
A Graph Partition Problem

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Abstract. Given a graph G on n vertices, for which m is it possible to partition the edge set of the m -fold complete graph mK_n into copies of G ? We show that there is an integer m_0 , which we call the *partition modulus* of G , such that the set $M(G)$ of values of m for which such a partition exists consists of all but finitely many multiples of m_0 . Trivial divisibility conditions derived from G give an integer m_1 that divides m_0 ; we call the quotient m_0/m_1 the *partition index* of G . It seems that most graphs G have partition index equal to 1, but we give two infinite families of graphs for which this is not true. We also compute $M(G)$ for various graphs and outline some connections between our problem and the existence of designs of various types.

1. INTRODUCTION. The problem of interest in this paper is the following.

Given a graph G on n vertices, is it possible to partition the edge set of the complete graph K_n into isomorphic copies of G ? If this is not possible, then we are interested in determining the set of integers m such that the edge set of the m -fold complete graph mK_n can be partitioned into copies of G .

We will see that this seemingly simple problem has connections to algebra, combinatorics, and geometry among others. Important open problems, such as the existence of finite projective planes, are equivalent to edge partition problems into certain specified graphs.

Historically, perhaps the first instance of this type problem goes back to Walecki. In [8], he showed that the edge set of the complete graph K_n can be partitioned into copies of the cycle C_n when $n \geq 3$ is odd, and the edge set of the complete graph K_n minus a perfect matching can be partitioned into copies of C_n when $n \geq 4$ is even. See Figure 1 for a partition of K_5 into edge disjoint copies of C_5 . This result shows the impossibility of decomposing the edge set of K_n into copies of C_n but also that the edge set of $2K_n$ can be partitioned into copies of C_n when $n \geq 4$ is even.

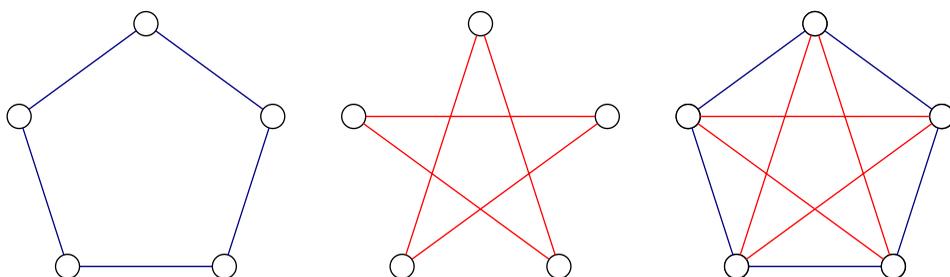


Figure 1. Two edge disjoint cycles on five vertices on the same vertex set

Figure 2 shows another instance of this problem where K_4 cannot be decomposed into edge disjoint copies of $K_{1,3}$, but $2K_4$ can be partitioned into 4 $K_{1,3}$ s.

Figure 3 shows that it is possible to find two edge-disjoint copies of the Petersen graph on 10 vertices. But can we find three edge-disjoint copies, that is, a partition of the edges of the complete graph on 10 vertices into three Petersen graphs?

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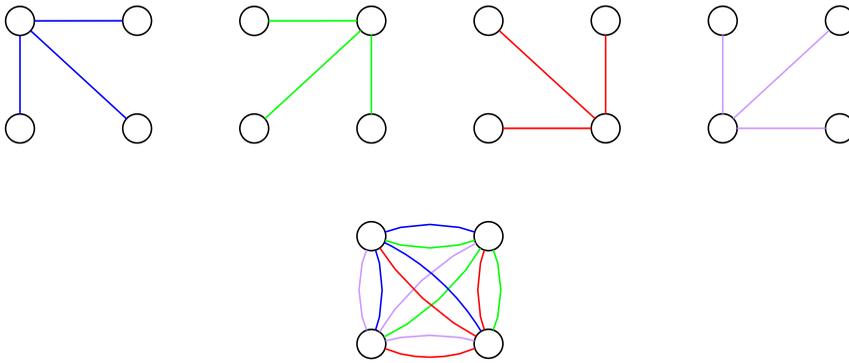


