A modified transmission eigenvalue problem for scattering by a partially coated crack

S. Cogar*

Abstract

We consider the detection of changes in the surface impedance of a partially coated crack, which we infer from the shift in a target signature arising from a modified interior transmission eigenvalue problem. We study this problem in a general setting in which the properties of the scattering medium are encoded in a Dirichlet-to-Neumann operator T, and we provide sufficient conditions for T that imply desirable properties of the eigenvalues of this problem. We conclude by placing scattering by a partially coated crack into this general framework and investigating the sensitivity of the associated eigenvalues to changes in the surface impedance with a series of numerical examples.

Key words. inverse scattering, nondestructive testing, modified transmission eigenvalues, generalized linear sampling method, non-selfadjoint eigenvalue problems

AMS subject classifications. 35J25, 35P05, 35P25, 35R30

1. Introduction

Many materials are covered in a thin coating in order to protect them from external factors, and the ability to evaluate the integrity of these coatings is an important problem in the field of nondestructive testing. In particular, we investigate the detection of changes in the material properties of an infinite cylinder with an open arc Γ in $\mathbb{R}^2$ as its cross section. We assume that Γ is a perfect conductor coated on one side by a material with surface impedance σ, and we collect far field data of the scattering by this cylinder of time-harmonic E-polarized electromagnetic incident fields. This scattering problem may be modeled as a mixed boundary value problem for the Helmholtz equation in $\mathbb{R}^2$ in the exterior of the open arc Γ, which is often referred to as a crack. We acknowledge the realistic possibility that this crack is embedded in a possibly inhomogeneous and anisotropic background medium [20], but for simplicity we assume that no such medium is present.

*Department of Mathematical Sciences, University of Delaware, Newark, DE 19716 (cogar@udel.edu).

This problem has received considerable attention for a variety of boundary conditions beyond those considered here, and we refer to [6] for further details on its history and appropriate references. Much of the previous work has involved reconstructing the shape of the crack by qualitative methods such as the linear sampling method [5] and the factorization method [16], and efforts have also focused on determining information on the surface impedance σ from far field data (cf. [19] and [20]). In our case we assume that the shape of the crack is known using one of the methods mentioned above, and rather than attempt to determine the surface impedance σ we aim only to detect changes in σ compared to some reference material. While this change could in principle be observed from the far field data directly, the presence of noise in realistic measurements makes this approach unlikely to reliably detect changes in a material.

Instead we use the idea of a target signature, and to this end we introduce an auxiliary scattering problem depending on a parameter $\eta \in \mathbb{C}$. Modifying the measured scattering data with this auxiliary data leads us to consider the eigenvalue problem of finding η and nontrivial fields w, v satisfying

$$\Delta w + k^2 w = 0 \text{ in } B \setminus \overline{\Gamma}, \tag{1.1a}$$

$$\gamma \Delta v + k^2 \eta v = 0 \text{ in } B, \tag{1.1b}$$

$$w - v = 0 \text{ on } \partial B, \tag{1.1c}$$

$$\frac{\partial w}{\partial \nu} - \gamma \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B, \tag{1.1d}$$

$$w^- = 0 \text{ on } \Gamma, \tag{1.1e}$$

$$\frac{\partial w^+}{\partial \nu} + i\sigma w^+ = 0 \text{ on } \Gamma, \tag{1.1f}$$

where B is a bounded Lipschitz domain with connected complement (e.g. a disk) chosen to include the closure of the crack Γ in its interior and $\gamma > 0$ is a fixed constant. We use the eigenvalue η as the target signature, and we would like to detect changes in σ from shifts in the eigenvalues compared to some known reference values. This method has been previously studied in the context of flaw detection in an inhomogeneous medium in [1], [11], and [12], and we refer to [8] and [9] for a similar method using a Stekloff auxiliary problem. However, rather than study this problem directly, we investigate a more general eigenvalue problem in which the physical scattering problem is encoded in a Dirichlet-to-Neumann operator T. This approach allows us to derive sufficient conditions on T which guarantee certain properties of the eigenvalues without specifying the scattering medium, and we use this general framework in order to deduce properties of (1.1a)–(1.1f), including the ability to detect the eigenvalues from far field data.

The outline of our paper is as follows. In the next section we introduce a general scattering problem in terms of an operator T and the aforementioned auxiliary problem, which leads us to consider an eigenvalue problem for the negative Laplacian with a generalized Robin boundary condition dependent upon T. In Section 3 we study a nonhomogeneous version of this boundary value problem and provide sufficient conditions on T which guarantee properties of the eigenvalues such as discreteness, existence, and their distribution, and in Section 4 we establish that under certain restrictions on T the eigenvalues may be computed from far field data using the generalized linear sampling method. We present our application of interest in Section 5, where we place scattering by a partially coated crack into the general framework developed in the previous sections, and in Section 6 we investigate the sensitivity of the eigenvalues to changes in the surface impedance σ with a series of numerical examples.

2. A general eigenvalue problem

In this section we introduce a generalized scattering problem in which the information on the particular scatterer is encoded by a bounded linear operator $T : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$, and in this framework we derive a general eigenvalue problem depending on T. We choose a bounded Lipschitz domain $B \subset \mathbb{R}^m$, $m = 2, 3$, with connected complement (e.g. a disk centered at the origin in $\mathbb{R}^2$) which is sufficiently large in the sense that the total field of the scattering problem of interest satisfies the Helmholtz equation in $\mathbb{R}^m \setminus \overline{B}$, and we let ν denote the outward unit normal vector to the boundary ∂B. Given an incident field $u^i \in H^1(\mathbb{R}^m)$ which satisfies the Helmholtz equation in $\mathbb{R}^m$, we consider the problem of finding a scattered field $u^s \in H^1_{loc}(\mathbb{R}^m \setminus \overline{B})$ satisfying

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^m \setminus \overline{B}, \tag{2.1a}$$

$$\frac{\partial u^s}{\partial \nu} - Tu^s = -\frac{\partial u^i}{\partial \nu} + Tu^i \text{ on } \partial B, \tag{2.1b}$$

$$\lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \tag{2.1c}$$

where the wave number $k > 0$ is fixed and the Sommerfeld radiation condition (2.1c) is assumed to hold uniformly in all directions. Writing an equivalent problem to (2.1a)–(2.1c) using an exterior Dirichlet-to-Neumann map and following the proof of Theorem 1.38 in [7] provides the following result.

Theorem 2.1. *The generalized scattering problem* (2.1a)–(2.1c) *is well-posed provided that*

(i) there exists an operator $\tilde{T} : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ with nonnegative real part for which $T - \tilde{T}$ is compact;

(ii) the imaginary part of T is nonpositive.

For the plane wave incident field $u^i = e^{ikx\cdot d}$ (with incident direction $d \in \mathbb{S}^{m-1}$) the scattered field u^s satisfying (2.1a)–(2.1c) has the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(m-1)/2}} u_\infty(\hat{x}, d) + O\left(|x|^{-(m+1)/2}\right),$$

where $\hat{x} = x/|x|$ and $u_\infty(\cdot, d)$ is the *far field pattern* (we have explicitly shown its dependence on the incident direction d). We define the *far field operator* $F : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$ as

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^{m-1}} u_\infty(\hat{x}, d) g(d) ds(d), \; \hat{x} \in \mathbb{S}^{m-1}. \tag{2.2}$$

A desirable property of the far field pattern is that it satisfies the *reciprocity principle*

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x}) \quad \forall \hat{x}, d \in \mathbb{S}^{m-1}, \tag{2.3}$$

and the following theorem establishes a sufficient condition for (2.3) to hold. We will see further importance of this assumption in Section 4.

Assumption 2.2. *Assume that the operator T satisfies*

$$\int_{\partial B} (g_1 T g_2 - g_2 T g_1) ds = 0 \quad \forall g_1, g_2 \in H^{1/2}(\partial B). \tag{2.4}$$

Theorem 2.3. *If the operator T satisfies Assumption 2.2, then the far field pattern u_∞ satisfies the reciprocity relation* (2.3).

Proof. Following the same lines as in the proof of the reciprocity principle for sound-soft obstacles (c.f. Theorem 3.15 in [13]), we see that

$$\gamma_m^{-1} \left[u_\infty(\hat{x}, d) - u_\infty(-d, -\hat{x}) \right] = \int_{\partial B} \Big[u(y, d) T u(y, -\hat{x}) - u(y, -\hat{x}) T u(y, d) \Big] ds(y),$$

where γ_m is a constant depending only on m, and consequently (2.3) follows from (2.4). ☐

Remark 2.4. Two examples in which (2.4) clearly holds are $Tg := 0$, in which case (2.1a)–(2.1c) is an exterior Neumann problem, and $Tg := -i\sigma g$ for some $\sigma > 0$, in which case (2.1a)–(2.1c) is an exterior impedance problem.

As the operator T encodes information about a scattering problem of interest, our aim is to detect changes in the operator T from its associated far field data, and in order to do so we introduce the following auxiliary problem. We let $\gamma > 0$ be a fixed constant not equal to one, and given a parameter $\eta \in \mathbb{C}$ we consider the *transmission auxiliary problem* of finding the scattered field $u_0^s \in H^1_{loc}(\mathbb{R}^m \setminus \overline{B})$ and the total field $u_0 \in H^1(B)$ which satisfy

$$\gamma \Delta u_0 + k^2 \eta u_0 = 0 \text{ in } B, \tag{2.5a}$$

$$\Delta u_0^s + k^2 u_0^s = 0 \text{ in } \mathbb{R}^m \setminus \overline{B}, \tag{2.5b}$$

$$u_0 - u_0^s = u^i \text{ on } \partial B, \tag{2.5c}$$

$$\gamma \frac{\partial u_0}{\partial \nu} - \frac{\partial u_0^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \text{ on } \partial B, \tag{2.5d}$$

$$\lim_{r \to \infty} r^{\frac{m-1}{2}} \left(\frac{\partial u_0^s}{\partial r} - ik u_0^s \right) = 0, \tag{2.5e}$$

where the Sommerfeld radiation condition (2.5e) is assumed to hold uniformly in all directions. This problem is well-posed provided that $\mathrm{Im}(\eta) \geq 0$ [7], and moreover it follows from an application of the analytic Fredholm theorem [13, Theorem 8.26] that (2.5a)–(2.5e) is well-posed for all η except in a discrete set (cf. [12] for a similar computation). We denote the far field pattern of the scattered field u_0^s from (2.5a)–(2.5e) with $u^i(x) = e^{ikx\cdot d}$ as $u_{0,\infty}(\cdot, d)$, and it has been shown that the auxiliary far field pattern satisfies the reciprocity principle $u_{0,\infty}(\hat{x}, d) = u_{0,\infty}(-d, -\hat{x})$ for all $\hat{x}, d \in \mathbb{S}^{m-1}$ [7]. The *auxiliary far field operator* F_0 is defined in the same manner as F with $u_{0,\infty}(\cdot, d)$ in place of $u_\infty(\cdot, d)$.

We define the *modified far field operator* $\mathcal{F} : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$ as the difference of the far field operators F and F_0, which may be written explicitly as

$$(\mathcal{F}g)(\hat{x}) := \int_{\mathbb{S}^{m-1}} \Big[u_\infty(\hat{x}, d) - u_{0,\infty}(\hat{x}, d)\Big] g(d)ds(d), \ \hat{x} \in \mathbb{S}^{m-1}, \tag{2.6}$$

and the following theorem provides a characterization of when $\mathcal{F}$ is injective with dense range. We first recall the definition of the *Herglotz wave function* with kernel $g \in L^2(\mathbb{S}^{m-1})$ as

$$u_g(x) := \int_{\mathbb{S}^{m-1}} e^{ikx\cdot d} g(d)ds(d), \ x \in \mathbb{R}^m, \tag{2.7}$$

and we observe by linearity that Fg is the far field pattern of (2.1a)–(2.1c) with $u^i = u_g$ and that the same relationship holds for the auxiliary far field operator F_0 and solutions of (2.5a)–(2.5e).

Theorem 2.5. *Under Assumption 2.2, the modified far field operator* $\mathcal{F} : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$ *is injective with dense range if and only if the* interior Robin problem

$$\gamma\Delta v + k^2\eta v = 0 \text{ in } B, \tag{2.8a}$$

$$\gamma\frac{\partial v}{\partial \nu} - Tv = 0 \text{ on } \partial B, \tag{2.8b}$$

has no nontrivial solutions of the form

$$v(x) = \int_{\mathbb{S}^{m-1}} u_0(x, d)g(d)ds(d), \ x \in B. \tag{2.9}$$

Proof. If $\mathcal{F}g = 0$ for some $g \in L^2(\mathbb{S}^{m-1})$, then $w_\infty = v_\infty$, where w_∞ and v_∞ are the far field patterns of the scattered fields w^s and v^s arising from the scattering problem (2.1a)–(2.1c) and the auxiliary problem (2.5a)–(2.5e), respectively, with the incident field $u^i = u_g$. Rellich's lemma implies that $w^s = v^s$ in $\mathbb{R}^m \setminus \overline{B}$, and in particular we see that w^s and v^s share Cauchy data on ∂B. If $v \in H^1(B)$

denotes the total field satisfying (2.5a)–(2.5e) with $u^i = u_g$, then it follows that

$$\begin{aligned}\gamma\frac{\partial v}{\partial \nu} - Tv &= \left(\frac{\partial v^s}{\partial \nu} - Tv^s\right) + \left(\frac{\partial u_g}{\partial \nu} - Tu_g\right) \\ &= \left(\frac{\partial w^s}{\partial \nu} - Tw^s\right) + \left(\frac{\partial u_g}{\partial \nu} - Tu_g\right),\end{aligned}$$

which vanishes due to the boundary condition (2.1b). We conclude that $v \in H^1(B)$ satisfies the interior Robin problem (2.8a)–(2.8b). If this problem admits only the trivial solution $v = 0$, then we must have $g = 0$ by well-posedness of the auxiliary problem and injectivity of the Herglotz mapping $g \mapsto u_g$ [13], and we obtain injectivity of $\mathcal{F}$.

