Graphs from Systems of Equations over Finite Fields

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Abstract

Beginning in the 1990s, graph theorists began to investigate the properties of graphs derived from systems of equations over finite fields. These graphs have vertex sets which are copies of the Cartesian product of a fixed number of copies of a field, i.e. each unique vector in the Cartesian product space is a vertex. Vertices are adjacent if and only if their coordinates satisfy a specified system of equations. Several families of such graphs are known to be mathematically interesting, including the Wenger and $D(k,q)$ graphs. We used SageMath to computationally generate these families of graphs, allowing their structure to be better understood. We investigated the behavior of the girth of the graphs $D(k,3)$, which attains a lower bound for some but not all values of k and is not well understood.

Graph Theory

A **graph** is a structure modeling a network, such as Facebook friends or computer servers. It consists of **vertices** and **edges**. Each vertex represents some object while each edge represents some sort of connection between two vertices.

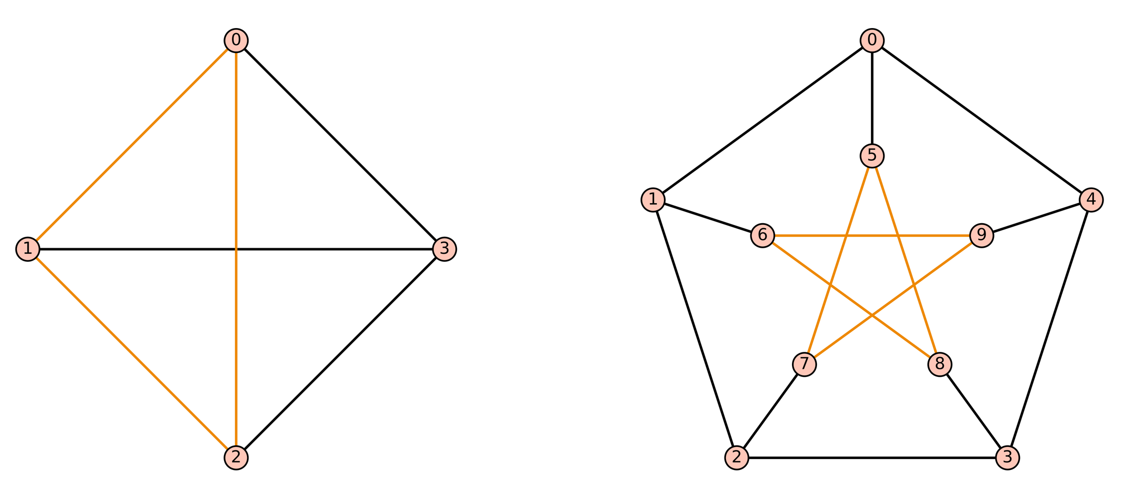


Figure 4: K_4 (left) and the Petersen Graph (right) with shortest cycles highlighted

The **girth** of a graph is defined as the length of the shortest cycle in the graph.

Systems of Equations

Graphs can be defined by **systems of equations** over **fields** $\mathbb{F}_q^k$. For example, take $W_1(3)$ (defined below). Every tuple between $(0,0)$ and $(2,2)$ appears in the graph twice, as both a line $[l]$ and a point (p). In this case only one equation must be satisfied: $p_2 + l_2 = p_1 l_1$.

