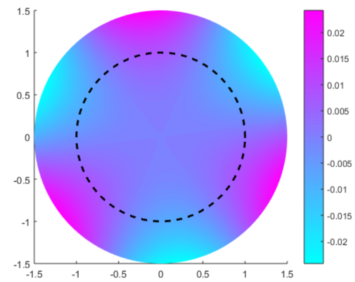


(d) L-shaped domain, $\gamma = 2$

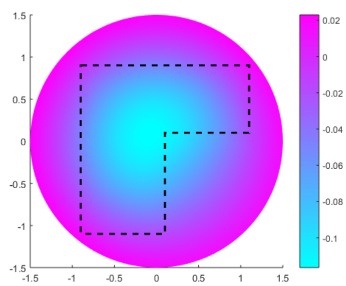
Figure 3: Sensitivity of modified transmission eigenvalues to changes in the size of a flaw for the case $B \neq D$ and both $\gamma = 0.5$ (left column) and $\gamma = 2$ (right column). We plot the change in the first few eigenvalues for each region due to the presence of the flaw. The numbers in each legend refer to the computed eigenvalue of the unflawed region. Note that the sensitivity of the first few eigenvalues to changes in the index of refraction due to a flaw is significantly increased if $\gamma = 0.5$ instead of $\gamma = 2$.



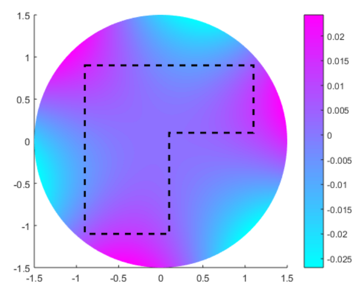
(a) disk, $\eta_1 = -7.1723$



(b) disk, $\eta_7 = 10.002$



(c) L-shaped domain, $\eta_1 = -12.3088$



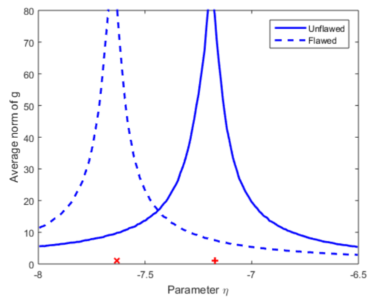
(d) L-shaped domain, $\eta_7 = 10.1031$

Figure 4: Plots of the eigenfunction w (scaled by $1/\|v\|_{\ell^2}$) for two eigenvalues for each region, where we have superimposed D onto each plot. Each eigenvalue in the left column is sensitive to flaws in the region, and each eigenvalue in the right column is insensitive to flaws due to the difference in the magnitude of w inside D.

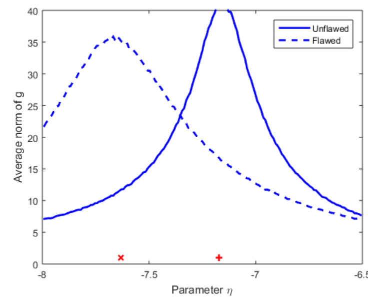
the eigenfunction w in a neighborhood of the flaw. Note that, to a first-order approximation, from (5.11) we have that this shift is not affected by multiplying w and v by a constant, and hence we have scaled w by $1/\|v\|_{\ell^2}$ in Figure 4. For an eigenvalue of multiplicity two, one count of the eigenvalue consistently shifts in the presence of a flaw away from the center of the domain, and the other count varies little. However, our choice of B as a disk places the region D in the center of the domain, and hence we do not observe significant shifts of eigenvalues with multiplicity.

7.2. Computing modified transmission eigenvalues from far field data

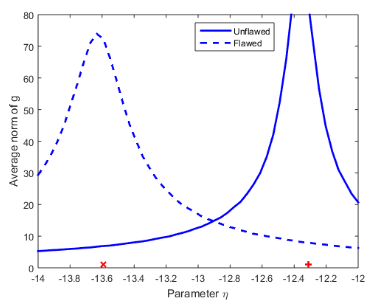
In Section 5 we showed that modified transmission eigenvalues may be computed from far field data by solving the far field equation (5.1). As described at the beginning of this section, we use Tikhonov regularization in order to solve the discrete version of this equation, a procedure which can be justified by using the generalized linear sampling method [2]. We focus our attention on the eigenvalue