Conversely, suppose that there exists a nontrivial solution v of the interior Robin problem (2.8a)–(2.8b) of the form (2.9) for some $g \in L^2(\mathbb{S}^{m-1})$, and define

$$v^s(x) := \int_{\mathbb{S}^1} u_0^s(x,d)g(d)ds(d),\ x \in \mathbb{R}^m \setminus \overline{B}.$$

By superposition we see that (v, v^s) satisfies the auxiliary problem (2.5a)–(2.5e) with incident field $u^i = u_g$, and in particular the Cauchy data of v and $v^s + u_g$ coincides on ∂B. As a result we see that

$$\frac{\partial v^s}{\partial \nu} - Tv^s = -\frac{\partial u_g}{\partial \nu} + Tu_g \text{ on } \partial B,$$

and it follows that $v^s \in H^1_{loc}(\mathbb{R}^m \setminus \overline{B})$ satisfies (2.1a)–(2.1c). Well-posedness of this problem and linearity of (2.5a)–(2.5e) imply that $Fg = v_\infty$ and $F_0 g = v_\infty$, respectively, and consequently we obtain $\mathcal{F}g = 0$ by definition. Since this g must be nonzero in order for the interior Robin problem (2.8a)–(2.8b) to have a nontrivial solution, we conclude that $\mathcal{F}$ is not injective.

It remains to show that $\mathcal{F}$ is injective if and only if it has dense range, but this holds as in the proof of Corollary 1.16 of [7] since both $u_\infty(\hat{x}, d)$ and $u_{0,\infty}(\hat{x}, d)$ satisfy the reciprocity principle. □

We see that (2.8a)–(2.8b) is an eigenvalue problem for the negative Laplacian with a nonlocal Robin boundary condition, which explains our choice of the name *interior Robin problem*. We refer to values of η for which the interior Robin problem has a nontrivial solution as *Robin eigenvalues associated with* T. When the choice of T is understood, we will simply use the term *Robin eigenvalue* to refer to such values of η. We remark that in the context of scattering by inhomogeneous media the eigenvalue problem (2.8a)–(2.8b) first appeared in [1], where it served to relate the structure of this problem to the Stekloff eigenvalue problem. In the next section we study this problem in greater detail and relate the properties of the Robin eigenvalues to those of the operator T. The specific properties of T we consider are motivated by different types of scattering problems, and as a result we stray somewhat from our intended application of scattering by a partially coated crack for purposes of generality.

3. Properties of the interior Robin problem

We begin by introducing the following nonhomogeneous version of (2.8a)–(2.8b). Given $f \in L^2(B)$ and $h \in H^{-1/2}(\partial B)$, we consider the problem of finding $v \in H^1(B)$ such that

$$\gamma \Delta v + k^2 \eta v = f \text{ in } B, \tag{3.1a}$$

$$\gamma \frac{\partial v}{\partial \nu} - Tv = h \text{ on } \partial B. \tag{3.1b}$$

In order to study this problem we consider the equivalent variational problem of finding $v \in H^1(B)$ such that

$$a_\eta(v, v') = \ell(v') \quad \forall v' \in H^1(B), \tag{3.2}$$

where the bounded sesquilinear form $a_\eta(\cdot,\cdot)$ is defined as

$$a_\eta(v, v') := \gamma(\nabla v, \nabla v')_B - k^2 \eta (v, v')_B - \langle Tv, v' \rangle_{\partial B} \quad \forall v, v' \in H^1(B),$$

and the bounded antilinear functional ℓ on $H^1(B)$ is defined as

$$\ell(v') := (f, v')_B + \langle h, v' \rangle_{\partial B}.$$

For a given open set $\mathcal{O}$ with boundary $\partial\mathcal{O}$, we have used $(\cdot,\cdot)_{\mathcal{O}}$ to denote the inner product on $L^2(\mathcal{O})$ and $\langle \cdot,\cdot \rangle_{\partial\mathcal{O}}$ to denote the duality pairing of $H^{-1/2}(\partial\mathcal{O})$ and $H^{1/2}(\partial\mathcal{O})$. Though not relevant to our present application of interest, in the following remark we discuss a desirable property of T which guarantees that this problem is of Fredholm type, and in fact this property is held by the examples given in Remark 2.4.

Remark 3.1. If $T : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ satisfies

$$-\operatorname{Re} \langle Tg, g \rangle_{\partial B} \geq 0$$

for all $g \in H^{1/2}(\partial B)$, then we see that

$$\begin{aligned} \operatorname{Re} a_{-1}(v, v) &= \gamma(\nabla v, \nabla v)_B + k^2 (v, v)_B - \operatorname{Re} \langle Tv, v \rangle_{\partial B} \\ &\geq \gamma(\nabla v, \nabla v)_B + k^2 (v, v)_B \end{aligned}$$

for all $v \in H^1(B)$ and hence $a_{-1}(\cdot,\cdot)$ is coercive. We also see that

$$a_\eta(v, v') - a_{-1}(v, v') = -k^2 (1 + \eta)(v, v')_B$$

for all $v, v' \in H^1(B)$, which due to the compact embedding of $H^1(B)$ into $L^2(B)$ represents a compact sesquilinear form. If we write

$$a_\eta(v, v') = a_{-1}(v, v') + [a_\eta(v, v') - a_{-1}(v, v')],$$

then we see that $a_\eta(\cdot,\cdot)$ is a compact perturbation of a coercive sesquilinear form, and it follows that the variational problem (3.2) and equivalently (3.1a)–(3.1b) satisfies the Fredholm property.

We now consider assumptions on T which include scattering by inhomogeneous media and scattering by a crack, which is our present application of interest. We recall the definition of the space $H^1_\Delta(B)$ as

$$H^1_\Delta(B) := \{\psi \in H^1(B) \mid \Delta\psi \in L^2(B)\}$$

equipped with the inner product

$$(\psi_1, \psi_2)_{H^1_\Delta(B)} := (\psi_1, \psi_2)_{H^1(B)} + (\Delta\psi_1, \Delta\psi_2)_B$$

and the usual induced norm $\|\cdot\|_{H^1_\Delta(B)}$.

Assumption 3.2. *Assume that $T : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ may be factorized as $T = N_{\partial B} S$, where the bounded linear operator $S : H^{1/2}(\partial B) \to H^1_\Delta(B)$ satisfies $(S\psi)|_{\partial B} = \psi$ for all $\psi \in H^{1/2}(\partial B)$ and the Neumann trace operator $N_{\partial B} : H^1_\Delta(B) \to H^{-1/2}(\partial B)$ is defined as*

$$N_{\partial B}\varphi := \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial B}.$$

In addition, we assume that there exists an operator $\tilde{T} : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ such that $\tilde{T} - T : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ is compact and which satisfies the following conditions.

Condition 1: If $0 < \gamma < 1$, then there exist positive constants $\alpha_0 > 0$ and $\epsilon_1, \epsilon_2 \in (0,1)$ for which

$$\mathrm{Re}\left\langle \tilde{T}g, g\right\rangle_{\partial B} \geq \gamma\epsilon_1^{-1}(\nabla(Sg), \nabla(Sg))_B + k^2\alpha_0\epsilon_2^{-1}(Sg, Sg)_B \quad \forall g \in H^{1/2}(\partial B). \tag{3.3}$$

Condition 2: If $\gamma > 1$, then there exist positive constants $\delta \in (0, \gamma)$ and $c > 0$ for which

$$-\mathrm{Re}\left\langle \tilde{T}v, v\right\rangle_{\partial B} \geq -\delta(\nabla v, \nabla v)_B - c(v, v)_B \quad \forall v \in H^1(B). \tag{3.4}$$

Given that Assumption 3.2 holds, we begin by defining the operators $\hat{\mathbb{A}}, \mathbb{B}_\eta : H^1(B) \to H^1(B)$ by means of the Riesz representation theorem such that

$$(\hat{\mathbb{A}}v, v')_{H^1(B)} = \gamma(\nabla v, \nabla v')_B + k^2\alpha(v, v')_B - \left\langle \tilde{T}v, v'\right\rangle_{\partial B},$$
$$(\mathbb{B}_\eta v, v')_{H^1(B)} = -k^2(\eta + \alpha)(v, v')_B + \left\langle (\tilde{T} - T)v, v'\right\rangle_{\partial B}$$

for all $v, v' \in H^1(B)$, where $\tilde{T}$ is the operator from Assumption 3.2 and $\alpha > 0$ is a constant to be determined later. We observe that

$$a_\eta(v, v') = ((\hat{\mathbb{A}} + \mathbb{B}_\eta)v, v')_{H^1(B)}$$

for all $v, v' \in H^1(B)$, and as a result our study of the solvability of (3.2) reduces to that of the operators $\hat{\mathbb{A}}$ and $\mathbb{B}_\eta$.

Lemma 3.3. *If $\gamma \neq 1$ and Assumption 3.2 holds, then the operator $\hat{\mathbb{A}} : H^1(B) \to H^1(B)$ is invertible.*

Proof. We begin with the case $0 < \gamma < 1$, and we assume that T satisfies Condition 1. In this case, we must make use of the idea of $\mathcal{T}$-coercivity (cf. [4] and [7]), and to this end we define the bounded linear operator $\mathcal{T} : H^1(B) \to H^1(B)$ by $\mathcal{T}v := v - 2Sv$ for all $v \in H^1(B)$, where S is the operator from the factorization in Condition 1 and we have written Sv rather than $S(v|_{\partial B})$ for convenience. Our assumption that $(Sg)|_{\partial B} = g$ for all $g \in H^{1/2}(\partial B)$ implies that $\mathcal{T}^2 = I$ and consequently $\mathcal{T}$ is an isomorphism. We define the bounded sesquilinear form $\hat{a}^{\mathcal{T}}(\cdot,\cdot)$ as

$$\hat{a}^{\mathcal{T}}(v,v') := (\hat{\mathbb{A}}v, \mathcal{T}v')_{H^1(B)} = (\hat{\mathbb{A}}v, v' - 2Sv')_{H^1(B)} \quad \forall v, v' \in H^1(B),$$

and we see that (since $v - 2Sv = -v$ on ∂B)

$$\begin{aligned}\operatorname{Re} \hat{a}^{\mathcal{T}}(v,v) = \gamma(\nabla v, \nabla v)_B + k^2\alpha(v,v)_B + \operatorname{Re}\left\langle \tilde{T}v, v\right\rangle_{\partial B} \\ - 2\gamma(\nabla v, \nabla(Sv))_B - 2k^2\alpha(v, Sv)_B\end{aligned}$$

for all $v \in H^1(B)$. Applying Young's inequality with the constants $\epsilon_1, \epsilon_2 \in (0,1)$ from Condition 1 yields the inequalities

$$\begin{aligned}2\mathrm{Re}(\nabla v, \nabla(Sv))_B &\leq \epsilon_1(\nabla v, \nabla v)_B + \epsilon_1^{-1}(\nabla(Sv), \nabla(Sv))_B, \\ 2\mathrm{Re}(v, Sv)_B &\leq \epsilon_2(v,v)_B + \epsilon_2^{-1}(Sv, Sv)_B\end{aligned}$$

for all $v \in H^1(B)$. It follows that

$$\begin{aligned}\operatorname{Re} \hat{a}^{\mathcal{T}}(v,v) \geq{}& \gamma(1-\epsilon_1)(\nabla v, \nabla v)_B + k^2\alpha(1-\epsilon_2)(v,v)_B \\ &+\Big[\operatorname{Re}\left\langle \tilde{T}v, v\right\rangle_{\partial B} - \gamma\epsilon_1^{-1}(\nabla(Sv), \nabla(Sv))_B - k^2\alpha\epsilon_2^{-1}(Sv,Sv)_B\Big].\end{aligned}$$

From the last part of Condition 1 we conclude that $\hat{a}^{\mathcal{T}}(\cdot,\cdot)$ is coercive for the choice $\alpha = \alpha_0$, and the fact that $\mathcal{T}$ is an isomorphism allows us to apply the Lax-Milgram lemma to conclude that $\hat{\mathbb{A}}$ is invertible with bounded inverse.