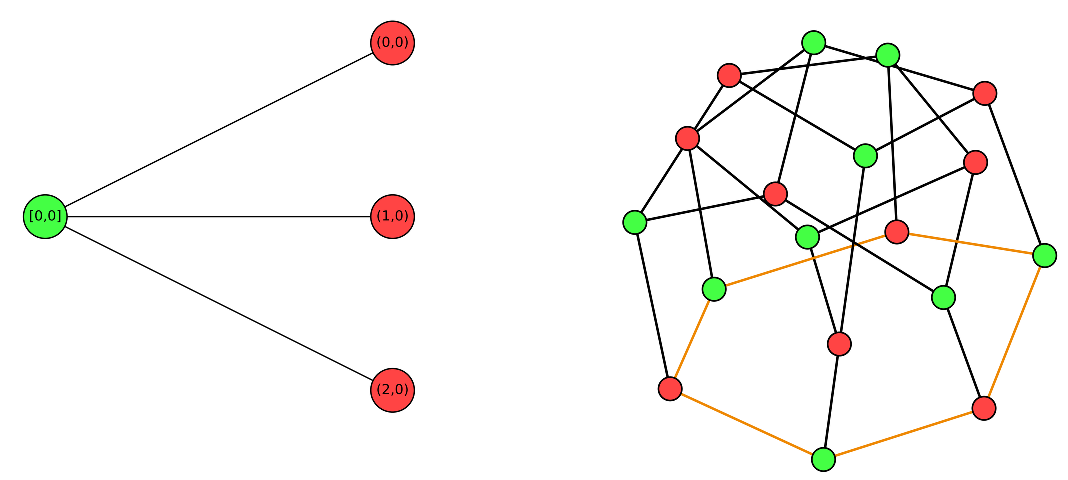


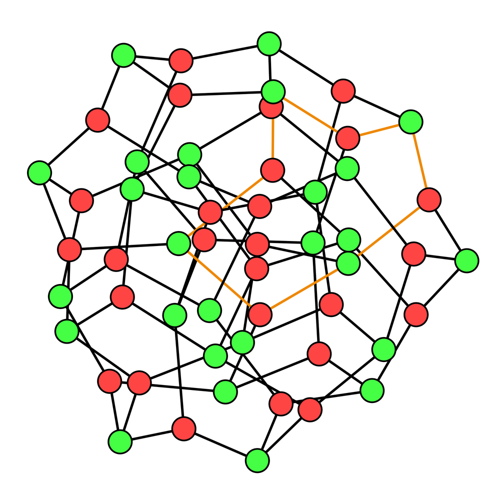
Figure 1: The neighbors of $[0,0]$ in $W_1(3)$ (left) and the graph itself (right)

Wenger Graph

The **Wenger Graphs** $W_m(q)$ are a family of expanders (meaning they are sparce and highly-connected) defined by the below system of equations over the field $\mathbb{F}_q$. $W_1(q)$, $W_2(q)$, and $W_4(q)$ are free of C_4, C_6, and C_{10}, respectively, while having a specific size and high order. This is shown by a shortest cycle displayed in orange.

Figure 5: $W_2(3)$ (right)

$$p_2 + l_2 = p_1^1 l_1$$
$$p_3 + l_3 = p_1^2 l_1$$
$$\vdots$$
$$p_n + l_n = p_1^m l_1$$



$D(k,q)$

The family of graphs $D(k,q)$ is associated with the field $\mathbb{F}_q$. The system of equations which defines this family starts with the following two:

$$p_2 + l_2 = p_1 l_1$$
$$p_3 + l_3 = p_1 l_2$$

The remaining equations follow the following pattern until there are $k-1$ equations.

$$p_i + l_i = -p_{i-2} l_1$$
$$p_j + l_j = -p_{j-2} l_1$$
$$p_k + l_k = p_1 l_{k-2}$$
$$p_m + l_m = p_1 l_{m-2}$$

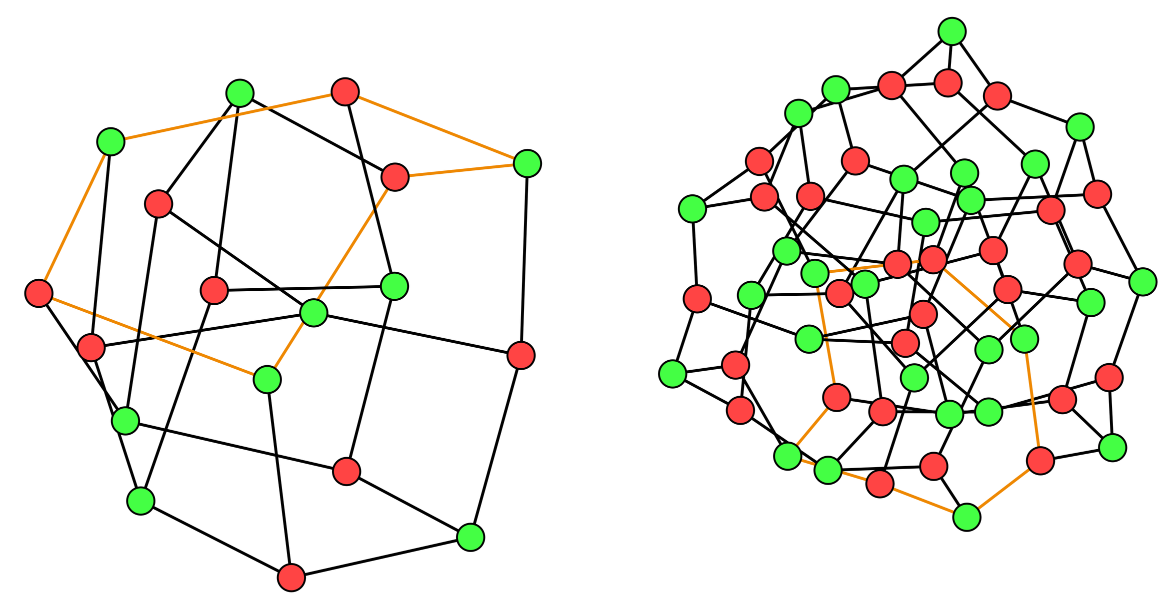


Figure 2: $D(2,3)$ with a cycle of minimum size (6) to demonstrate the girth (left) and $D(3,4)$ with girth 8 (right)

