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PARTITION OF UNITY FINITE ELEMENT METHOD IMPLEMENTATION FOR POISSON EQUATION

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ABSTRACT. Partition of Unity Finite Element Method (PUFEM) is a very powerful tool to deal overlapping grids. It is flexible and keeps the global continuity. In this paper, we consider PUFEM for Poisson equation for minimal overlapping grids. We present details of the implementation of Poisson equation in 2D for two overlapping domains using triangular meshes are chosen.

1. **Introduction.** The main idea of overlapping grids is to divide a physical domain into overlapping regions which can accommodate smooth, simple, easily generated grids. Objects with complex geometry can be divided into simple domains and then meshes for each of them are generated separately. Refinement of the grids in one or part of domains can be done without affecting other domains. Furthermore, overlapping grids are suited to parallelization.

The study of finite element method applied to overlapping grids is done mainly in the framework of mortar method or Lagrange multiplier [5, 6, 8]. Partition of unity method [1] is used implicitly or explicitly to develop so-called generalized finite element methods. The study for the development of conforming finite element methods for overlapping and nonmatching grid can be found in [2]. Bacuta (et.al) study partition of unity method on nonmatching grids for the stokes problem [3]. Parallel partition of unity method is considered by Holst [4]. In this paper, we focus on the implementation aspects of PUFEM for Poisson equation and numerical consequences.

The rest of the paper is organized as following. In Section 2, we shall give a review of the standard finite element method for Poisson equation and various implementation aspects for it. In Section 3, we present the theoretic results of partition of unity method of Melenk and Babuška [1] and Huang and Xu's approach for overlapping and nonmathing grid [2]. In Section 4, we deal with the implementation of PUFEM for Poisson equation in two dimensions with minimal overlapping region. Numerical results are also presented. Finally in Section 5, we make some conclusions of the implementation and problems that might be interesting.

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2. **Standard FEM for Poisson Equation.** The problem we deal with is the following Poisson equation

$$-\triangle u \ = \ f \quad \text{in} \quad \Omega, \tag{2.1}$$
$$u \ = \ 0 \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

where $\Omega = (0,1) \times (0,1)$. Its weak formulation is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega)$$

where

$$a(u,v) = \int \nabla u \cdot \nabla v dx.$$

The corresponding finite element method formulation can be expressed as: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

where V_h is a finite dimension subspace of $H_0^1(\Omega)$. For elliptic problems of the second order, the error bounds can be found in [7].

In the implementation, we discretize the domain by uniform triangles. Figure 1 shows the generated mesh and numbering.
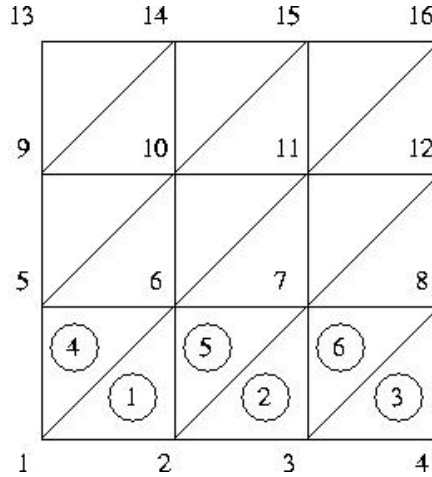

FIGURE 1. Mesh generated for the unit square.

For overlapping grids, we would like to generate meshes on the overlapping domains separately. Figure 2 shows the meshes on two overlapping domains for the unit square and their combination.

Remark 2.1. In the implementation of the PUFEM, we need to know if a triangle is in the overlapping region or not. It is not efficient to loop through all the triangles. When we generate the mesh in each subdomain, we put the triangles in the overlapping region together sequentially.

We use the standard finite element method to solve the following model problem for the purpose of later comparison.

$$-\triangle u \ = \ 2(x+y) - 2(x^2 + y^2) \quad \text{in} \quad \Omega \tag{2.3}$$
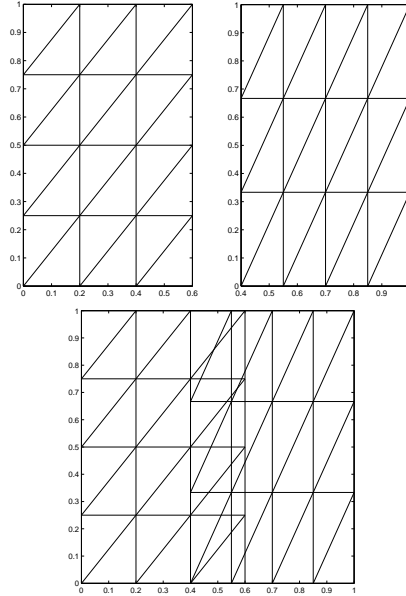$$u \ = \ 0 \quad \text{on} \quad \partial\Omega \tag{2.4}$$