We now consider the case $\gamma > 1$, and we assume that T satisfies Condition 2. Though the idea of $\mathcal{T}$-coercivity is not explicitly required in this case, we remark that it is implicitly built into Condition 2. The last part of Condition 2 implies that

$$\mathrm{Re}(\hat{\mathbb{A}}v, v)_{H^1(B)} \geq (\gamma - \delta)(\nabla v, \nabla v)_B + (k^2\alpha - c)(v,v)_B$$

for all $v \in H^1(B)$, and taking α to be sufficiently large yields coercivity of $\hat{\mathbb{A}}$. An application of the Lax-Milgram lemma implies that $\hat{\mathbb{A}}$ is invertible with bounded inverse. $\square$

Since compactness of $\mathbb{B}_\eta : H^1(B) \to H^1(B)$ follows easily from compactness of $\tilde{T} - T$ and the compact embedding of $H^1(B)$ into $L^2(B)$, we conclude that $\hat{\mathbb{A}} + \mathbb{B}_\eta$ is a Fredholm operator of index zero, and we have proved the following theorem.

Theorem 3.4. *If $\gamma \neq 1$ and Assumption 3.2 holds, then* (3.1a)–(3.1b) *satisfies the Fredholm property.*

An immediate corollary of the Fredholm property of (3.2) is that (3.1a)–(3.1b) is well-posed provided that η is not a Robin eigenvalue. We now proceed to establish sufficient conditions for T to guarantee certain properties of its associated Robin eigenvalues.

Theorem 3.5. *If the operator $T : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ satisfies*

$$-\mathrm{Im}\,\langle Tg, g\rangle_{\partial B} \geq 0 \quad \forall g \in H^{1/2}(\partial B),$$

then every Robin eigenvalue associated with T has nonnegative imaginary part.

Proof. If (η, v) is a nontrivial Robin eigenpair, then taking the imaginary part of the equation $a_\eta(v, v) = 0$ yields

$$-k^2\mathrm{Im}(\eta)(v, v)_B - \mathrm{Im}\,\langle Tv, v\rangle_{\partial B} = 0.$$

Since $v \neq 0$, we may solve for $\mathrm{Im}(\eta)$ to obtain

$$\mathrm{Im}(\eta) = -\frac{\mathrm{Im}\,\langle Tv, v\rangle_{\partial B}}{k^2(v, v)_B} \geq 0.$$

□

Theorem 3.6. *If there exists $\eta_0 \in \mathbb{C}$ which is not a Robin eigenvalue associated with T, then the set of Robin eigenvalues is discrete without finite accumulation point.*

Proof. Define $\Psi_{\eta_0} : L^2(B) \to L^2(B)$ such that $\Psi_{\eta_0} f := v$, where $v \in H^1(B)$ satisfies

$$a_{\eta_0}(v, v') = k^2(f, v')_B \quad \forall v' \in H^1(B).$$

The choice of η_0 guarantees that this variational problem possesses a unique solution satisfying the estimate

$$\|v\|_{H^1(B)} \leq C\,\|f\|_B$$

with C independent of f, which implies that

$$\|\Psi_{\eta_0} f\|_{H^1(B)} \leq C\,\|f\|_B \quad \forall f \in L^2(B).$$

It follows that Ψ_{η_0} is bounded as a map from $L^2(B)$ into $H^1(B)$, and consequently Ψ_{η_0} is compact by the compact embedding of $H^1(B)$ into $L^2(B)$. From the definition of Ψ_{η_0} we see that η is a Robin eigenvalue associated with T if and only if $(\eta - \eta_0)^{-1}$ is an eigenvalue for Ψ_{η_0}. The spectral theorem for compact operators asserts that the eigenvalues of Ψ_{η_0} are discrete in the complex plane with the origin as the only possible accumulation point, from which we conclude that the set of Robin eigenvalues is discrete without finite accumulation point.
□

Combining these two theorems, we immediately see that if $-\text{Im}T$ is nonnegative definite, then choosing any $\eta_0 \in \mathbb{C}$ such that $\text{Im}(\eta_0) < 0$ satisfies the assumptions of Theorem 3.6 and hence the set of eigenvalues is discrete without finite accumulation point. While the following result does not apply to our present application of scattering by a partially coated crack, it is of interest in its own right and includes the result found in [11] as a special case.

Theorem 3.7. *If there exists $\eta_0 \in \mathbb{R}$ which is not a Robin eigenvalue associated with T and T satisfies the symmetry relation*

$$\overline{\langle Tg_2, g_1 \rangle_{\partial B}} = \langle Tg_1, g_2 \rangle_{\partial B} \quad \forall g_1, g_2 \in H^{1/2}(\partial B),$$

then all of its associated Robin eigenvalues are real, and the set of interior Robin eigenvalues is infinite.

Proof. We recall the compact operator $\Psi_{\eta_0} : L^2(B) \to L^2(B)$ defined such that

$$a_{\eta_0}(\Psi_{\eta_0} f, v') = k^2 (f, v')_B \quad \forall v' \in H^1(B)$$

for a given $f \in L^2(B)$. For any $f_1, f_2 \in L^2(B)$, if we let $v_i := \Psi_{\eta_0} f_i$, $i = 1, 2$, then we see that

$$\begin{aligned} k^2(\Psi^*_{\eta_0} f_1, f_2)_B &= k^2(f_1, v_2)_B \\ &= \gamma(\nabla v_1, \nabla v_2)_B - k^2 \eta_0 (v_1, v_2)_B - \langle Tv_1, v_2 \rangle_{\partial B} \\ &= \gamma\overline{(\nabla v_2, \nabla v_1)_B} - k^2 \eta_0 \overline{(v_2, v_1)_B} - \overline{\langle Tv_2, v_1 \rangle_{\partial B}} \\ &= k^2 \overline{k^2 (f_2, v_1)_B} \\ &= k^2 (\Psi_{\eta_0} f_1, f_2)_B, \end{aligned}$$

and we conclude that Ψ_{η_0} is a self-adjoint operator. Thus, the Hilbert-Schmidt theorem implies that all of the eigenvalues of Ψ_{η_0} are real and infinitely many eigenvalues exist. Since $\Psi_{\eta_0} : L^2(B) \to L^2(B)$ is clearly injective, it follows that the set of Robin eigenvalues corresponding to T is infinite as well. □

We remark that if any $\eta_0 \in \mathbb{C}$ exists which is not a Robin eigenvalue, then discreteness of the eigenvalues in the complex plane implies the existence of some real η which is not an eigenvalue, and as a result η_0 may be chosen on the real line.

Remark 3.8. Though our present application for this general framework lies in scattering by a partially coated crack, we remark that it may be used to prove the results in the context of scattering by an inhomogeneous medium (represented by a function $n \in L^\infty(B)$ with contrast $1 - n$ supported in a bounded domain D contained in B) found in [11]. Indeed, in this case we define the operator $T_n : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ as $T_n g := \frac{\partial w}{\partial \nu}|_{\partial B}$, where $w \in H^1(B)$ satisfies

$$\Delta w + k^2 n w = 0 \text{ in } B, \tag{3.5a}$$

$$w = g \text{ on } \partial B, \tag{3.5b}$$

and we assume that this problem with $g = 0$ has only the trivial solution $w = 0$ in order to guarantee that T_n is well-defined. The choice $T = T_n$ satisfies Assumption 3.2 and the hypothesis of Theorem 3.7, which implies that the so-called *modified transmission eigenvalues* are real and that infinitely many exist.

A persistent difficulty in the study of eigenvalue problems in scattering theory is establishing the existence of eigenvalues, and as such we now briefly present a generalization of the existence results shown in [11] (for the case $T = T_n$ from Remark 3.8) to our more general eigenvalue problem. We remark that this theory does not apply to our intended application of scattering by a partially coated crack.

It was shown in [11] that under certain conditions there exist infinitely many Robin eigenvalues corresponding to $T = T_n$ even in the case of complex-valued n, and we generalize these results to provide a sufficient condition in order to guarantee existence for other choices of T. We content ourselves with stating the main assumption and theorem, as the proofs of the necessary lemmas follow exactly along the lines of Section 6 in [11] except with the more general source problem defined shortly in place of the modified interior transmission problem considered in that work. We note that in our analysis we have replaced γ^{-1} with γ, and we still require $\gamma \neq 1$.

For a given $z \in \mathbb{C}$ and $f \in L^2(B)$ we consider the source problem of finding $v \in H^1(B)$ satisfying

$$\gamma\Delta v + k^2 zv = k^2 f \text{ in } B, \tag{3.6a}$$

$$\gamma\frac{\partial v}{\partial \nu} - Tv = 0 \text{ on } \partial B, \tag{3.6b}$$

which was already introduced in an equivalent variational form in the definition of the operator $\Psi_z : L^2(B) \to L^2(B)$. From that definition, we see that if (3.6a)–(3.6b) is well-posed with solution v then $\Psi_z f = v$. Throughout this section we assume that T is such that (3.6a)–(3.6b) satisfies the Fredholm property (e.g. the assumption of Theorem 3.1 or Assumption 3.2) along with the assumption of Theorem 3.5 (which implies that the associated Robin eigenvalues have nonnegative imaginary part). The following additional assumption on T provides the *a priori* estimate required in order to carry out the remainder of the analysis.

Assumption 3.9. *We assume that the operator T is such that if $\arg z$ is fixed, $z \notin [0, \infty)$, and $|z|$ is sufficiently large, then for $v \in H^1(B)$ and $f \in L^2(B)$ satisfying* (3.6a)–(3.6b) *it follows that $v \in H^2(B)$ and*

$$\|v\|_B \leq c\frac{1}{|z|}\|f\|_B, \tag{3.7}$$

where the constant c is independent of f.

Remark 3.10. This assumption was shown to hold for the choice $T = T_n$ whenever $n \in C^\infty(\overline{B})$ [11] using the theory of pseudodifferential operators and semiclassical analysis [21], and in particular the result of Lemma 6.9 in [11] implies in general that Assumption 3.9 holds whenever T is of the form $\text{op}(a_1)+\text{op}(a_0)$ for a certain symbol a_1 of order one and a symbol a_0 of order zero. We refer to Section 6.1 of [11] for details.

We choose z to satisfy the conditions of Assumption 3.9, and we recall that η is a Robin eigenvalue if and only if $(z-\eta)^{-1}$ is an eigenvalue of Ψ_z. As a result, our study of the Robin eigenvalues associated with T reduces to an investigation of the spectral properties of Ψ_z. We arrive at the following theorem which is identical to Theorem 6.6 in [11], the proof of which is also identical once the minor adjustments have been made to the preceding lemmas.

Theorem 3.11. *If T satisfies Assumption 3.9, then there exist infinitely many Robin eigenvalues associated with T, and the space spanned by the nonzero generalized eigenfunctions is dense in $L^2(B)$. Moreover, for any positive ϵ there exist only finitely many eigenvalues lying outside the wedge $\{\eta \in \mathbb{C} \mid 0 \leq \arg \eta < \epsilon\}$.*

In the next section we show that Robin eigenvalues may be computed from far field data associated with T.

4. Determination of Robin eigenvalues from far field data

A necessary property of a target signature is that it may be computed from measured scattering data. In this paper we focus on far field data, but many of the following results are independent of the type of data collected and the remainder may be easily modified for the case of near field data [3]. We begin by defining the generalized Herglotz operator $H : L^2(\mathbb{S}^{m-1}) \to H^1(B)$ as $Hg := v_g$, where (v_g, v_g^s) is the solution of the auxiliary problem (2.5a)–(2.5e) with $u^i = u_g$, and the solution operator $G : \overline{R(H)} \to L^2(\mathbb{S}^{m-1})$ as $G\varphi := w_\infty^*$, where $w^* \in H^1_{loc}(\mathbb{R}^m \setminus \overline{B})$ is the unique radiating solution of

$$\Delta w^* + k^2 w^* = 0 \text{ in } \mathbb{R}^m \setminus \overline{B}, \tag{4.1a}$$

$$\frac{\partial w^*}{\partial \nu} - T w^* = -\gamma \frac{\partial \varphi}{\partial \nu} + T\varphi \text{ on } \partial B. \tag{4.1b}$$

We recall from [1] that the closure of the range of H is given by

$$\overline{R(H)} = \{v \in H^1(B) \mid \gamma \Delta v + k^2 \eta v = 0 \text{ in } B\},$$

and it easily follows that the modified far field operator may be factorized as $\mathcal{F} = GH$. We also recall the definition of the radiating fundamental solution of the Helmholtz equation in $\mathbb{R}^m$ as

$$\Phi(x,z) := \begin{cases} H_0^{(1)}(k\,|x-y|) \text{ in } \mathbb{R}^2, \\ \dfrac{e^{ik|x-z|}}{4\pi\,|x-z|} \quad \text{in } \mathbb{R}^3, \end{cases}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero, and we provide a characterization of Robin eigenvalues in terms of the range of the solution operator G.