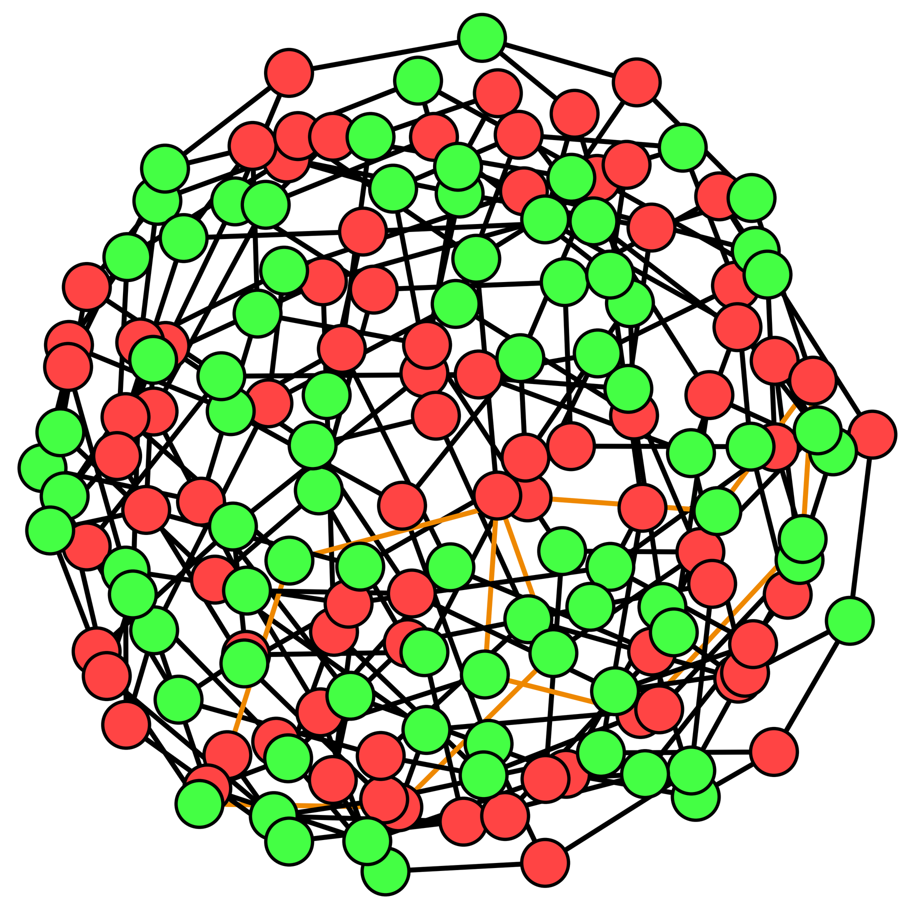


Figure 3: $D(4,3)$ with girth 12

Girth of $D(k,q)$

The girth of $D(k,q)$ is conjectured to behave as follows if $q > 3$: $D(k,q)$ has girth $k+5$ for odd k and $k+4$ for even k for all prime powers $q \geq 4$. This has been proven for infinitely many choices of k and q and has been computationally confirmed for small values of the parameters. $D(k,3)$, however, behaves differently, with computation revealing the following behavior:

k	2	3	4	5	6	7	8
girth	6	8	12	12	12	12	12
k	9	10	11	12	13	14	15
girth	18	18	18	18	18	18	20
k	16	17	18	19	20	21	22
girth	20	24	24	24	28	28	28
k	23	24	25	26			
girth	28	28	34	34			

Open Problems: Some open problems include calculating the eigenvalues of $D(k,q)$, proving its girth for both $k=3$ and $k \neq 3$, and showing that it is an expander for fixed q.
Families of graphs are known as **expanders** if they are sparce and highly-connected. These properties are extremely useful when designing efficient networks.

References

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Acknowledgments

We thank the Summer Scholars Program, run by the Undergraduate Research Program, for the funding provided for this research.