FIGURE 2. Two overlapping meshes and their combination.

where $\Omega = [0,1] \times [0,1]$. The exact solution is

$$u(x,y) = xy(1-x)(1-y).$$

Table 1 is the L_∞ norm of the error $u - u_h$ for different values of h. Actually it is the maximum of all values evaluated at grid points.

TABLE 1. Numerical results of FEM for Poisson Equation ($u = xy(1-x)(1-y)$).

Element Diameter	L_∞ Error
1/10	4.873229035610716e-04
1/20	1.225445153886506e-04
1/40	3.068129255456853e-05
1/80	7.673154667112159e-06
1/160	1.918465776740153e-06

Other than the L_∞ norm, we may also look at the H^1 semi-norm. Let u be the solution of the continuous problem and u_h be the solution of the discrete problem, then

$$
\begin{aligned}
a(u - u_h, u - u_h) &= a(u, u - u_h) - a(u_h, u - u_h) \\
&= a(u, u) - a(u, u_h) \\
&= a(u, u) - (f, u_h).
\end{aligned}
$$

where we have already used the fact that $a(u_h, u - u_h) = 0$. Hence if we know u_h and the exact solution u, we can use above formula to calculate the H^1 semi-norm. For our first model problem, $u = xy(1-x)(1-y)$, we have $a(u,u) = \frac{1}{45}$. We use a

accurate quadrature formula to evaluate (f, u_h). Table 2 shows the H^1 semi-norm of $u - u_h$ for the model problem.

TABLE 2. Numerical results of FEM for Poisson equation ($u = xy(1-x)(1-y)$). The first column is the element size. The second column is the H^1 semi-norm of $u - u_h$.

Element Size	H^1 Semi-norm of the Error.
1/10	0.02420557358585
1/20	0.01215431899870
1/40	0.00608364175173
1/80	0.00304263245469
1/160	0.00152141771552

3. PUFEM for Overlapping Grids.

3.1. **Basic Theory of PUFEM.** We present the preliminaries of the PUFEM which can be found in detail in [1].

Definition 3.1. Let $\Omega \in R^n$ be an open set, $\{\Omega_i\}$ be an open cover of Ω satisfying a pointwise overlap condition:

$$\exists M \in N \quad \forall x \in \Omega \quad \text{card}\{i | x \in \Omega_i\} \leq M.$$

Let $\{\phi_i\}$ be a Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying

$$\text{supp}\phi_i \subset \text{closure}(\Omega_i) \quad \forall i,$$
$$\sum_i \phi_i \equiv 1 \quad \text{on} \quad \Omega,$$
$$\|\phi_i\|_{L^\infty(R^n)} \leq C_\infty,$$
$$\|\nabla \phi_i\|_{L^\infty(R^n)} \leq \frac{C_G}{\text{diam}\Omega_i},$$

where C_∞, C_G are two constants. Then $\{\phi_i\}$ is called a (M, C_∞, C_G) partition of unity subordinate to the cover $\{\Omega_i\}$. The partition of unity $\{\phi_i\}$ is said to be of degree $m \in N_0$ if $\{\phi_i\} \subset C^m(R^n)$. The covering sets $\{\Omega_i\}$ are called *patches*.

Definition 3.2. Let $\{\Omega_i\}$ be an open cover of $\Omega \subset R^n$ and let $\{\phi_i\}$ be a (M, C_∞, C_G) partition of unity subordinate to $\{\Omega_i\}$. Let $V_i \subset H^1(\Omega_i \cap \Omega)$ be given. Then the space

$$V := \sum_i \phi_i V_i = \{\sum_i \phi_i v_i | v_i \in V_i\} \subset H^1(\Omega)$$

is called the *PUFEM space*. The PUFEM space V is said to be of degree m if $V \subset C^m(\Omega)$. The space V_i are referred to as the *local approximation spaces*.