Theorem 4.1. *Let $z \in B$. If η is not a Robin eigenvalue, then $\Phi_\infty(\cdot, z) \in R(G)$.*

Proof. Since η is not a Robin eigenvalue associated with T, there exists a unique $v_z \in H^1(B)$ satisfying

$$\gamma \Delta v_z + k^2 \eta v_z = 0 \text{ in } B, \tag{4.2a}$$

$$\gamma \frac{\partial v_z}{\partial \nu} - T v_z = -\frac{\partial \Phi(\cdot, z)}{\partial \nu} + T\Phi(\cdot, z) \text{ on } \partial B. \tag{4.2b}$$

Then $w^* := \Phi(\cdot, z)$ in $\mathbb{R}^m \setminus \overline{B}$ satisfies (4.1a)–(4.1b) with $\varphi = v_z \in \overline{R(H)}$ and consequently $G v_z = w^*_\infty = \Phi_\infty(\cdot, z)$. Thus, we conclude that $\Phi_\infty(\cdot, z) \in R(G)$. □

Theorem 4.2. *Assume that T satisfies Assumption 2.2. If η is a Robin eigenvalue, then the set of $z \in B$ for which $\Phi_\infty(\cdot, z) \in R(G)$ is nowhere dense in B.*

Proof. We suppose to the contrary that $\Phi_\infty(\cdot, z) \in R(G)$ for z in a dense subset of a ball $B_\rho \subset B$, and for each such z it follows that $G v_z = \Phi_\infty(\cdot, z)$ for some $v_z \in \overline{R(H)}$. If we let w_z^* be the unique radiating solution of (4.1a)–(4.1b) with $\varphi = v_z$, then the definition of G implies that $w^*_{z,\infty} = \Phi_\infty(\cdot, z)$. Rellich's lemma implies that $w_z^* = \Phi(\cdot, z)$ in $\mathbb{R}^m \setminus \overline{B}$, and in particular we see that the Cauchy data of w_z^* and $\Phi(\cdot, z)$ coincide on ∂B. As a consequence we see that $v_z \in H^1(B)$ satisfies (4.2a)–(4.2b). Since η is a Robin eigenvalue associated with T, there exists a nontrivial eigenpair (η, v_η) satisfying the homogeneous interior Robin problem (2.8a)–(2.8b), and we observe from Green's second identity that

$$\int_{\partial B} \left(\gamma \frac{\partial v_z}{\partial \nu} v_\eta - \gamma \frac{\partial v_\eta}{\partial \nu} v_z \right) ds = 0. \tag{4.3}$$

Applying the boundary conditions for v_z and v_η along with (4.3) implies that

$$\int_{\partial B} \left(\gamma \frac{\partial v_\eta}{\partial \nu} \Phi(\cdot, z) - \frac{\partial \Phi(\cdot, z)}{\partial \nu} v_\eta \right) ds = \int_{\partial B} \Big(\Phi(\cdot, z) T v_\eta - v_\eta T \Phi(\cdot, z) \Big) ds + \int_{\partial B} \Big(v_z T v_\eta - v_\eta T v_z \Big) ds,$$

and by Assumption 2.2 both of the integrals on the right-hand side vanish. We define

$$v_\eta^i(z) := \int_{\partial B} \left(\gamma \frac{\partial v_\eta}{\partial \nu} \Phi(\cdot, z) - \frac{\partial \Phi(\cdot, z)}{\partial \nu} v_\eta \right) ds, \; z \in B,$$

and the observation that v_η^i satisfies the Helmholtz equation in B and vanishes in a dense subset of a ball in B allows us to apply the unique continuation principle to conclude that $v_\eta^i(z) = 0$ for all $z \in B$. If we define

$$v_\eta^s(z) := -\int_{\partial B} \left(\gamma \frac{\partial v_\eta}{\partial \nu} \Phi(\cdot, z) - \frac{\partial \Phi(\cdot, z)}{\partial \nu} v_\eta \right) ds, \ z \in \mathbb{R}^m \setminus \overline{B},$$

then we see from the jump properties of the single and double layer potentials that (v_η, v_η^s) satisfies the auxiliary problem (2.5a)–(2.5e) with $u^i = v_\eta^i = 0$, and since this problem is well-posed we conclude that $v_\eta = 0$. This result contradicts the assumption that η is a Robin eigenvalue, and it follows that the set of $z \in B$ for which $\Phi_\infty(\cdot, z) \in R(G)$ is nowhere dense in B. □

In order to relate the range of G to some convenient indicator function, we require a different type of factorization of the auxiliary near field operator $F_0 : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$. For convenience we define the space

$$\mathcal{L}(B) := (L^2(B))^m \times L^2(B),$$

and we begin by recalling the definition of the standard Herglotz operator $U : L^2(\mathbb{S}^{m-1}) \to \mathcal{L}(B)$ as $Ug := (\nabla u_g, u_g)$, where u_g is the Herglotz wave function with kernel g defined in (2.7). It is known (c.f. [7, Lemma 2.38]) that the range of U is dense in the space

$$H_{inc}(B) := \{(\nabla\psi, \psi) \mid \psi \in H^1(B), \ \Delta\psi + k^2\psi = 0 \text{ in } B\}.$$

In a similar manner to our factorization of $\mathcal{F}$, given $(\varphi, \psi) \in \overline{R(U)}$ we let $v^* \in H^1_{loc}(\mathbb{R}^2)$ be the unique radiating solution of

$$\nabla \cdot A_0 \nabla v^* + k^2 n_0 v^* = \nabla \cdot (I - A_0)\varphi + k^2(1 - n_0)\psi \text{ in } \mathbb{R}^m, \tag{4.4}$$

where n_0 is equal to η in B and one otherwise and A_0 is the 2×2 matrix function given by $A_0 = \gamma I$ in B and $A_0 = I$ otherwise. By Green's formula and the fact that v^* is a radiating solution of the Helmholtz equation in $\mathbb{R}^m \setminus \overline{B}$ we may write

$$v^*(x) = -\int_B \Phi(x, y)\big[\Delta v^*(y) + k^2 v^*(y)\big] dy, \ x \in \mathbb{R}^m \setminus \overline{B},$$

which implies that

$$\begin{aligned} v^*_\infty(x) = -\gamma_m \int_B \Big[& ik\hat{x} \cdot (I - A_0)(\nabla v^*(y) + \varphi(y)) \\ & + k^2(1 - n_0)(v^*(y) + \psi(y)) \Big] e^{-ik\hat{x}\cdot y} dy, \ \hat{x} \in \mathbb{S}^{m-1}. \end{aligned} \tag{4.5}$$

Since the adjoint $U^* : \mathcal{L}(B) \to L^2(\mathbb{S}^{m-1})$ is given by

$$(U^*(\varphi, \psi))(\hat{x}) = -\int_B \big[ik\hat{x} \cdot \varphi(y) - \psi(y)\big] e^{-ik\hat{x}\cdot y} dy, \ \hat{x} \in \mathbb{S}^{m-1}, \tag{4.6}$$

we arrive at the factorization $F_0 = \gamma_m U^* T_0 U$, where the middle operator $T_0 : \mathcal{L}(B) \to \mathcal{L}(B)$ is given by

$$T_0(\varphi, \psi) := ((I - A_0)(\nabla v^* + \varphi), k^2(n_0 - 1)(v^* + \psi)) \tag{4.7}$$

with v^* the unique radiating solution of (4.4). We make the following necessary assumption.

Assumption 4.3. *We assume that k, γ, and η are such that there exist no nontrivial solutions (w, v) of the homogeneous interior transmission problem*

$$\Delta w + k^2 w = 0 \text{ in } B \tag{4.8a}$$

$$\gamma \Delta v + k^2 \eta v = 0 \text{ in } B \tag{4.8b}$$

$$w - v = 0 \text{ on } \partial B \tag{4.8c}$$

$$\frac{\partial w}{\partial \nu} - \gamma \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B. \tag{4.8d}$$

We remark that this assumption implies that η is not a Robin eigenvalue associated with the standard interior Dirichlet-to-Neumann map for the Helmholtz equation in B. For a given scattering medium of interest (represented by the choice of T), this assumption has its own physical interpretation. In the particular case of scattering by a partially coated crack which we will study in the next section, this assumption implies that η is not a crack transmission eigenvalue (a value of η for which nontrivial solutions of (1.1a)–(1.1f) exist) for the case in which no crack exists in the medium. It is known that the set of η for which this problem has nontrivial solutions is discrete [11]. The importance of this assumption is that the middle operator T_0 is coercive on $\overline{R(U)}$ [7, Lemma 2.42], which is a fundamental property for the application of the generalized linear sampling method. We refer to [1] for further discussion on this assumption.

We now recall the generalized linear sampling method (GLSM) as given in the appendix of [1] (see also [3]). We begin by defining $\mathcal{B} : L^2(\mathbb{S}^{m-1}) \to \mathbb{R}$ as

$$\mathcal{B}(g) := |(F_0 g, g)_{\mathbb{S}^{m-1}}|$$

for $g \in \mathbb{S}^{m-1}$. For this choice of auxiliary near field operator F_0, it was shown in [1] that $\mathcal{B}$ has the following relationship with the generalized Herglotz operator H.

Lemma 4.4. *If Assumption 4.3 holds, then given a sequence $\{g_n\}$ in $L^2(\mathbb{S}^{m-1})$, the sequence $\{\mathcal{B}(g_n)\}$ is bounded if and only if the sequence $\{\|Hg_n\|_{H^1(B)}\}$ is bounded.*

For fixed $\phi \in L^2(\mathbb{S}^{m-1})$ and $\alpha > 0$ we define the GLSM cost functional as

$$J_\alpha(\phi; g) := \alpha \mathcal{B}(g) + \|\mathcal{F} g - \phi\|^2_{\mathbb{S}^{m-1}},$$

and though this cost functional may not have a minimizer, by nonnegativity we may define

$$j_\alpha(\phi) := \inf_{g \in L^2(\mathbb{S}^{m-1})} J_\alpha(\phi; g).$$

The following central theorem in GLSM relates the range of G to the functional $\mathcal{B}$ and the modified far field operator $\mathcal{F}$, and we refer to [1] for a proof.

Theorem 4.5. *Assume that $\mathcal{F}$ has dense range. Let $C > 0$ be a given constant independent of α and consider a minimizing sequence $\{g_\alpha\}$ of $J_\alpha(\phi; \cdot)$ such that*

$$J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + C\alpha.$$

Then $\phi \in R(G)$ if and only if the sequence $\{\mathcal{B}(g_\alpha)\}$ is bounded as $\alpha \to 0$.

We remark that by Theorem 2.5 the modified far field operator $\mathcal{F}$ has dense range provided that there exist no nontrivial solutions of the homogeneous interior Robin problem (2.8a)–(2.8b) of the form (2.9). By choosing $\phi_z = \Phi_\infty(\cdot, z)$ for some $z \in B$, we may combine Theorems 4.1, 4.2, and 4.5 to obtain the following characterization of the Robin eigenvalues associated with T.

Theorem 4.6. *Assume that $\mathcal{F}$ has dense range and that Assumption 2.2 is satisfied. Let $C > 0$ be a given constant independent of α and consider a minimizing sequence $\{g_\alpha^z\}$ of $J_\alpha(\phi_z; \cdot)$ such that*

$$J_\alpha(\phi_z; g_\alpha^z) \leq j_\alpha(\phi_z) + C\alpha.$$

Then η is a Robin eigenvalue if and only if the set of $z \in B$ for which $\{\mathcal{B}(g_\alpha^z)\}$ is bounded as $\alpha \to 0$ is nowhere dense in B.

In practice the measured scattering data represented by the far field operator F will be subject to some noise, and as a result we must consider a regularized cost functional defined as

$$J_\alpha^\delta(\phi_z; g) := \alpha\mathcal{B}^\delta(g) + \left\|\mathcal{F}^\delta g - \phi_z\right\|_{\mathbb{S}^{m-1}}^2,$$

where $\mathcal{F}^\delta = F^\delta - F_0$, F^δ is the noisy far field operator, and we define

$$\mathcal{B}^\delta(g) := |(F_0 g, g)_{\mathbb{S}^{m-1}}| + \delta \|g\|_{\mathbb{S}^{m-1}}^2$$

for $g \in L^2(\mathbb{S}^{m-1})$. The noise constant $\delta > 0$ is such that $\left\|F^\delta - F\right\| \leq \delta$. The regularized cost functional now has a minimizer $g_{\alpha,\delta}^z$, and the above theorem holds with the appropriate modifications. We refer the reader to the discussion in [1, Section 3] for further details and references.

5. Application to scattering by a partially coated crack

We now apply the theory developed above to the case of scattering by a partially coated crack. Let $\Gamma \subset \mathbb{R}^2$ be a smooth, open, nonintersecting arc, and moreover assume that Γ is a subset of a smooth curve ∂D that encloses a region D in $\mathbb{R}^2$. In this case we must choose the region B such that $D \subsetneq B$. We choose the unit normal ν on Γ to coincide with the outward normal to ∂D, and for a function u with sufficient regularity for the trace on ∂D to be well-defined we denote the trace from the exterior and the interior of D as u^+ and u^-, respectively.