Figure 2. Four stars $K_{1,3}$ decomposing $2K_4$

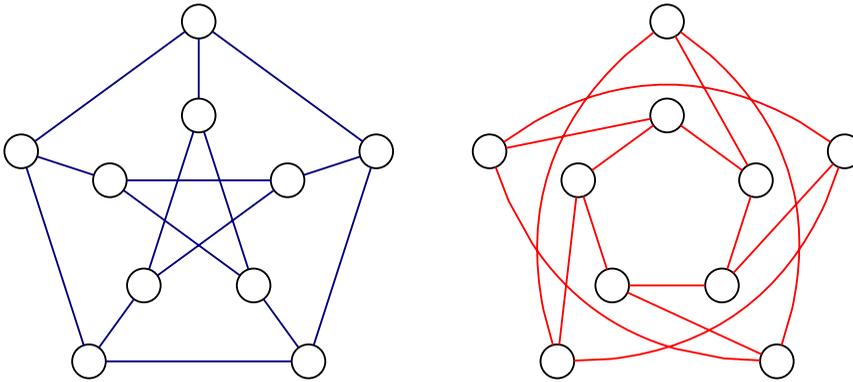


Figure 3. Two edge disjoint Petersen graphs on the same vertex set

This problem was proposed by Allen Schwenk in the *American Math Monthly* in 1983. Elegant negative solutions by Schwenk and O. P. Lossers appeared in the same journal in 1987 (see [13] for references to Schwenk’s problem and its solutions). This result is described in many books on algebraic graph theory, including Godsil and Royle [7, Section 9.2], and Brouwer and Haemers [3, Section 1.5.1]. Schwenk’s argument plus some simple combinatorial ideas can be used to show that whenever we can arrange two edge disjoint Petersen graphs on the same vertex set, then the complement of their union must be the bipartite cubic graph on 10 vertices that is the bipartite complement of C_{10} . In Figure 3, every missing edge goes from the set of five outer vertices to the set of five inner vertices, so the complement of the union of the two Petersen graphs is visibly bipartite (see [9] for a proof).

A generalization was posed by Rowlinson [12], and further variants have also been studied. Šiagiová and Meszka [14] obtained a packing of five Hoffman–Singleton graphs in the complete graph K_{50} . At the present time, it is not known if it is possible to decompose K_{50} into seven Hoffman–Singleton graphs. Van Dam [5] showed that if the edge set of the complete graph of order n can be partitioned into three (not necessarily isomorphic) strongly regular graphs of order n , then this decomposition forms an amorphic association scheme (see also van Dam and Muzychuk [6]).

At the Durham Symposium on Graph Theory and Interactions in 2013, the authors amused themselves by showing that, for every $m > 1$, the m -fold complete graph mK_{10} (with m edges between each pair of vertices) can be partitioned into $3m$ copies of the Petersen graph. Our purpose in this paper is to extend this investigation by replacing the Petersen graph by an arbitrary graph.

In fact, a stronger result for the Petersen graph was found by Adams, Bryant, and Khodkar [1]; these authors allow n to be arbitrary. In other words, they allow adding arbitrary many isolated vertices to the Petersen graph. We do not consider this more general problem.

2. PARTITION MODULUS.

Definition. For a graph G on n vertices, we let

$$M(G) = \{m : mK_n \text{ can be partitioned into copies of } G\}.$$

Example. As mentioned in the Introduction, one of the earliest results concerning this concept is that of Walecki [8] that states that the complete graph K_n can be partitioned into $(n - 1)/2$ Hamiltonian cycles if and only if n is odd. If n is even, then $2K_n$ can be partitioned into Hamiltonian cycles. However, when n is even and m is odd, mK_n cannot be partitioned into Hamiltonian cycles. This is evident because n does not divide $mn(n - 1)/2$. So

$$M(C_n) = \begin{cases} \mathbb{N} & \text{if } n \text{ is odd,} \\ 2\mathbb{N} & \text{if } n \text{ is even.} \end{cases}$$

Proposition 2.1. For any graph G , the set $M(G)$ is nonempty. In fact, if G has e edges and $\text{Aut}(G)$ is its automorphism group, then

$$2(n - 2)!e/|\text{Aut}(G)| \in M(G).$$

Proof. The graph G has $n!/|\text{Aut}(G)|$ images under the symmetric group S_n since S_n acts transitively on the set of graphs on vertex set $\{1, \dots, n\}$ that are isomorphic to G , and $\text{Aut}(G)$ is the stabilizer of one of these graphs. Each of the $n(n - 1)/2$ pairs of points is covered equally often by an edge in one of these images since S_n is doubly transitive; double counting gives this number to be $(n!/|\text{Aut}(G)|)e / (n(n - 1)/2)$, as required. ■

Our main result is a description of the set $M(G)$.