Theorem 3.3. *Let $\Omega \in R^n$ be given. Let $\{\phi_i\}$, $\{\Omega_i\}$ and $\{V_i\}$ be as in the definitions above. Let $u \in H^1(\Omega)$ be the function to be approximated. Assume that the local approximation spaces V_i have the following approximation properties: On each patch $\Omega_i \cap \Omega$, u can be approximated by a function $v_i \in V_i$ such that*

$$\|u - v_i\|_{L^2(\Omega_i \cap \Omega)} \leq \epsilon_1(i),$$
$$\|\nabla(u - v_i)\|_{L^2(\Omega_i \cap \Omega)} \leq \epsilon_2(i).$$

Then the function

$$u_{ap} = \sum_i \phi_i v_i \in V \subset H^1(\Omega)$$

satisfies

$$\|u - u_{ap}\|_{L^2(\Omega)} \leq \sqrt{M} C_\infty \left(\sum_i \epsilon_1^2(i) \right)^{1/2},$$

$$\|\nabla(u - u_{ap})\|_{L^2(\Omega)} \leq \sqrt{2M} \left(\sum_i \left(\frac{C_G}{diam\Omega_i} \right)^2 \epsilon_1^2(i) + C_\infty^2 \epsilon_2^2(i) \right)^{1/2}.$$

3.2. **Huang and Xu's Approach.** Let each Ω_i is partitioned by quasi-uniform triangulation τ^{h_i} of maximal mesh size h_i. With each triangulation τ^{h_i}, associate a finite element spaces $V_i \subset H^r(\Omega_i)$. Let $u \in H^r(\Omega)$, and let $m_i \geq 1$ denote the additional degree of smoothness of u on Ω_i. Assume optimal approximation property on subdomains:

For any $u \in H^{m_i+r}(\Omega_i)$, there exist a $v_h \in V_i$ such that

$$\sum_{k=0}^r h_i^k |u - v_h|_{k,\Omega_i} \leq c h_i^{m_i+r} \|u\|_{m_i+r,\Omega_i}.$$

Also assume that

$$|\nabla^k \phi_i| \leq c d_i^{-k}, 1 \leq k \leq r$$

where d_i is the minimal overlapping size of Ω_i with its neighboring subdomains.

Theorem 3.4. *(Huang and Xu) If the overlapping size $d_i \geq c h_i$, then for $0 \leq k \leq r$,*

$$\inf_{v_h \in V} \|u - v_h\|_{k,\Omega} \leq C \sum_{i=1}^p h_i^{m_i+r-k} \|u\|_{m_i+r,\Omega_i},$$

*for any $u \in H^r(\Omega) \cap_{i=1}^p H^{m_i+r}(\Omega_i)$.*

For $u \in H^2(\Omega)$ and H^1 conforming finite element space, we have

$$\inf_{v_h \in V} \|u - v_h\|_{0,\Omega} \leq C \sum_{i=1}^p h_i^2 \|u\|_{2,\Omega_i},$$

$$\inf_{v_h \in V} \|u - v_h\|_{1,\Omega} \leq C \sum_{i=1}^p h_i^1 \|u\|_{2,\Omega_i},$$

where we set $k = 0$ and $k = 1$ in the above theorem and $m_i = 1$, $r = 1$.

3.3. **Two Overlapping Domains.** We consider a simple case for overlapping grids. Let $\Omega = (0,1) \times (0,1)$, $\Omega_1 = (0,0.6) \times (0,1)$ and $\Omega_2 = (0.5,1) \times (0,1)$. The overlapping region is $\Omega_o = (0.5,0.6) \times (0,1)$. We generate meshes on Ω_1 and Ω_2 and end up with an overlapping mesh on Ω (Figure 3). Notice that the meshes in Ω_o overlap. To be precise, $\{\Omega_1, \Omega_2\}$ is an open cover of Ω satisfying a point wise overlap condition (See [1]). Let