Given an entire solution u^i of the Helmholtz equation in $\mathbb{R}^2$ representing an incident field, the scattering of u^i by a thin, infinitely long, cylindrical obstacle with cross section Γ which is coated on one side by a material with surface impedance σ (which is assumed to be positive and bounded away from zero) is given by the problem of finding the total field $u = u^s + u^i$ satisfying

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Gamma}, \tag{5.1a}$$

$$u^- = 0 \text{ on } \Gamma, \tag{5.1b}$$

$$\frac{\partial u^+}{\partial \nu} + i\sigma u^+ = 0 \text{ on } \Gamma, \tag{5.1c}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{5.1d}$$

The Sommerfeld radiation condition (5.1d) is assumed to hold uniformly in all directions, and the same arguments as in Section 8.7 of [6] (where constant $\sigma > 0$ was assumed) show that this problem is well-posed. In order to place this problem into the framework we introduced in Section 2, we define the operator $T_\Gamma : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ such that $T_\Gamma g := \frac{\partial w}{\partial \nu}|_{\partial B}$, where $w \in H^1(B \setminus \overline{\Gamma})$ is the unique solution of

$$\Delta w + k^2 w = 0 \text{ in } B \setminus \overline{\Gamma}, \tag{5.2a}$$

$$w^- = 0 \text{ on } \Gamma, \tag{5.2b}$$

$$\frac{\partial w^+}{\partial \nu} + i\sigma w^+ = 0 \text{ on } \Gamma, \tag{5.2c}$$

$$w = g \text{ on } \partial B. \tag{5.2d}$$

Writing an equivalent variational formulation of this Dirichlet problem clearly shows that it is of Fredholm type, and consequently the following lemma guarantees that it is well-posed.

Lemma 5.1. *The Dirichlet problem* (5.2a)–(5.2d) *with* $g = 0$ *has only the trivial solution* $w = 0$*.*

Proof. If $w \in H^1(B \setminus \overline{\Gamma})$ satisfies (5.2a)–(5.2d) with $g = 0$, then it satisfies the variational problem

$$(\nabla w, \nabla w')_{B\setminus\overline{\Gamma}} - k^2 (w, w')_{B\setminus\overline{\Gamma}} - i\left\langle \sigma w^+, (w')^+\right\rangle_\Gamma = 0 \tag{5.3}$$

for all $w' \in H^1(B \setminus \overline{\Gamma})$ such that $(w')^- = 0$ on Γ and $w' = 0$ on ∂B. Taking the imaginary part of (5.3) with $w' = w$ yields

$$\left\langle \sigma w^+, w^+\right\rangle_\Gamma = 0,$$

from which strict positivity of σ implies that $w^+ = 0$ on Γ. The boundary condition (5.2c) implies that $\frac{\partial w^+}{\partial \nu} = 0$ on Γ as well, and an application of Holmgren's theorem [13] implies that $w = 0$ in $B \setminus \overline{D}$. Since w and $\frac{\partial w}{\partial \nu}$ are continuous across $\partial D \setminus \overline{\Gamma}$ it follows that both vanish on this portion of ∂D, and another application of Holmgren's theorem implies that $w = 0$ in D as well. □

We observe that T_Γ is the Dirichlet-to-Neumann map for the interior Dirichlet crack problem (5.2a)–(5.2d) and that (5.1a)–(5.1d) is equivalent to (2.1a)–(2.1b) with $T = T_\Gamma$. We remark that we may also conclude well-posedness of (5.1a)–(5.1d) from Theorem 2.1 using the operator $\tilde{T} = \tilde{T}_\Gamma$ we will introduce shortly. We now proceed to verify that $T = T_\Gamma$ possesses the required properties described in the previous sections. First, by well-posedness and linearity of (5.2a)–(5.2d), we observe that T_Γ is a bounded linear operator, and in the following lemma we show that Assumption 2.2 is satisfied by $T = T_\Gamma$.

Lemma 5.2. *The operator T_Γ satisfies Assumption 2.2.*

Proof. For given $g_1, g_2 \in H^{1/2}(\partial B)$, let $w_i \in H^1(B \setminus \overline{\Gamma})$ satisfy (5.2a)–(5.2d) for $g = g_i$, $i = 1, 2$. Observing that $w_i|_{\partial B} = g_i$, $i = 1, 2$, and applying Green's second identity on $B \setminus \overline{D}$ yields

$$\begin{aligned}\int_{\partial B} (g_1 T g_2 - g_2 T g_1) ds &= \int_{\partial B} \left(w_1 \frac{\partial w_2}{\partial \nu} - w_2 \frac{\partial w_1}{\partial \nu} \right) ds \\ &= \int_{\partial D} \left(w_1^+ \frac{\partial w_2^+}{\partial \nu} - w_2^+ \frac{\partial w_1^+}{\partial \nu} \right) ds\end{aligned}$$

since both w_1 and w_2 satisfy the Helmholtz equation in $B \setminus \overline{D}$. The boundary condition (5.2c) implies that the portion of this integral over Γ vanishes, and since w_i, $i = 1, 2$, has continuous values and normal derivatives across $\partial D \setminus \overline{\Gamma}$ and satisfies $w_i^- = 0$ on Γ we see that

$$\int_{\partial B} (g_1 T g_2 - g_2 T g_1) ds = \int_{\partial D} \left(w_1^- \frac{\partial w_2^-}{\partial \nu} - w_2^- \frac{\partial w_1^-}{\partial \nu} \right) ds.$$

Since both w_1 and w_2 satisfy the Helmholtz equation in D, another application of Green's identity implies that this integral vanishes, providing the desired result. ☐

By Theorem 2.3 it follows that the far field pattern corresponding to (5.1a)–(5.1d) satisfies the reciprocity principle (2.3), and applying Theorem 2.5 along with the definition of T_Γ implies that the modified far field operator $\mathcal{F}$ is injective with dense range if and only if the *interior crack transmission problem* of finding $(w, v) \in H^1(B \setminus \overline{\Gamma}) \times H^1(B)$ satisfying

$$\Delta w + k^2 w = 0 \text{ in } B \setminus \overline{\Gamma}, \tag{5.4a}$$

$$\gamma \Delta v + k^2 \eta v = 0 \text{ in } B, \tag{5.4b}$$

$$w - v = 0 \text{ on } \partial B, \tag{5.4c}$$

$$\frac{\partial w}{\partial \nu} - \gamma \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B, \tag{5.4d}$$

$$w^- = 0 \text{ on } \Gamma, \tag{5.4e}$$

$$\frac{\partial w^+}{\partial \nu} + i\sigma w^+ = 0 \text{ on } \Gamma, \tag{5.4f}$$

has no nontrivial solutions with v of the form (2.9). In order to emphasize the specific application of interest, we call a value of η for which the interior crack transmission problem (5.4a)–(5.4f) has nontrivial solutions a *crack transmission eigenvalue*. It is known that the far field operator F corresponding to (5.1a)–(5.1d) is always injective with dense range [6], and as a result no standard transmission eigenvalues exist. Thus, although this type of eigenvalue has been referred to as a *modified transmission eigenvalue* in [10], [11], and [12] we do not introduce any confusion in our naming convention.

In order to study the solvability of the interior crack transmission problem (5.4a)–(5.4f), we consider the following nonhomogeneous version. Given $f \in L^2(B)$, $g \in H^{1/2}(\partial B)$, and $h \in H^{-1/2}(\partial B)$, we seek $(w, v) \in H^1(B \setminus \overline{\Gamma}) \times H^1(B)$ satisfying

$$\Delta w + k^2 w = 0 \text{ in } B \setminus \overline{\Gamma}, \tag{5.5a}$$

$$\gamma \Delta v + k^2 \eta v = f \text{ in } B, \tag{5.5b}$$

$$w - v = g \text{ on } \partial B, \tag{5.5c}$$

$$\frac{\partial w}{\partial \nu} - \gamma \frac{\partial v}{\partial \nu} = -h \text{ on } \partial B, \tag{5.5d}$$

$$w^- = 0 \text{ on } \Gamma, \tag{5.5e}$$

$$\frac{\partial w^+}{\partial \nu} + i\sigma w^+ = 0 \text{ on } \Gamma. \tag{5.5f}$$

By defining a lifting function $\varphi_g \in H^1_\Delta(B)$ such that $\varphi_g|_{\partial B} = g$, replacing v with $v + \varphi_g$, and appropriately modifying the right-hand sides f and h, we may assume without loss of generality that $g = 0$ in (5.5a)–(5.5f). With this simplification we see that (5.5a)–(5.5f) is equivalent to the nonhomogeneous interior Robin problem (3.1a)–(3.1b) with $T = T_\Gamma$, and as a result we may apply the theory developed in Section 3 to obtain results on this problem. We begin by verifying that T_Γ satisfies Assumption 3.2.

Before we factorize T_Γ, we first choose a relative neighborhood Ω of ∂B in B for which $\overline{\Omega}$ is disjoint from $\overline{\Gamma}$, and we choose a smooth cutoff function χ such that $\chi = 1$ on a relative neighborhood of ∂B contained in Ω, $0 \le \chi \le 1$, and χ is supported in Ω. In particular, we see that $\chi = 1$ near ∂B and $\chi = 0$ in a neighborhood of $\overline{\Gamma}$. The purpose of introducing this cutoff function is to remedy the mismatched spaces $H^1(B)$ and

$$H^1_{\Gamma^-} := \{ w \in H^1(B \setminus \overline{\Gamma}) \mid w^- = 0 \text{ on } \Gamma \}$$

in which solutions of (5.5a)–(5.5f) lie. Indeed, the mapping $X : \psi \mapsto \chi\psi$ is well-defined from $H^1(B)$ into $H^1_{\Gamma^-}(B \setminus \overline{\Gamma})$ and from $H^1_{\Gamma^-}(B \setminus \overline{\Gamma})$ into $H^1(B)$.

If we define the cutoff solution operator $S_\Gamma : H^{1/2}(\partial B) \to H^1_\Delta(B)$ such that $S_\Gamma g := \chi w_g$, where w_g satisfies (5.2a)–(5.2d) for this choice of g, then we obtain the factorization $T_\Gamma = N_{\partial B} S_\Gamma$, where we recall that $N_{\partial B} : H^1_\Delta(B) \to H^{-1/2}(\partial B)$ is the Neumann trace operator on ∂B. From Green's first identity it follows that for each $g \in H^{1/2}(\partial B)$ the operator T_Γ satisfies

$$\langle T_\Gamma g, g' \rangle_{\partial B} = (\nabla w_g, \nabla w')_{B \setminus \overline{\Gamma}} - k^2 (w_g, w')_{B \setminus \overline{\Gamma}} - i \left\langle \sigma w_g^+, (w')^+ \right\rangle_\Gamma \tag{5.6}$$

for all $w' \in H^1_{\Gamma^-}(B \setminus \overline{\Gamma})$ such that $w'|_{\partial B} = g'$. For a constant $\beta > 0$ to be determined later, we define the operator $\tilde{T}_\Gamma : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ such that

$$\left\langle \tilde{T}_\Gamma g, g' \right\rangle_{\partial B} = (\nabla w_g, \nabla w_{g'})_{B\setminus\overline{\Gamma}} + k^2\beta(w_g, w_{g'})_{B\setminus\overline{\Gamma}} - i\left\langle \sigma w_g^+, w_{g'}^+ \right\rangle_\Gamma. \quad (5.7)$$

With this definition, we verify in the following two lemmas that $\tilde{T} = \tilde{T}_\Gamma$ possesses the desired properties described in Assumption 3.2, and we first remark that even though $B \setminus \overline{\Gamma}$ is not a Lipschitz domain due to the presence of the crack Γ, the space $H^1(B \setminus \overline{\Gamma})$ is nevertheless compactly embedded into $L^2(B \setminus \overline{\Gamma})$, as can be seen by writing $B \setminus \partial D = (B \setminus \overline{D}) \cup D$ and applying the standard compactness result for each component.