Theorem 2.2. For any graph G , there is a positive integer m_0 and a finite set F of multiples of m_0 such that $M(G) = m_0\mathbb{N} \setminus F$.

We call the number m_0 the *partition modulus* of G and denote it by $\text{pm}(G)$.

The theorem follows immediately from a couple of simple lemmas.

Lemma 2.3. The set $M(G)$ is additively closed.

Proof. Superimposing partitions of the edges of aK_n and bK_n gives a partition of $(a + b)K_n$. ■

Lemma 2.4. An additively closed subset M of \mathbb{N} has the form $m\mathbb{N} \setminus F$, where F is a finite set of multiples of m .

Proof. We have no convenient reference (though [11] is related), so we sketch the proof. We let $m = \gcd(M)$. By dividing through by m , we obtain a set with gcd equal to 1, so it suffices to prove the result in this case.

First, we observe that M is finitely generated, that is, there is a finite subset K such that any element M is a linear combination, with nonnegative integer coefficients, of elements of K . Then we proceed by induction on $|K|$. It is well known that, if $\gcd(a, b) = 1$, then all but finitely many positive integers have the form $xa + yb$ for some $x, y \geq 0$. Assume that the result holds for generating sets smaller than K . Take $a \in K$, and let $b = \gcd(K \setminus \{a\})$. By induction, $K \setminus \{a\}$ generates all but finitely many multiples of b . Also, $\gcd(a, b) = 1$ so that the result for sets of size 2 finishes the argument. ■

We have not tried to get an explicit bound here since for most graphs the excluded set F seems to be much smaller than our general argument suggests.

Example. As mentioned in Section 1, if P is the Petersen graph, then $M(P) = \mathbb{N} \setminus \{1\}$ so that the partition modulus of the Petersen graph is 1. It suffices to show that $2, 3 \in M(P)$.

That $2 \in M(P)$ follows from a generalisation of Proposition 2.1.

Proposition 2.5. *Suppose that G has n vertices and e edges and that there is a doubly transitive group H of degree n for which $|H : H \cap \text{Aut}(G)| = r$. Then $2re/n(n-1) \in M(G)$.*

Proof. The graph G has r images under H , whose re edges cover all pairs $2re/n(n-1)$ times. ■

Now $\text{Aut}(P) \cong S_5$, a subgroup of index 6 in S_6 (which acts as a 2-transitive group on the vertex set of P). So $6 \cdot 15/45 = 2 \in M(P)$.

A direct construction shows that $3 \in M(P)$. We do this by means of a 9-cycle on the vertex set of P , fixing a point ∞ and permuting the remaining points as $(0, 1, 2, \dots, 8)$. It is clear that the images of the three edges of P containing ∞ cover all pairs of the form $\{\infty, x\}$ three times. For the remaining pairs, we need to choose a drawing of $P - \infty$ on the vertices $\{0, \dots, 8\}$ in such a way that each of the distances 1, 2, 3, 4 in the 9-cycle is represented three times by an edge since these distances index the orbits of the cycle on 2-sets. It takes just a moment by computer to find dozens of solutions. For example, the edges

$$\{\infty, 1\}, \{\infty, 4\}, \{\infty, 8\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{1, 3\}, \{1, 5\}, \\ \{2, 5\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5, 6\}, \{6, 7\}, \{7, 8\}$$

have the required properties.

3. PARTITION INDEX. As is common in problems of this kind, there are some divisibility conditions that are necessary for a partition to exist.