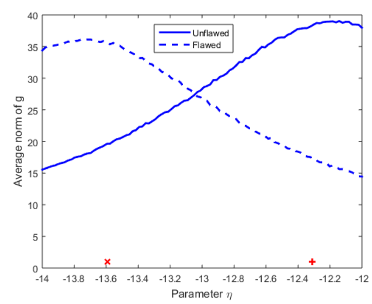
(a) disk, $\epsilon = 0.01$



(b) disk, $\epsilon = 0.05$



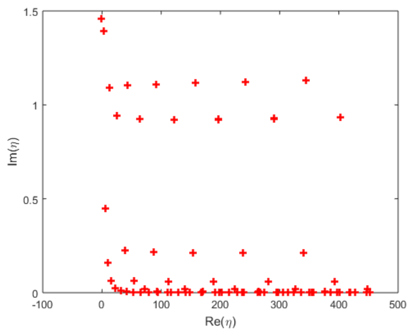
(c) L-shaped domain, $\epsilon = 0.01$



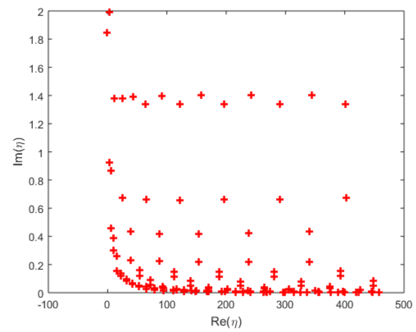
(d) L-shaped domain, $\epsilon = 0.05$

Figure 5: Shift in the peaks of the norm of g_z^ϵ due to the presence of a flaw in the case $B \neq D$ with $\gamma = 0.5$. The noise is set at $\epsilon = 0.01$ (left column) and $\epsilon = 0.05$ (right column). Each cross refers to the eigenvalue for the unflawed domain, and each $\times$ symbol refers to the eigenvalue for the flawed domain. The peaks in the norm of g_z^ϵ clearly shift due to the flaw in each domain, even in the presence of noise.

corresponding to $\gamma = 0.5$ which displays the greatest sensitivity to the presence of flaws for each region ($\eta_1 = -7.1723$ for the disk and $\eta_1 = -12.3088$ for the L-shaped domain), and we introduce noise to the scattering data as described above. In Figure 5 we present our results for detecting a flaw of radius $r_c = 0.1$ in the disk and the L-shaped domain using far field data. For the disk, we choose $(x_c, y_c) = (0.3, 0.2)$, and for the L-shaped domain, we choose $(x_c, y_c) = (0.1, 0.4)$. We note that when we compute the approximation of h_∞ for each value of η, we do not include the presence of the flaw in the region, as in practice we would not have any a priori information about the size or location of the flaw. We note that for both regions, the peaks in the norm of g_z^ϵ correspond to the eigenvalues computed using our finite element eigensolver, both with and without a flaw. Even in the presence of noise, we are able to distinguish between the two peaks and clearly see the shift caused the flaw in each region. Further testing shows that other eigenvalues may computed as well with similar results. However, higher eigenvalues may appear with their peaks slightly shifted.



(a) disk

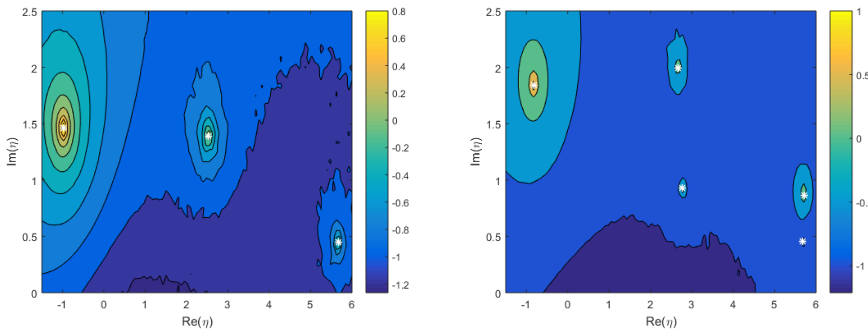


(b) L-shaped domain

Figure 6: Modified transmission eigenvalues for $n = 4+4i$, $\gamma = 0.5$, and $B \neq D$. Many eigenvalues have small imaginary part, but none are real as predicted. The eigenvalues in these plots appear to lie in a strip along the real axis, which conforms to our prediction from Theorem 6.6 even though $n(x)$ is not infinitely differentiable in B. Note that complex modified transmission eigenvalues behave differently than Stekloff eigenvalues [6].

7.3. Complex modified transmission eigenvalues

In this section, we investigate the case of complex-valued n. From Section 6, we have that all of the modified transmission eigenvalues have positive imaginary part whenever n has nonzero imaginary part on an open subset of D. In Figure 6 we plot the eigenvalues in the complex plane for $n = 4 + 4i$ and both the disk and the L-shaped domain with $\gamma = 0.5$ and $B \neq D$, and we see that all eigenvalues in this plot have positive imaginary part as expected, though many have small imaginary part. Moreover, the eigenvalues appear to lie in a strip along the real axis, which conforms to our result in Theorem 6.6 even though $n(x)$ is not infinitely differentiable in B (as it is discontinuous across ∂D). Note that the behavior is different than for Stekloff eigenvalues [6, Figure 11]. In Figure 7, we provide two examples which support our claim that modified transmission eigenvalues may be detected from far field data in the case of absorbing media. We perform the same procedure described previously, but now we choose $n = 4 + 4i$ and we vary η in a region of the complex plane. For the disk (Figure 7a), we see that all three eigenvalues in this region are detected, and for the L-shaped domain, all but one eigenvalue is detected. For each region, the leftmost peak is more prominent than the others, suggesting that this eigenvalue may be useful in detecting flaws as in the case of real n.