Lemma 5.3. *The operator $\tilde{T}_\Gamma - T_\Gamma : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ is compact.*

Proof. We first observe that for all $g, g' \in H^{1/2}(\partial B)$ we have

$$\left\langle (\tilde{T}_\Gamma - T_\Gamma)g, g' \right\rangle_{\partial B} = k^2(1+\beta)(w_g, w_{g'})_{B\setminus\overline{\Gamma}}.$$

The definition of the operator norm and the Cauchy-Schwarz inequality imply that

$$\begin{aligned} \left\| (\tilde{T}_\Gamma - T_\Gamma)g \right\|_{H^{-1/2}(\partial B)} &= \sup_{g'\neq 0} \frac{\left| \left\langle (\tilde{T}_\Gamma - T_\Gamma)g, g' \right\rangle_{\partial B} \right|}{\|g'\|_{H^{1/2}(\partial B)}} \\ &= \sup_{g'\neq 0} \frac{\left| k^2(1+\beta)(w_g, w_{g'})_{B\setminus\overline{\Gamma}} \right|}{\|g'\|_{H^{1/2}(\partial B)}} \\ &\le C_1 \sup_{g'\neq 0} \frac{\|w_g\|_{B\setminus\overline{\Gamma}} \|w_{g'}\|_{B\setminus\overline{\Gamma}}}{\|g'\|_{H^{1/2}(\partial B)}}. \end{aligned}$$

From well-posedness of (5.2a)–(5.2d) it follows that $\|w_{g'}\|_{H^1(B\setminus\overline{\Gamma})} \le C_2 \|g'\|_{H^{1/2}(\partial B)}$ for some constant $C_2 > 0$ in dependent of g' and we obtain

$$\left\| (\tilde{T}_\Gamma - T_\Gamma)g \right\|_{H^{-1/2}(\partial B)} \le C \|w_g\|_{B\setminus\overline{\Gamma}}, \quad (5.8)$$

where the constant $C > 0$ is independent of g. If a sequence $\{g_j\}$ in $H^{1/2}(\partial B)$ converges weakly to some $g_0 \in H^{1/2}(\partial B)$, then we see from well-posedness of (5.2a)–(5.2d) that the sequence $\{w_{g_j}\}$ converges weakly to w_{g_0} in $H^1_\Delta(B \setminus \overline{\Gamma})$. The compact embedding of this space into $L^2(B \setminus \overline{\Gamma})$ implies that $w_{g_j} \to w_{g_0}$ in $L^2(B \setminus \overline{\Gamma})$, and consequently from the inequality (5.8) we obtain $(\tilde{T}_\Gamma - T_\Gamma)g_j \to (\tilde{T}_\Gamma - T_\Gamma)g_0$ in $H^{-1/2}(\partial B)$. It follows that the operator $\tilde{T}_\Gamma - T_\Gamma$ maps weakly convergent sequences in $H^{1/2}(\partial B)$ to strongly convergent sequences in $H^{-1/2}(\partial B)$, and we conclude that $\tilde{T}_\Gamma - T_\Gamma : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ is compact. □

Lemma 5.4. *The operator $\tilde{T}_\Gamma : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ satisfies Conditions 1 and 2 in Assumption 3.2.*

Proof. We first assume that $0 < \gamma < 1$, and we aim to show that (3.3) in Condition 1 holds. We let $g \in H^{1/2}(\partial B)$, and we recall that w_g is the solution of (5.2a)–(5.2d). We first see that by definition of the cutoff function χ we obtain

$$(Sg, Sg)_B = (\chi w_g, \chi w_g)_B = (|\chi|^2 w_g, w_g)_\Omega \leq (w_g, w_g)_\Omega, \tag{5.9}$$

and we observe that

$$(\nabla(Sg), \nabla(Sg))_B = (|\chi|^2 \nabla w_g, \nabla w_g)_\Omega + (|\nabla\chi|^2 w_g, w_g)_\Omega + 2\mathrm{Re}(\chi\nabla w_g, (\nabla\chi)w_g)_\Omega.$$

For any $\epsilon_0 > 0$ Young's inequality asserts that

$$2\mathrm{Re}(\chi\nabla w_g, (\nabla\chi)w_g)_\Omega \leq \epsilon_0(\nabla w_g, \nabla w_g)_\Omega + \epsilon_0^{-1} \sup_B |\nabla\chi|^2 (w_g, w_g)_\Omega,$$

from which we obtain

$$(\nabla(Sg), \nabla(Sg))_B \leq (1+\epsilon_0)(\nabla w_g, \nabla w_g)_\Omega + (1+\epsilon_0^{-1}) \sup_B |\nabla\chi|^2 (w_g, w_g)_\Omega. \tag{5.10}$$

Combining (5.9) and (5.10) and defining $\Omega' := (B \setminus \overline{\Omega}) \setminus \overline{\Gamma}$, we see that for any $\alpha_0 > 0$ and $\epsilon_1, \epsilon_2 > 0$ we have

$$\begin{aligned}
\mathrm{Re}\left\langle \tilde{T}_\Gamma g, g \right\rangle_{\partial B} &- \gamma\epsilon_1^{-1}(\nabla(Sg), \nabla(Sg))_B - k^2\alpha_0\epsilon_2^{-1}(Sg, Sg)_B \\
&\geq (\nabla w_g, \nabla w_g)_{\Omega'} + k^2\beta(w_g, w_g)_{\Omega'} \\
&\quad + \Big[1 - \gamma\epsilon_1^{-1}(1+\epsilon_0)\Big](\nabla w_g, \nabla w_g)_\Omega \\
&\quad + \Big[k^2(\beta - \alpha_0\epsilon_2^{-1}) - \gamma\epsilon_1^{-1}(1+\epsilon_0^{-1}) \sup_B |\nabla\chi|^2\Big](w_g, w_g)_\Omega,
\end{aligned}$$

and the expression on the right-hand side is nonnegative if we choose $0 < \epsilon_0 < \gamma^{-1} - 1$, $\gamma(1+\epsilon_0) < \epsilon_1 < 1$, $0 < \epsilon_2 < 1$, and $\beta > 0$ sufficiently large.

We now assume that $\gamma > 1$, and we show that (3.4) in Condition 2 holds. Let $v \in H^1(B)$, and let $g = v|_{\partial B}$. Then we see that $v|_{\partial B} = (2\chi v - w_g)|_{\partial B}$ and we may write

$$\begin{aligned}
\mathrm{Re}\left\langle \tilde{T}_\Gamma v, v \right\rangle_{\partial B} &= \mathrm{Re}\left\langle \tilde{T}_\Gamma, 2\chi v - w_g \right\rangle_{\partial B} \\
&= 2\mathrm{Re}(\nabla w_g, \nabla(\chi v))_\Omega + 2k^2\beta\mathrm{Re}(w_g, \chi v)_\Omega \\
&\quad - (\nabla w_g, \nabla w_g)_{B\setminus\overline{\Gamma}} - k^2\beta(w_g, w_g)_{B\setminus\overline{\Gamma}}.
\end{aligned}$$

For any $\delta_1, \delta_2, \delta_3 > 0$ we obtain from Young's inequality that

$$\begin{aligned}
2\mathrm{Re}(\nabla w_g, \nabla(\chi v))_\Omega &\leq 2\left|(\nabla w_g, \chi\nabla v)_\Omega\right| + 2\left|(\nabla w_g, (\nabla\chi)v)_\Omega\right| \\
&\leq (\delta_1 + \delta_2)(\nabla w_g, \nabla w_g)_\Omega + \delta_1^{-1}(\nabla v, \nabla v)_\Omega \\
&\quad + \delta_2^{-1} \sup_B |\nabla\chi|^2 (v, v)_\Omega
\end{aligned}$$

and

$$\mathrm{Re}(w_g, \chi v)_\Omega \leq \delta_3(w_g, w_g)_\Omega + \delta_3^{-1}(v, v)_\Omega.$$

Combining these estimates we observe that

$$\operatorname{Re}\left\langle \tilde{T}_\Gamma v, v\right\rangle_{\partial B} \leq \delta_1^{-1}(\nabla v, \nabla v)_\Omega + \left[k^2\beta\delta_3^{-1} + \delta_2^{-1}\sup_B |\nabla\chi|^2\right](v,v)_\Omega \\ + (\delta_1+\delta_2-1)(\nabla w_g, \nabla w_g)_{B\setminus\overline{\Gamma}} + k^2\beta(\delta_3-1)(w_g,w_g)_{B\setminus\overline{\Gamma}},$$

and the result holds with $\delta = \delta_1^{-1}$ and $c = k^2\beta\delta_3^{-1} + \delta_2^{-1}\sup_B |\nabla\chi|^2$ provided that we choose $0 < \delta_2 < 1-\gamma^{-1}$, $\gamma^{-1} < \delta_1 < 1-\delta_2$, and $0 < \delta_3 < 1$. □

Combining Lemmas 5.3–5.4 and their preceding discussion, we see that Assumption 3.2 holds for $T = T_\Gamma$ and consequently we obtain from Theorem 3.4 that the interior crack transmission problem (5.5a)–(5.5f) satisfies the Fredholm property. In particular, this result implies that if η is not a crack transmission eigenvalue then (5.5a)–(5.5f) is well-posed. From the representation (5.6) of T_Γ we see that

$$-\operatorname{Im}\langle T_\Gamma g, g\rangle_{\partial B} = \left\langle \sigma w_g^+, w_g^+\right\rangle_\Gamma \geq 0$$

for all $g \in H^{1/2}(\partial B)$ and as a result Theorem 3.5 implies that every crack transmission eigenvalue has nonnegative imaginary part. Thus, the remark after Theorem 3.5 implies that the set of crack transmission eigenvalues is discrete without finite accumulation point. Unfortunately, we cannot conclude existence of crack transmission eigenvalues from Theorem 3.7 since this problem is not self-adjoint, and it is unlikely that the existence theory from the end of Section 3 applies due to the lack of interior regularity of the domain $B \setminus \overline{\Gamma}$.

We have also verified the assumptions required for application of the generalized linear sampling method to compute crack transmission eigenvalues from far field data corresponding to (5.1a)–(5.1d), and in the next section we provide numerical examples in which we investigate both the effectiveness of eigenvalue detection and the sensitivity of crack transmission eigenvalues to changes in the surface impedance σ of the partially coated crack.

6. Numerical examples for the case of scattering by a partially coated crack

We use the finite element software `FreeFem++` [15] in order to generate simulated scattering data and to compute eigenvalues for validating our method. We first describe how we compute eigenvalues and eigenfunctions using finite elements. It is tempting to use the variational formulation (3.2) with an explicit form of T_Γ in order to compute eigenvalues, in which we seek η and nonzero $(w,v) \in \mathcal{H} := \{(\psi,\varphi) \in H^1(B\setminus\overline{\Gamma}) \times H^1(B) \mid \psi - \varphi = 0 \text{ on } \partial B,\ \psi^- = 0 \text{ on } \Gamma\}$ such that

$$\gamma(\nabla v, \nabla v')_B - (\nabla w, \nabla w')_{B\setminus\overline{\Gamma}} \\ - k^2(w,w')_{B\setminus\overline{\Gamma}} + i\left\langle \sigma w^+, (w')^+\right\rangle_\Gamma = k^2\eta(v,v')_B \quad \forall (w',v') \in \mathcal{H}.$$

Using finite element basis functions, we would construct a matrix $\mathbf{A}$ from the left-hand side and a matrix $\mathbf{B}$ from the right-hand side (without η) and solve the generalized eigenvalue problem $\mathbf{Au} = \eta\mathbf{Bu}$. However, we would also need to take into account the boundary condition $w - v = 0$ on ∂B built into the space $\mathcal{H}$, which is difficult to implement directly. In [11] this issue was overcome by rewriting the problem in terms of $u_1 := w - v$ and enforcing a homogeneous Dirichlet boundary condition on the outer boundary; in the present case, the fields w and v do not lie in the same space, and as a result we must find an alternative manner in which to enforce this boundary condition. Our choice is to use a mixed finite element method in which we enforce the boundary condition variationally. In particular, we define the sesquilinear forms $\tilde{a}(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot)$ as

$$\tilde{a}((w,v),(w',v')) := \gamma(\nabla v, \nabla v')_B - (\nabla w, \nabla w')_{B\setminus\overline{\Gamma}} - k^2(w,w')_{B\setminus\overline{\Gamma}} \\ + i\left\langle \sigma w^+, (w')^+ \right\rangle_\Gamma$$

for all $(w,v),(w',v') \in \mathcal{H}_\Gamma := \{(\psi,\varphi) \in H^1(B\setminus\overline{\Gamma}) \times H^1(B) \mid \psi^- = 0 \text{ on } \Gamma\}$,

$$b((w,v),\xi') := \langle \xi', w - v\rangle_{\partial B}$$

for all $(w,v) \in \mathcal{H}_\Gamma$ and $\xi' \in H^{-1/2}(\partial B)$, and

$$c((w,v),(w',v')) := k^2(v,v')_B$$

for all $(w,v),(w',v') \in \mathcal{H}_\Gamma$. It follows that the interior crack transmission problem (5.4a)–(5.4f) is equivalent to the mixed variational problem of finding $((w,v),\xi) \in \mathcal{H}_\Gamma \times H^{-1/2}(\partial B)$ such that

$$\tilde{a}((w,v),(w',v')) + b((w',v'),\xi) = \eta c((w,v),(w',v')) \quad \forall (w',v') \in \mathcal{H}_\Gamma, \quad (6.1)$$

$$b((w,v),\xi') = 0 \quad \forall \xi' \in H^{-1/2}(\partial B). \quad (6.2)$$

We use $\mathbb{P}_1$ elements to discretize both components of $\mathcal{H}_\Gamma$, and we discretize $H^{-1/2}(\partial B)$ using piecewise constant elements on the boundary mesh induced by the interior mesh. Representing the sesquilinear forms $\tilde{a}(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot)$ as finite element matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, respectively, and enforcing the homogeneous Dirichlet condition $w^- = 0$ on Γ, we solve the generalized eigenvalue problem

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{B}^* \\ \hline \mathbf{B} & \mathbf{0} \end{array}\right)\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \eta \left(\begin{array}{c|c} \mathbf{C} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}$$

using the `eigs` command in `MATLAB`. We note that due to the lack of positive-definiteness of the matrix on the right-hand side we slightly perturb the diagonal by 10^{-15} before using `eigs`.