Proposition 3.1. *Let G have n vertices and e edges and let d be the greatest common divisor of all the vertex degrees of G . Then every element of $M(G)$ has the property that e divides $mn(n-1)/2$ and d divides $m(n-1)$.*

Proof. If l copies of G partition mK_n , then $mn(n-1)/2 = le$, proving the first assertion; and the $m(n-1)$ edges incident at a vertex in mK_n are partitioned by the vertex stars in copies of G , from which the second assertion follows. ■

Let m_1 be the number for which these divisibility conditions are equivalent to the assertion that $m_1 \mid m$ for all $m \in M(G)$. Thus,

$$m_1 = \text{lcm} \left(\frac{e}{\gcd(e, n(n-1)/2)}, \frac{d}{\gcd(d, n-1)} \right).$$

We have that $m_1 \mid m_0 = \text{pm}(G)$. We define the *partition index* $\text{pi}(G)$ to be the quotient m_0/m_1 .

Proposition 3.2. *Let G have n vertices and e edges, and let \bar{G} denote its complement. Then $m \in M(G)$ if and only if $m \binom{n(n-1)/2 - e}{e} \in M(\bar{G})$.*

Proof. If l copies of G cover kK_n , then l copies of \bar{G} cover $(l-k)K_n$. Using $l = kn(n-1)/(2e)$ from the preceding proposition gives the result. ■

Corollary 3.3. *If G has n vertices and e edges, then*

$$\text{pm}(\bar{G}) = \left(\frac{n(n-1)/2 - e}{e} \right) \text{pm}(G).$$

The *triangular graph* $T(l)$ is the line graph of K_l , that is, its vertices are the edges of K_l and two vertices of $T(l)$ are adjacent when the edges in K_l share a vertex.

Example. It happens that $T(5)$ (the line graph of K_5) is the complement of the Petersen graph and we have $M(T(5)) = 2\mathbb{N} \setminus \{2\}$.

Remark. If the same relation held between the numbers m_1 for G and \bar{G} defined earlier as for the partition moduli in Corollary 3.3, then we would have $\text{pi}(G) = \text{pi}(\bar{G})$. But this is not true, as we will show using the graphs in Figure 4.

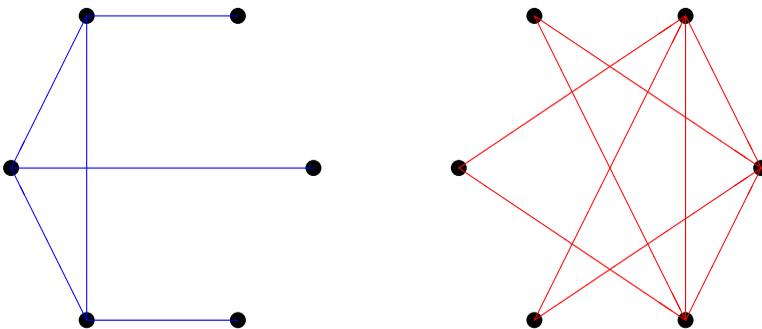


Figure 4. An example

Let G be the graph with six vertices and six edges consisting of a triangle with a pendant edge at each vertex. Then G has six edges, and the gcd of the vertex degrees

is 1, so $m_1(G)$ is the least common multiple of $6/\gcd(6, 15)$ and $1/\gcd(1, 5)$, that is, $m_1(G) = 2$. However, \overline{G} has nine edges and all vertex degrees even, so $m_1(\overline{G})$ is the least common multiple of $9/\gcd(9, 15)$ and $2/\gcd(2, 5)$. That is, $m_1(\overline{G}) = 6$.

We have $4 \in M(G)$. This follows from Proposition 2.5 since $\text{Aut}(G)$, which is dihedral of order 6, is a subgroup of index 10 in the 2-transitive group $\text{PSL}(2, 5)$. So $6 \in M(\overline{G})$. It follows that $M(\overline{G}) = 6\mathbb{N}$, and $\text{pm}(\overline{G}) = 6$ and $\text{pi}(\overline{G}) = 1$. But, by Corollary 3.3, we have $M(G) = 4\mathbb{N}$ so that $\text{pm}(G) = 4$ and $\text{pi}(G) = 2$.

4. EXAMPLES. In this section, we construct two families of examples of graphs that have partition index greater than 1. We are grateful to Mark Walters for the first of these.