$$\phi_1 = \begin{cases} 1 & 0 \leq x \leq 0.5, \quad 0 \leq y \leq 1, \\ \frac{0.6-x}{0.6-0.5} & 0.5 < x < 0.6, \quad 0 \leq y \leq 1, \\ 0 & 0.6 \leq x \leq 1, \quad 0 \leq y \leq 1, \end{cases} \tag{3.1}$$
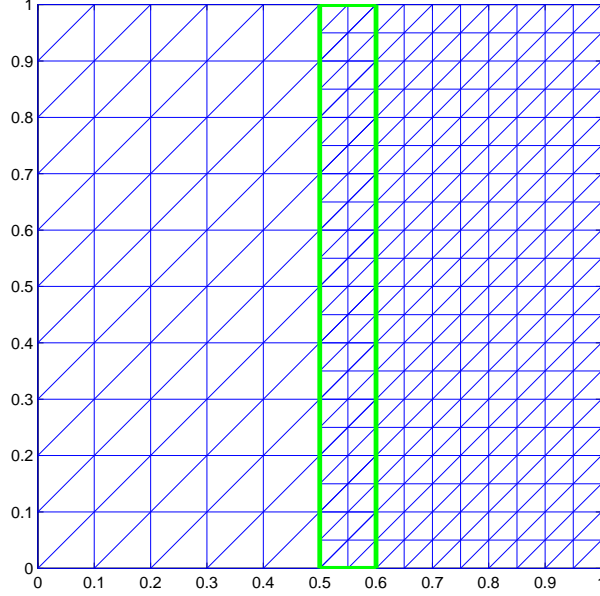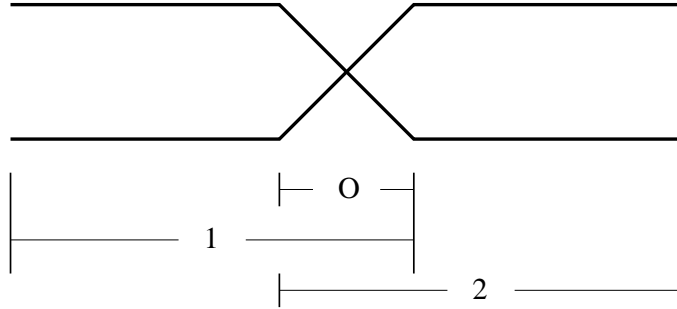
FIGURE 3. Overlapping Meshes in Ω.

and

$$\phi_2 = \begin{cases} 0 & 0 \leq x \leq 0.5, & 0 \leq y \leq 1, \\ \frac{x-0.5}{0.6-0.5} & 0.5 < x < 0.6, & 0 \leq y \leq 1, \\ 1 & 0.6 \leq x \leq 1, & 0 \leq y \leq 1. \end{cases} \tag{3.2}$$

Then $\{\phi_1, \phi_2\}$ is a Lipschitz partition of unity subordinate to the cover $\{\Omega_1, \Omega_2\}$, i.e.,

$$\phi_1 + \phi_2 = 1, \quad 0 \leq \phi_1, \phi_2 \leq 1, \quad \|\nabla\phi_i\|_{L^\infty} \leq 1/d$$

and $\phi_i \equiv 1, i = 1, 2$ on $\Omega_i \backslash \Omega_o$, and $\phi_i \equiv 0$ on $\Omega_j, j \neq i$. In the numerical tests, the

FIGURE 4. Partition of unity functions ϕ_1 and ϕ_2 in x-direction.

mesh in each domain will be refined and the width of the overlapping region also decreases. We set $\Omega_1 = (0.5, 0.5 + h_1) \times (0, 1)$, where h_1 is the width of one element of the mesh for Ω_1, and $\Omega_2 = (0.5, 1) \times (0, 1)$. Hence the width of the overlapping region Ω_o is h_1.

Let V_1 and V_2 be the local approximation spaces corresponding to Ω_1 and Ω_2. Then the PUFEM space is given by

$$V := \sum_{i=1,2} \phi_i V_i = \{\sum_{i=1,2} \phi_i v_i | v_i \in V_i\} \subset H^1(\Omega).$$

Thus V is a conforming subspaces of $H^1(\Omega)$. The discrete problem is to find u_h of the form

$$u_h = \phi_1 v_i + \phi_2 w_j$$

where $v_i \in V_1$ and $w_j \in V_2$, such that

$$a(u_h, v_h) = (f, v_h) \quad \forall \quad v_h \in V.$$

To numerically solve the above discrete problem we need to find a basis for V. The next theorem is of crucial importance in our paper.

Theorem 3.5. *Suppose we have the regular triangulation meshes for Ω_1 and Ω_2, which are defined as before and the overlapping region is a strip-type rectangle domain. Suppose that $V_k, k = 1, 2$ are linear finite element spaces with base functions $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$. Then the set*

$$\{\phi_1 v_i, \phi_2 w_j\}_{i \in I, j \in J}$$

forms a basis for the PUFEM space V.