In order to construct the approximation of the far field operator F, we use N_{inc} incident directions d_j, $j = 1,\ldots,N_{inc}$, distributed uniformly on the unit circle, and we compute an $N_{inc} \times N_{inc}$ matrix $\mathbf{F}$ with $\mathbf{F}_{i,j} \approx u_\infty(d_i,d_j)$. By the same process we may approximate the auxiliary far field operator F_0 as an $N_{inc} \times N_{inc}$ matrix $\mathbf{F}_0$ with $(\mathbf{F}_0)_{i,j} \approx u_{0,\infty}(d_i,d_j)$, but for simplicity we choose B to be a disk centered at the origin and compute $\mathbf{F}_0$ analytically using separation of variables. In order to add noise to the data, we choose $\delta_{noise} > 0$ and set

$$\mathbf{F}^{\delta}_{i,j} = \mathbf{F}_{i,j}\left(1 + \delta_{noise}\frac{\zeta_{i,j} + i\mu_{i,j}}{\sqrt{2}}\right),\ i, j = 1, \dots, N_{inc},$$

where $\zeta_{i,j}$ and $\mu_{i,j}$ are uniformly distributed random numbers in $[-1, 1]$ computed using the `rand` command in `MATLAB`. Once the simulated data has been computed with suitable noise added, we compute the data vector $\boldsymbol{\phi}_z$ with ith entry given by $(\boldsymbol{\phi}_z)_i = \Phi_\infty(d_i, z)$, $i = 1, \dots, N_{inc}$, for some $z \in B$. In each example we use $N_{inc} = 51$.

We now describe our implementation of the generalized linear sampling method to compute interior crack transmission eigenvalues given the noisy far field matrix $\mathbf{F}^\delta$, which is similar to that of [12] with the exception that we may now use the simpler symmetric version of GLSM. Since the interior crack transmission problem is not self-adjoint, we begin by choosing a region in the upper half-plane in which to sample values of the eigenparameter η in a Cartesian grid. For each sampled value of η we compute the auxiliary far field matrix $\mathbf{F}_0$ and construct the approximation of the noisy modified far field operator $\mathcal{F}^\delta$ as the matrix $\boldsymbol{\mathcal{F}}^\delta := \mathbf{F}^\delta - \mathbf{F}_0$. We then construct the discrete regularized cost functional as

$$\mathbf{J}^\delta_\alpha(\mathbf{g}) := \alpha\left|\mathbf{g}^*(\mathbf{F}_0\mathbf{g})\right| + \alpha\delta\mathbf{g}^*\mathbf{g} + (\boldsymbol{\mathcal{F}}^\delta\mathbf{g} - \boldsymbol{\phi}_z)^*(\boldsymbol{\mathcal{F}}^\delta\mathbf{g} - \boldsymbol{\phi}_z) \tag{6.3}$$

for all $\mathbf{g} \in \mathbb{C}^{N_{inc}}$, where * refers to the Hermitian transpose and $\delta > 0$ is an upper bound on the noise level. This cost functional is difficult to minimize in $\mathbb{C}^{N_{inc}}$ as it is neither differentiable nor convex [2], and we follow similar procedures to those found in [3]. We choose a starting point $\mathbf{g}_0$ as

$$\mathbf{g}_0 = \underset{\mathbf{g}\in\mathbb{C}^{N_{inc}}}{\arg\min}\left(\beta_0\left\|\mathbf{g}\right\|_2^2 + \left\|\boldsymbol{\mathcal{F}}^\delta\mathbf{g} - \boldsymbol{\phi}_z\right\|_2^2\right)$$

for a regularization parameter β_0. The parameter β_0 may be allowed to vary with η and z if we choose it using the Morozov discrepancy principle, but we have observed no additional benefit from this approach and we instead use Tikhonov regularization with a fixed choice of $\beta_0 = 10^{-6}$. We carry out this minimization procedure with the `regtools` package in `MATLAB` [14], and we note that $\left\|\mathbf{g}_0\right\|_2$ provides the indicator function for the linear sampling method.

With the parameter $\beta_0 > 0$ and the initial guess $\mathbf{g}_0$ chosen, we must choose the value of α and a suitable optimization algorithm in order to minimize the cost functional $\mathbf{J}^\delta_\alpha(\cdot)$. We adopt the heuristic $\alpha = \beta_0/\left\|\mathbf{F}_0\right\|_2$, and we use limited memory BFGS from the Complex Optimization Toolbox [18] described in [17] in order to compute the minimizer $\mathbf{g}^{\text{glsm}}_\eta$. We then evaluate the indicator function (the discrete version of $\mathcal{B}$)

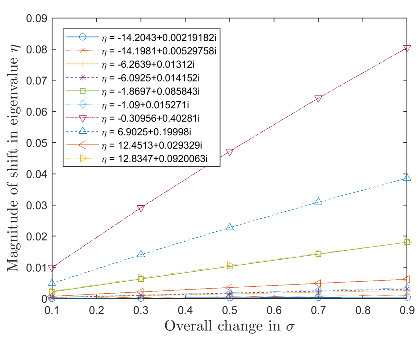
$$\boldsymbol{\mathcal{B}}(\mathbf{g}) := \left|\mathbf{g}^*(\mathbf{F}_0\mathbf{g})\right| + \delta\mathbf{g}^*\mathbf{g} \tag{6.4}$$

at $\mathbf{g} = \mathbf{g}^{\text{glsm}}_\eta$. For each sampled value of η we repeat this process for 5 random choices of z in B. By plotting the values of $\boldsymbol{\mathcal{B}}(\mathbf{g}^{\text{glsm}}_\eta)$ (averaged over the randomly chosen z) against η in the sampled region of the complex plane, we obtain a contour map whose peaks should correspond to the crack transmission eigenvalues. We use the eigensolver described at the beginning of this section in order to test our ability to compute eigenvalues, and we investigate the sensitivity of the eigenvalues to changes in the surface impedance σ of the partially coated crack.

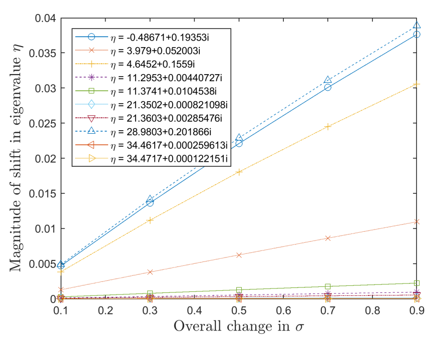
We begin by examining the sensitivity of crack transmission eigenvalues to overall changes in a constant surface impedance $\sigma = 4$. We consider the case when Γ is a circular arc defined parametrically as

$$\mathbf{r}_\Gamma(t) = (\cos t + x_0, \sin t + y_0),\ 0 \le t \le \pi/4, \tag{6.5}$$

where $(x_0, y_0) = (-\cos(\pi/4), -\sin(\pi/4))$ is defined such that the midpoint of the arc Γ lies is at the origin (see Figure 6). We choose B to be a disk of radius $R = 1$ or $R = 0.5$ centered at the origin, and we consider both $\gamma = 0.5$ and $\gamma = 2$. We note that either choice of R guarantees that $\overline{\Gamma}$ is contained in B. In Figure 1 we plot the magnitude of the shifts in the crack transmission eigenvalues resulting from overall changes in the surface impedance σ with $R = 1$ for both $\gamma = 0.5$ (left) and $\gamma = 2$ (right), and in Figure 2 we show the same plots except with $R = 0.5$. The eigenvalues appear to have a higher sensitivity to changes in σ for $\gamma = 0.5$ compared to $\gamma = 2$ in both figures, with the maximum sensitivity doubled in this example. Moreover, if we compare Figures 1 and 2 for each value of γ, then we find that the choice $R = 0.5$ increases the sensitivity of the eigenvalues by an order of magnitude. However, in the next example we will see that while the choice $R = 0.5$ greatly improves the sensitivity of the eigenvalues to flaws, it drastically decreases the accuracy of their detection from far field data.

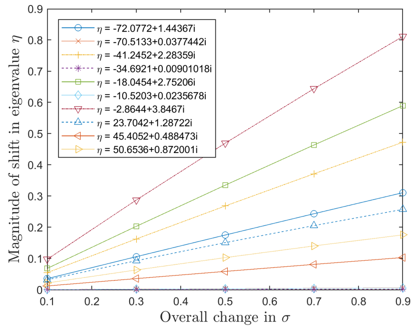


(a) $R = 1, \gamma = 0.5$

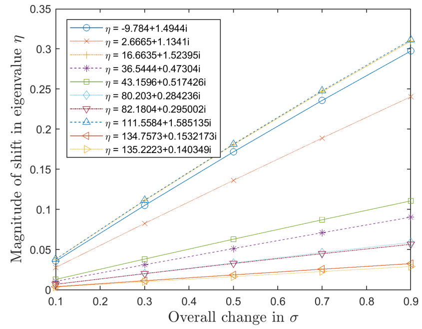


(b) $R = 1, \gamma = 2$

Figure 1: Plots showing the magnitude of the shift in the crack transmission eigenvalues resulting from an overall change in σ for $R = 1$ and two different values of the parameter γ. We observe that the eigenvalues exhibit an increase in sensitivity for the choice $\gamma = 0.5$ compared to $\gamma = 2$.



(a) $R = 0.5, \gamma = 0.5$



(b) $R = 0.5, \gamma = 2$

Figure 2: Plots showing the magnitude of the shift in the crack transmission eigenvalues resulting from an overall change in σ for $R = 0.5$ and two different values of the parameter γ. We observe that the eigenvalues exhibit an increase in sensitivity for the choice $\gamma = 0.5$ compared to $\gamma = 2$.

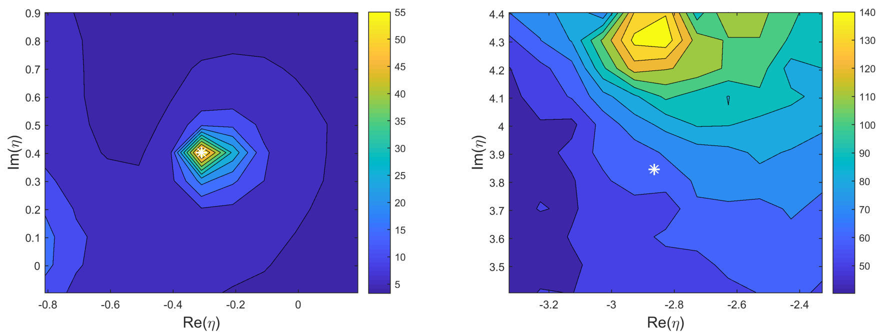
In order to test the detection of crack transmission eigenvalues from far field data, we use the same example and we focus on detecting the eigenvalues which displays the greatest sensitivity. We add approximately 1.5% noise to the data, and we use the same noisy far field data in each example. Since $\gamma = 0.5$ appears to improve sensitivity of the eigenvalues to changes in σ, we restrict our attention to this choice and compare the two choices of R in Figure 3. We see that with the choice $R = 1$ we successfully detect the eigenvalue $\eta = -0.30956 + 0.40281i$, whereas with $R = 0.5$ we fail to detect the eigenvalue $\eta = -2.8644 + 3.8467i$. These examples suggest that a good choice of R must balance sensitivity of the eigenvalues to changes in σ with the ability to accurately detect them from far field data, and further experimentation for this example has shown that $R = 1$ is a good choice to satisfy both of these criteria. We remark that this phenomenon is not a unique feature of the present problem; it has been observed for scattering by inhomogeneous media (c.f. [8], [9], and [11]) that domains with corners often prevent the accurate detection of eigenvalues from far field data unless B is chosen with smooth boundary ∂B which is sufficiently far from the corners.

We now provide a more realistic scenario in which we choose a nonconstant σ defined as

$$\sigma(\mathbf{x}) = 1 + e^{-\sigma_0 |\mathbf{x}|^2}, \; \mathbf{x} \in \Gamma, \tag{6.6}$$

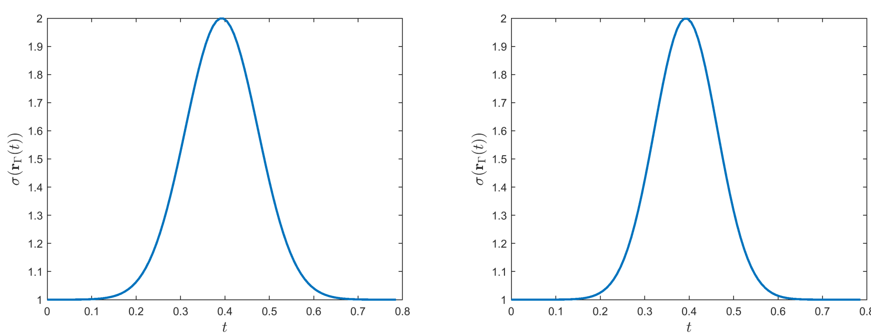
where $\sigma_0 > 0$ is some constant. In Figure 4 we plot σ as a function of t using the parametric definition (6.5) of the curve Γ with $\sigma_0 = 75$ (left) and $\sigma_0 = 100$ (right). We see that the curve with $\sigma_0 = 75$ is slightly wider than the curve with $\sigma_0 = 100$, and in fact the relative error in the ℓ^2-norm between these two functions is approximately 4%.