Proposition 4.1. *Let G be a star $K_{1,n-1}$, with $n > 2$. Then $\text{pm}(G) = 2$ (and indeed $M(G) = 2\mathbb{N}$); so*

$$\text{pi}(G) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. $m_1(G)$ is the least common multiple of $(n-1)/\gcd(n-1, n(n-1)/2)$ (which is equal to 1 if n is even and 2 if n is odd) and $1/\gcd(1, n-1) = 1$.

Suppose that mK_n is covered with copies of the star; let the vertex set be $\{1, 2, \dots, n\}$, and let x_i be the number of stars with center at the vertex i . Then the edge $\{i, j\}$ is covered $x_i + x_j$ times, so $x_i + x_j = m$ for all $i \neq j$. This forces x_i to have a constant value x , and $m = 2x$. But we can achieve $m = 2$ by taking one star with each possible center ($x_i = 1$ for all i).

So the partition modulus of the star is 2, and the partition index is as claimed. ■

For the second construction, we observe that, if n is a multiple of 4, then the number $n(n-1)/2$ of edges of the complete graph is even, so there are graphs G and \overline{G} each having $n(n-1)/4$ edges. So the first term in the lcm for both $m_1(G)$ and $m_1(\overline{G})$ is 1. Suppose we arrange that G has all degrees odd, then \overline{G} will have all degrees even. If d and \overline{d} are the least common multiples of these degrees, then $d/\gcd(d, n-1)$ is odd, but $\overline{d}/\gcd(\overline{d}, n-1)$ is even (since $n-1$ is odd). Thus, $m_1(G)$ is odd, but $m_1(\overline{G})$ is even.

On the other hand, Proposition 3.3 shows that the partition moduli of the two graphs are equal and so necessarily even. Thus, certainly, $\text{pi}(G) > 1$.

The smallest example of this construction, for $n = 4$, has for G the star $K_{1,3}$ and for \overline{G} the graph $K_3 \cup K_1$. As we saw, the partition indices of these graphs are 2 and 1, respectively. For larger n , the construction gives many examples.

Problem. Is there a simple method of calculating the partition modulus (and hence the partition index) of a graph G ? Is it true that almost all graphs have partition index 1?

5. CONNECTION WITH DESIGN THEORY. For some special graphs, our partition problem is equivalent to the existence of certain 2-designs. A 2 -(n, k, λ) design consists of a set of n points and a collection of k -element subsets called blocks, such that any two points lie in exactly λ blocks. The design is resolvable if the blocks can be partitioned into classes of size n/k , each class forming a partition of the point set.

The existence of 2-designs has received an enormous amount of study; we refer the reader to [2] for some results.

Now the following result is clear.

- Theorem 5.1.** (a) Let G_1 be the graph consisting of a k -clique and $n - k$ isolated vertices. Then $m \in M(G_1)$ if and only if there exists a 2 - (n, k, m) design.
- (b) Let $k \mid n$, and let G_2 be the graph consisting of n/k disjoint k -cliques. Then $m \in M(G_2)$ if and only if there exists a resolvable 2 - (n, k, m) design.

Figure 5 shows K_9 decomposed into four graphs, each the union of three disjoint triangles, otherwise known as the affine plane $AG(2, 3)$. To reduce clutter, we adopt the convention that a line through three collinear points represents a triangle.

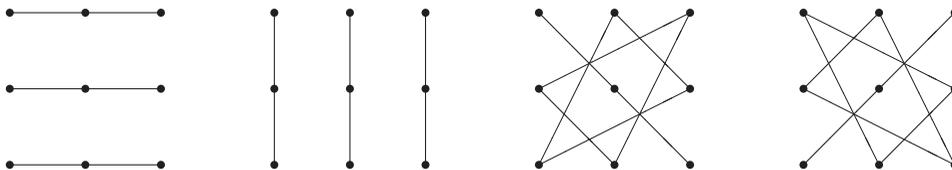


Figure 5. The affine plane $AG(2, 3)$

In fact, a more general version of this theorem is true.

Proposition 5.2. Let G be a graph on n vertices whose edge set can be partitioned into s complete graphs on k vertices. Then a necessary condition for $\lambda \in M(G)$ is that there exists a 2 - (n, k, λ) design.

This class of graphs includes, for example, the point graphs of partial geometries and generalized polygons.