Proof. Since $\phi_1 \equiv 1, \phi_2 \equiv 0$ in domain $\Omega_1 \setminus \Omega_o$ and $\phi_2 \equiv 1, \phi_1 \equiv 0$ in domain $\Omega_2 \setminus \Omega_o$, the linear independence is obvious. We will show the linear independence in the overlapping region.
Suppose

$$\phi_1 \sum_i \alpha_i v_i + \phi_2 \sum_j \beta_j w_j = 0.$$

Write $\phi_2 = 1 - \phi_1$ to obtain

$$\phi_1 \left(\sum_i \alpha_i v_i - \sum_j \beta_j w_j\right) = -\sum_j \beta_j w_j$$

where the right hand side is piece-wise linear at most. Since ϕ_1 is linear, we can conclude that

$$\sum_i \alpha_i v_i - \sum_j \beta_j w_j$$

is constant in Ω_o. Using the zero boundary condition, the above expression must be zero. Then we have $-\sum_j \beta_j w_j =$ and $\beta_j = 0$ for all j because of the linear independence of w_j's. Thus $\sum_i \alpha_i v_i = 0$ and we conclude that $\alpha_i = 0$ for all i because of the linear independence of v_i's. Hence $\{\phi_1 v_i, \phi_2 w_j\}_{i \in I, j \in J}$ are linearly independent and forms a basis for V.

$\square$

Remark 3.6. To show the linear independence of the base functions, we use the zero boundary condition. If ϕ_1 and ϕ_2 are not linear in the overlapping region, the linear independence can be shown without the application of the boundary condition.

4. Implementation and Numerical Results.

4.1. **Meshing.** We generate the mesh on Ω_1 and Ω_2 and then combine the meshes together to obtain the mesh on Ω. Since Ω_1 and Ω_2 overlap, we need to be deliberate when we generate meshes. To simplify coding, we make the following assumptions:

1. The boundary of Ω_o aligns with the two meshes on Ω_1 and Ω_2.
2. In the overlapping region, a triangle in the finer mesh (smaller in size) is contained entirely in a triangle in the coarser mesh. In other words, it can be thought as the refinement of the coarser mesh in the overlapping region.

4.2. **Stiffness Matrix.** As usual, we would like to setup the local stiffness matrix for every element e and then assemble the global stiffness matrix. If the element e is inside the non-overlapping regions, $\Omega_1 \setminus \Omega_o$ or $\Omega_2 \setminus \Omega_o$, the setup of the local stiffness matrix is as usual. But if the element e is inside the overlapping region, we have two cases.

Case 1: Suppose e is a triangle in the fine mesh, then e is contained in a triangle e' in the coarse mesh (see Figure 5). We will have two kinds of entries in the local stiffness matrix K_e. The first kind is due to the $e - e$ connection:

$$a(\phi_2\psi_i, \phi_2\psi_j)|_e, \quad i,j = 1,2,3$$

where ψ_i and ψ_j are nodal base functions on e. The second kind is due to the $e - e'$ connection:

$$a(\phi_2\psi_i, \phi_1\varphi_j)|_e, \quad i,j = 1,2,3$$

where $\phi_i, i = 1,2,3$ are nodal base functions on e and $\varphi_j, j = 1,2,3$ are nodal base functions on e'. Hence in this case, the local stiffness matrix has 18 entries.
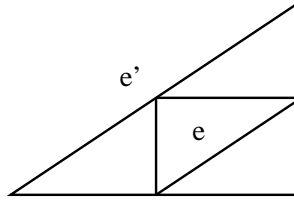

FIGURE 5. The figure shows that the relations of the triangles in the overlapping region. e' is the large triangle in the coarse mesh in Ω_1, which contains four small triangles in the fine mesh of Ω_2. e is one of the small triangles in e'.

Case 2: Suppose e' is a triangle in the coarse mesh, then it contains a couple of fine triangles, denoted by $e_k, k = 1, \ldots, N_{e'}$. The local stiffness matrix K_e also contains two kinds of entries. The first kind is due to the $e' - e'$ connection:

$$a(\phi_1\varphi_i, \phi_1\varphi_j)|_{e'}, \quad i,j = 1,2,3.$$

where φ_i and φ_j are nodal base functions on e'. The second kind is due to the $e' - e_k, k = 1, \ldots, N_{e'}$ connection:

$$a(\phi_1\varphi_i, \phi_2\psi_j)|_{e'}, \quad i,j = 1,2,3$$

where $\varphi_i, i = 1,2,3$ are nodal base functions on e' and $\psi_j, j = 1,2,3$ are nodal base functions on e_k. Hence in this case, the local stiffness matrix has $9 + 9 \times N_{e'}$ entries.