(a) disk (b) L-shaped domain

Figure 7: Contour plot of the base 10 logarithm of the averaged norm of g_z^ϵ for $n = 4 + 4i$, $\gamma = 0.5$, and $B \neq D$. The peaks in the norm of g_z^ϵ correspond to the modified transmission eigenvalues, shown as the white $*$ symbols. Note that the leftmost peak in each plot is more prominent than the others.

8. Conclusion

We have considered scattering of an incident plane wave by an isotropic inhomogeneous medium of compact support with (possibly complex-valued) refractive index n. By introducing an artificial scattering problem depending on a (possibly complex) parameter η and a fixed parameter $\gamma > 0$, $\gamma \neq 1$, we have constructed a modified far field operator which may be used to detect values of the parameter η for which nontrivial solutions exist to a modified interior transmission problem, i.e. for which η is a modified transmission eigenvalue. We have shown that such computation is possible from given far field data, and numerical testing has indicated that changes in the refractive index (including the presence of cavities) result in shifts of the modified transmission eigenvalues. This shift depends on the values of the associated eigenfunctions in a neighborhood of the flaw, and the fixed parameter γ may be tuned to improve sensitivity of the eigenvalues. As with the case of Stekloff eigenvalues, this class of eigenvalues has many advantages over classical transmission eigenvalues in the context of nondestructive testing using target signatures. In particular, modified transmission eigenvalues may be computed from far field data measured at a single frequency independent of the tested material, and they may even be computed for absorbing media. A significant practical advantage is that the artificial problem may be solved in a ball B with the support of $1 - n$ in its interior, which allows for the computation and storage of the far field pattern for the artificial problem with many different values of η in advance of testing.

9. Acknowledgments

We thank Irene de Teresa for sharing her `FreeFem++` code used to simulate scattering data.

References

[1] S. Agmon. *Lectures on Elliptic Boundary Value Problems.* VanNostrand Mathematical Studies, Vol. 2, 1965.

[2] L. Audibert, F. Cakoni, and H. Haddar. New sets of eigenvalues in inverse scattering for inhomogeneous media and their determination from scattering data, *to appear.*

[3] C. Baum. *The Singularity Expansion Method in Electromagnetics: a Summary Survey and Open Questions.* Lulu Enterprises, 2012.

[4] A. S. Bonnet-Ben Dhia, L. Chesnel, and H. Haddar. On the use of T-coercivity to study the interior transmission eigenvalue problem. *C. R. Math. Acad. Sci. Paris*, 349:647–651, 2011.

[5] F. Cakoni, D. Colton, and H. Haddar. *Inverse Scattering Theory and Transmission Eigenvalues.* CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 88, SIAM, Philadelphia, 2016.

[6] F. Cakoni, D. Colton, S. Meng, and P. Monk. Stekloff eigenvalues in inverse scattering. *SIAM Journal on Applied Mathematics*, 76(4):1737–1763, 2016.

[7] J. Camaño, C. Lackner, and P. Monk. Electromagnetic Stekloff eigenvalues in inverse scattering. *SIAM J. Math. Anal.*, to appear.

[8] S. Cogar, D. Colton, and Y.J. Leung. The inverse spectral problem for transmission eigenvalues. *Inverse Problems*, 33:055015, 2017.

[9] D. Colton and R. Kress. *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer, New York, 3rd edition, 2013.

[10] N. Dunford and J.T. Schwartz. *Linear Operators.* Part I-III. Interscience Publishers, New York, 1958.

[11] S. Dyatlov and M. Zworski. *Mathematical Theory of Scattering Resonances.* book in preparation.

[12] F. Gylys-Colwell. An inverse problem for the Helmholtz equation. *Inverse Problems*, 12:139–156, 1996.

[13] P.C. Hansen. Regularization tools: a MATLAB package for analysis and solution of discrete ill-posed problems. *Numerical Algorithms*, 6:1–35, 1994.

[14] I. Harris, F. Cakoni, and J. Sun. Transmission eigenvalues and non-destructive testing of anisotropic magnetic materials with voids. *Inverse Problems*, 30:035016, 2014.

[15] F. Hecht. New development in `FreeFem++`. *J. Numer. Math.*, 20:251–265, 2012.

[16] J.M. Lee and G. Uhlmann. Determining anisotropic real-analytic conductivities by boundary measurements. *Communications on Pure and Applied Mathematics*, 42:1097–1112, 1989.

[17] L. Robbiano. Spectral analysis of the interior transmission eigenvalue problem. *Inverse Problems*, 29:104001, 2013.

[18] M. Zworski. *Semiclassical Analysis*. Graduate Studies in Mathematics, Vol. 138. American Mathematical Society, Providence, RI, 2012.