This result gives a proof that $1 \notin M(T(l))$ (see Theorem 6.1 for another proof). For $T(l)$ is the edge-disjoint union of l cliques of size $l - 1$, and the resulting 2 - $(l(l - 1)/2, l - 1, 1)$ design would have

$$l(l - 1)/2 \cdot (l + 1)(l - 2)/2 \Big/ (l - 1)(l - 2) = l(l + 1)/4$$

blocks and so would violate Fisher's inequality (asserting that a 2 -design has at least as many blocks as points; see also [2, Section 2.2] for more details).

In some cases, the converse is true.

Proposition 5.3. Let $L_2(q)$ denote the line graph of $K_{q,q}$, the $q \times q$ square lattice graph. Then $1 \in M(L_2(q))$ if and only if q is odd and there exists a projective plane of order q .

Proof. We begin by noting that the existence of a projective plane of order q is equivalent to that of an affine plane of order q (a 2 - $(q^2, q, 1)$ design); such a design is necessarily resolvable, with $q + 1$ parallel classes.

Now the necessity of the condition follows from our general results. If the affine plane exists, then partition the parallel classes into $(q + 1)/2$ sets of size 2; each set, regarded as a set of $2q$ complete graphs of size q , gives a copy of $L_2(q)$. ■

Figure 5 illustrates the proof of the previous proposition when $q = 3$ and gives a decomposition of K_9 into two copies of $L_2(3)$.

The problem of determining $M(L_2(q))$ in cases not covered by this result, especially those when the required plane does not exist, is open. For example, what is $M(L_2(6))$?

It is worth mentioning that the problem is solved for the unique strongly regular graph with the same parameters as $L_2(q)$ but not isomorphic to it. This is the 16-vertex *Shrikhande graph*. Darryn Bryant [4] found five copies of the Shrikhande graph S that cover the edges of K_{16} twice; so $M(S) = 2\mathbb{N}$, and $\text{pi}(S) = 1$.

6. TRIANGULAR GRAPHS. Recall that the *triangular graph* $T(l)$ is the line graph of K_l , that is, its vertices are the edges of K_l , and two vertices of $T(l)$ are adjacent if the corresponding edges in K_l share a vertex.

Now $T(l)$ has $l(l-1)/2$ vertices and valency $2(l-2)$ with $l(l-1)(l-2)/2$ edges. Thus, $e/\text{gcd}(e, n(n-1)/2) = 4, 2$ or 1 according as the power of 2 dividing $l+1$ is $1, 2$ or at least 4 . Also, $d = l-2$, so $d/\text{gcd}(d, n-1)$ is 1 or 2 depending on whether $l+1$ is even or odd. So

$$m_1(T(l)) = \begin{cases} 1 & \text{if } l \equiv 3 \pmod{4}, \\ 2 & \text{if } l \equiv 1 \pmod{4}, \\ 4 & \text{if } l \text{ is even.} \end{cases}$$

We conjecture that these are also the partition moduli of the triangular graphs so that the partition indices are all 1 . We saw this already for $T(5)$.

Theorem 6.1. (a) *If $l \geq 4$, then $1 \notin M(T(l))$.*

(b) *If l is odd, then $2 \notin M(T(l))$.*

Proof. (a) The graph $T(l)$ has clique number $l-1$ and independence number $\lfloor l/2 \rfloor$ so cannot be embedded into its complement.

(b) The proof is by contradiction and generalizes Schwenk's argument [13] showing that three Petersen graphs cannot partition K_{10} . The distinct eigenvalues of the adjacency matrix of $T(l)$ are $2(l-2), l-4$, and -2 (see [3, Chapter 9]). Assume that $2 \in M(T(l))$ and consider a decomposition of $2K_{\binom{l}{2}}$ into $2\left(\binom{l}{2} - 1\right)/2(l-2) = (l+1)/2$ copies of $T(l)$. Let $A_1, \dots, A_{(l+1)/2}$ denote the adjacency matrices of these copies of $T(l)$. For $1 \leq i \leq (l+1)/2$, denote by \mathcal{E}_i the eigenspace of A_i corresponding to -2 . Each \mathcal{E}_i is contained in the orthogonal complement of the all-one vector in $\mathbb{R}^{\binom{l}{2}}$ and $\dim(\mathcal{E}_i) = \binom{l}{2} - l$ (see [3, Chapter 9]). Since the intersection of m subspaces each of codimension n in a vector space has codimension at most mn , we have

$$\begin{aligned} \dim\left(\bigcap_{i=1}^{(l-1)/2} \mathcal{E}_i\right) &\geq \binom{l}{2} - 1 - (l-1) \cdot (l-1)/2 \\ &= (l-3)/2 \\ &> 0. \end{aligned}$$