Note that by the symmetry of the bilinear form $a(\cdot, \cdot)$, we need to compute either case 1 or case 2. In our implementation, we compute case 1.

Remark 4.1. The setup of the right hand side of the linear system is similar to the above situation for the stiffness matrix.

4.3. **Numerical Result.** Let Ω be the unit square and $\Omega_1 = (0, 0.6) \times [0, 1]$ and $\Omega_2 = (0.5, 1) \times (0, 1)$. The mesh is exactly like that in Figure 3. The solution of the model problem by PUFEM is plotted (Figure 6). Since in the overlapping region,
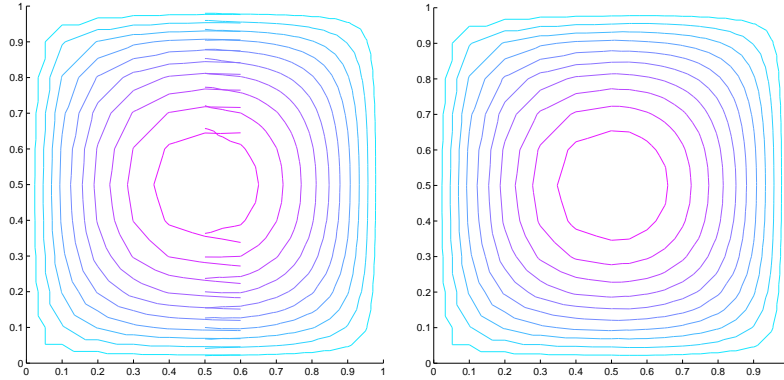

FIGURE 6. The figure on the left hand side is the solution of the problem. The figure on the right hand side is the contour plot of solution after adjustment. On Ω_1, $h = 1/10$. On Ω_2, $h = 1/20$.

the value of the function is given by

$$\alpha_i \phi_1 v_i + \beta_j \phi_2 w_j,$$

we need to combine the value in the overlapping region to get the solution. We can see the solution is smoother in the right part of the plot than that in the left part of the plot where the mesh are coarser.

We decrease the mesh size in both Ω_1 and Ω_2. We also decrease the width of the overlapping region. Hence we will have $\Omega_1 = (0, 0.5 + h_1) \times (0, 1)$ where h_1 is the width of one grid in Ω. We can get a series of the results and calculate the H^1 semi-norm as in the standard FEM. The H^1 semi-norm of the error is evaluated (Table 3).

TABLE 3. Numerical results of PUFEM for Poisson equation ($u = xy(1-x)(1-y)$). The first column is the element size in Ω_1 and the second is the element size in Ω_2. The third column is the H^1 semi-norm of $u - u_h$.

Element Size in Ω_1	Element Size in Ω_2	H^1 Semi-norm of the Error
1/10	1/20	0.01821113459601
1/20	1/40	0.00865699798935
1/40	1/80	0.00380050384524
1/80	1/160	0.00120187622777

Compare with Table 2, the results verifies the error estimate.

We also look at the condition number of the stiffness matrix. We first calculate the condition number of the stiffness matrix for the standard FEM (Table 4). We see that the condition number is of $O(h^{-2})$.

TABLE 4. Condition number of the stiffness matrix for standard finite element method.

Element Size in Ω	Condition number of the stiffness matrix.
1/10	58.4787
1/20	235.2855
1/40	942.5293
1/80	3.7715e+03

The condition number of the stiffness matrix for PUFEM is given is Table 5. We see that we end up with much larger condition numbers. But fortunately, it is still of $O(h^{-2})$.

TABLE 5. Condition number of the stiffness matrix for PUFEM.

h_1 in Ω_1	h_2 in Ω_2	Condition number of the stiffness matrix.
1/10	1/20	4.2017e+03
1/20	1/40	1.6950e+04
1/40	1/80	6.8245e+04
1/80	1/160	2.7403e+05

5. **Conclusions and Future Work.** In this paper, we implement PUFEM for Poisson equation in two dimensions. Numerical results verify the error estimates for PUFEM. We also show the condition number of the stiffness matrix for PUFEM is of $O(h^{-2})$ numerically.

We are considering to implement smooth partition of unity functions in the overlapping regions and study how it affects the stiffness matrix. Implementation of PUFEM for other equations, such as Helmholtz's Equation, on a more complex geometry is definitely more interesting.

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