We first investigate the sensitivity of the crack transmission eigenvalues to changes in the nonconstant surface impedance σ given by (6.6), and in Figure



(a) $R = 1, \gamma = 0.5$ (b) $R = 0.5, \gamma = 0.5$

Figure 3: The detection of a crack transmission eigenvalue from far field data with $\gamma = 0.5$ and either $R = 1$ (left) or $R = 0.5$ (right). The white stars represent the exact eigenvalues computed from finite elements for each example. The choice $R = 1$ permits an accurate detection of the eigenvalue $\eta = -0.30956 + 0.40281i$, whereas the choice $R = 0.5$ appears to be ineffective in detecting the eigenvalue $\eta = -2.8644 + 3.8467i$.

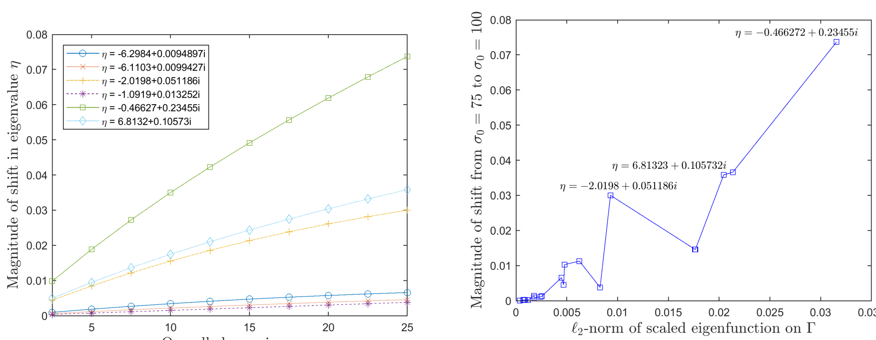


(a) $\sigma_0 = 75$ (b) $\sigma_0 = 100$

Figure 4: A plot of the function σ as an argument of t using the parametric definition of Γ for $\sigma_0 = 75$ (left) and $\sigma_0 = 100$ (right). The relative error in the ℓ^2-norm betwen these two functions is approximately 4%.

5a we plot the magnitude of the shift of each eigenvalue due to overall changes in the constant $\sigma_0 = 75$. In [11] it was found that for scattering by inhomogeneous media the sensitivity of modified transmission eigenvalues can sometimes be predicted by the magnitude of the scaled eigenfunction $w / \|v\|_{L^2(B)}$ in a neighborhood of where the change occurs. Though the lack of self-adjointness of the interior crack transmission problem prevents any analytic expression of this fact, we may investigate a similar connection for crack transmission eigenvalues numerically. In Figure 5b each data point represents a crack transmission eigenvalue (more are shown in this plot than in Figure 5a), and we have labeled the eigenvalues from Figure 5a with the greatest sensitivity. The x-coordinate of each point is the ℓ^2-norm of the scaled eigenfunction $w_{sc} := w / \|v\|_{L^2(B)}$ restricted to Γ (with the restriction from the exterior of D, i.e. w_{sc}^+), and the y-coordinate is the magnitude of the shift in the eigenvalue from $\sigma_0 = 75$ to $\sigma_0 = 100$. Though the relationship is far from monotonic, the trend is that the sensitivity increases with the magnitude of w_{sc}^+ on Γ.

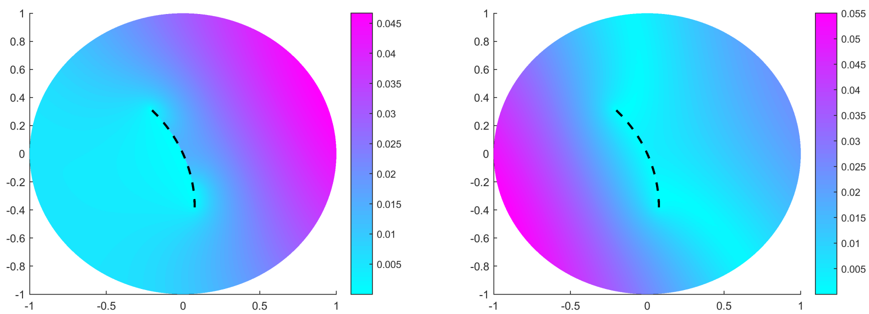
Though this observation is certainly useful in identifying eigenvalues with high sensitivity, we remark that the generalized linear sampling method does not provide any information on the eigenfunctions. Thus, a detailed knowledge of the geometry and material properties of the reference configuration of the crack is required in order to take advantage of this relationship. In Figure 6 we plot the modulus of two eigenfunctions corresponding to $\eta = -0.46627 + 0.23455i$ (the eigenvalue in the above example with the greatest sensitivity) and $\eta = -1.0919 + 0.013252i$ (the eigenvalue with the least sensitivity), respectively, in order to visualize the magnitude of the scaled eigenfunction w_{sc} on Γ.



(a) sensitivity of the eigenvalues

(b) relationship between sensitivity and the eigenfunctions

Figure 5: An investigation of the sensitivity of the crack transmission eigenvalue to changes in the parameter σ_0. The left figure shows the sensitivity of a few eigenvalues to changes in σ_0, and the right figure shows the relationship between sensitivity of each eigenvalue and the ℓ^2-norm of the trace w_{sc}^+ of an associated eigenfunction. We see a general trend that a higher magnitude of w_{sc}^+ results in greater sensitivity of that eigenvalue to changes in σ_0.



(a) $\eta = -0.46627 + 0.23455i$ (b) $\eta = -1.0919 + 0.013252i$

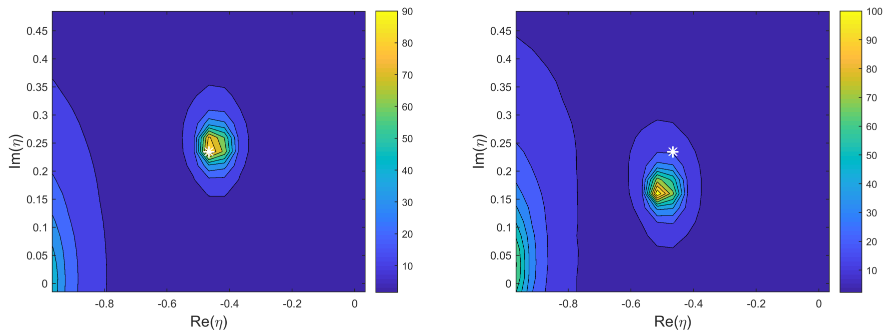
Figure 6: A plot of the modulus of the scaled eigenfunction w_{sc} corresponding to $\eta = -0.46627 + 0.23455i$ (left) and $\eta = -1.0919 + 0.013252i$ (right). The dashed line represents the crack Γ. We observe from the color map that w_{sc}^+ restricted to Γ has a higher magnitude for $\eta = -0.46627 + 0.23455i$ than for $\eta = -1.0919 + 0.013252i$.

We now focus our attention on the crack transmission eigenvalue $\eta = -0.46627 + 0.23455i$, and in Figure 7 we plot the indicator function $\boldsymbol{\mathcal{B}}(\mathbf{g}_\eta^{\text{glsm}})$ for both $\sigma_0 = 75$ (left) and $\sigma_0 = 100$ (right) with approximately 0.8% noise. In both plots the white star represents the exact eigenvalue computed from finite elements for $\sigma_0 = 75$ in order to more clearly show the shift in the eigenvalue to the change in σ to $\sigma_0 = 100$. We see from this example that it is indeed possible to detect crack transmission eigenvalues from far field data even for a complicated surface impedance σ, and we may also detect small changes (in this case 4%) in σ from shifts in its associated crack transmission eigenvalues.

7. Conclusion

We began by considering a general scattering problem in terms of a given operator T which may be regarded as an interior Dirichlet-to-Neumann map, and upon introducing an auxiliary scattering problem depending on a parameter η we developed and investigated a generalized Robin eigenvalue problem associated with the operator T. We derived sufficient conditions for T in order to guarantee certain properties of the associated Robin eigenvalues, including discreteness of the eigenvalues, their distribution in the complex plane, and a general existence theory. With the goal of detecting changes in the operator T (in turn arising from changes to the material properties of the scattering medium of interest), we showed that the generalized linear sampling method may be applied in this general framework in order to compute Robin eigenvalues from far field data associated with T.

After working in generality for the bulk of our analysis, we introduced the direct problem for scattering by a partially coated crack in $\mathbb{R}^2$ with surface



(a) $\sigma_0 = 75$ (b) $\sigma_0 = 100$

Figure 7: Detection of a crack transmission eigenvalue for $\sigma_0 = 75$ (left) and its observed shift for $\sigma_0 = 100$ (right). In both plots the white star represents the exact eigenvalue computed from finite elements for the case $\sigma_0 = 75$ in order to clearly show the shift in the eigenvalue due to the change in σ.

impedance σ as an application of our general results, and we derived various properties of the so-called crack transmission eigenvalues – the name given to the Robin eigenvalues for this particular choice of T. Unfortunately, due to the lack of interior regularity of solutions to this eigenvalue problem we were unable to apply the general existence theory to conclude that crack transmission eigenvalues exist, but we continued with a numerical investigation in which their existence was seen numerically. We ended with numerical examples in which we showed that crack transmission eigenvalues may indeed be computed from far field data in the presence of noise and that they shift due to changes in the surface impedance σ.

Two interesting conclusions from these examples are that the fixed parameter γ appearing in the auxiliary scattering problem may be tuned to improve sensitivity of the crack transmission eigenvalues to changes in σ and that the choice of the domain B must be a balance of sensitivity of the eigenvalues and the ability to detect them from far field data. For any choice of the operator T, an interesting open question is the precise effect of the choice of γ and B on the distribution and sensitivity of the eigenvalues to changes in T.

Acknowledgments: This material is based upon work supported by the Army Research Office through the National Defense Science and Engineering Graduate (NDSEG) Fellowship, 32 CFR 168a. The author would like to thank his advisors, Professors David Colton and Peter Monk, for their continued support and their assistance in proofreading the manuscript.

References

[1] L. Audibert, F. Cakoni, and H. Haddar, New sets of eigenvalues in inverse scattering for inhomogeneous media and their determination from scattering data, *Inverse Problems*, 33(12):125011, 28, 2017.

[2] L. Audibert and H. Haddar, A generalized formulation of the linear sampling method with exact characterization of targets in terms of farfield measurements, *Inverse Problems*, 30(3):035011, 20, 2014.

[3] L. Audibert and H. Haddar, The generalized linear sampling method for limited aperture measurements, *SIAM J. Imaging Sci.*, 10(2):845–870, 2017.

[4] A. S. Bonnet-Ben Dhia, L. Chesnel, and H. Haddar, On the use of T-coercivity to study the interior transmission eigenvalue problem, *C. R. Math. Acad. Sci. Paris*, 349:647–651, 2011.

[5] F. Cakoni and D. Colton, The linear sampling method for cracks, *Inverse Problems*, 19(2):279–295, 2003.

[6] F. Cakoni and D. Colton, *A Qualitative Approach to Inverse Scattering Theory*, volume 188 of *Applied Mathematical Sciences*, Springer, New York, 2014.

[7] F. Cakoni, D. Colton, and H. Haddar, *Inverse Scattering Theory and Transmission Eigenvalues*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 88, SIAM, Philadelphia, 2016.

[8] F. Cakoni, D. Colton, S. Meng, and P. Monk, Stekloff eigenvalues in inverse scattering, *SIAM Journal on Applied Mathematics*, 76(4):1737–1763, 2016.

[9] J. Camaño, C. Lackner, and P. Monk, Electromagnetic Stekloff eigenvalues in inverse scattering, *SIAM J. Math. Anal.*, 49(6):4376–4401, 2017.

[10] S. Cogar, D. Colton, and Y.J. Leung, The inverse spectral problem for transmission eigenvalues, *Inverse Problems*, 33(5):055015, 15, 2017.

[11] S. Cogar, D. Colton, S. Meng, and P. Monk, Modified transmission eigenvalues in inverse scattering theory, *Inverse Problems*, 33(12):125002, 2017.

[12] S. Cogar, D. Colton, and P. Monk, The detection of anomalies in an anisotropic medium surrounding a cavity, *submitted*.

[13] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer, New York, 3rd edition, 2013.

[14] P.C. Hansen, Regularization tools: a MATLAB package for analysis and solution of discrete ill-posed problems, *Numerical Algorithms*, 6:1–35, 1994.

[15] F. Hecht, New development in `FreeFem++`, *J. Numer. Math.*, 20:251–265, 2012.

[16] A. Kirsch and S. Ritter, A linear sampling method for inverse scattering from an open arc, *Inverse Problems*, 16(1):89–105, 2000.

[17] L. Sorber, M. Van Barel, and L. De Lathauwer, Unconstrained optimization of real functions in complex variables, *SIAM J. Optim.*, 22(3):879–98, 2012.

[18] L. Sorber, M. Van Barel, and L. De Lathauwer, Complex Optimization Toolbox v1.0, http://esat.kuleuven.be/stadius/cot/, 2013.

[19] N. Zeev, Identification of surface impedance of thin dielectric objects from far-field data, *Inverse Problems*, 27(2):025011, 12, 2011.

[20] N. Zeev and F. Cakoni, The identification of thin dielectric objects from far field or near field scattering data, *SIAM J. Appl. Math.*, 69(4):1024–1042, 2009.

[21] M. Zworski, *Semiclassical Analysis*, Graduate Studies in Mathematics, Vol. 138, American Mathematical Society, Providence, RI, 2012.