Let x be a nonzero vector in $\bigcap_{i=1}^{(l-1)/2} \mathcal{E}_i$. Since $A_1 + \dots + A_{(l-1)/2} + A_{(l+1)/2} = 2(J - I)$,

we deduce that

$$\begin{aligned}A_{(l+1)/2}x &= (2J - 2I - A_1 - \cdots - A_{(l-1)/2})x \\ &= -2x + ((l-1)/2)2x \\ &= (l-3)x.\end{aligned}$$

This implies that $l-3$ is an eigenvalue of $T(l)$, contradiction. ■

Proposition 6.2. $M(T(6)) = 4\mathbb{N}$ so that $\pi(T(6)) = 1$.

Proof. The automorphism group $\text{Aut}(T(6))$ is S_6 and has a subgroup A_6 that has index 7 in the 2-transitive group A_7 of degree 15. So seven copies of $T(6)$ cover the edges of K_{15} four times. ■

Thus, 7 is the smallest value of m for which we don't know $M(T(m))$. Here are a few comments on this case. Let $G = T(7)$.

- (a) By Theorem 6.1, we see that $1, 2 \notin M(G)$.
- (b) There does exist a $2-(21, 6, 4)$ design, namely the point residual of the Witt design on 22 points. However, this design is not possible in our situation for it has the property that any two blocks meet in 0 or 2 points, whereas $T(7)$ has six cliques meeting in one point.
- (c) Applying Proposition 2.5, with $H = PSL(3, 4)$, we obtain $1440 \in M(G)$. This leaves a very big gap.

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All Triangles at Once

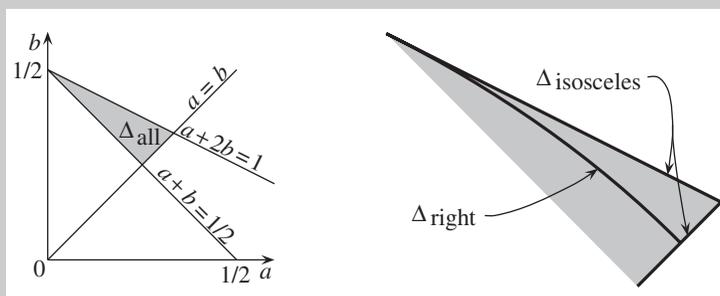
A triangle is determined modulo similarity. This is done by reordering the sides a , b , and c so that $a \leq b \leq c$ and rescaling so that the perimeter $a + b + c$ is 1, where $(a, b) \in \mathbb{R}^2$ such that

$$\left. \begin{array}{l} 0 \leq a \leq b \leq c \\ a + b + c = 1 \\ a \leq b + c, b \leq a + c, c \leq a + b \end{array} \right\} \Leftrightarrow (*) \left\{ \begin{array}{l} 0 \leq a \leq b \\ a + 2b \leq 1 \\ 1/2 \leq a + b \\ c = 1 - a - b \end{array} \right.$$

The set $\Delta_{\text{all}} = \{(a, b) \in \mathbb{R}^2 : (*)\}$ of all “triangles” is itself a triangle. We invite the reader to find Δ_{acute} , Δ_{obtuse} , and $\Delta_{\text{equilateral}}$. We see below where the following classes of triangles lie in Δ_{all} :

$$\Delta_{\text{right}} = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = c^2\} = \{(a, b) \in \mathbb{R}^2 : a + b - ab = 1/2\},$$

$$\Delta_{\text{isosceles}} = \{(a, b) \in \mathbb{R}^2 : a = b \vee b = c\} = \{(a, b) \in \mathbb{R}^2 : a = b \vee a + 2b = 1\}.$$



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