Li-Yorke Chaos in Models with Backward Dynamics

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Abstract

Some economic models like the cash-in-advance model of money, the overlapping generations model and a model of credit with limited commitment may have the property that the dynamical system characterizing equilibria in the model are multi-valued going forward in time, but single-valued going backward in time, i.e., the model or dynamical system has *backward dynamics*. In such instances, what does it mean for a dynamical system (set-valued) to be chaotic? Furthermore, under what conditions are such dynamical systems chaotic? In this paper, I provide a definition of chaos that is in the spirit of Li and Yorke for a dynamical system with backward dynamics. I utilize the theory of *inverse limits* to provide sufficient conditions for such a dynamical system to be Li-Yorke chaotic.

Keywords: cash-in-advance, overlapping generations, limited commitment, Li-Yorke chaos, inverse limits, backward dynamics.

JEL: C6, E3, and E4.

1 Introduction

The equilibrium of a dynamic economic model can often be characterized as a trajectory generated by a dynamical system. Many nonlinear dynamical systems are single-valued moving forward, but multi-valued going backward, i.e., the system is not invertible. However, in economics there are dynamical systems with the opposite property, namely dynamics that are multi-valued going forward, but are single-valued going backward. When this phenomenon occurs, Medio and Raines (2007) say the model or dynamical system has *backward dynamics*. Three such models that may have backward dynamics include the overlapping generations (OG) model, the cash-in-advance (CIA) model and models of credit with limited commitment.[1] Typically, the problem of backward dynamics is either ignored by using a local analysis or avoided by analyzing the model with the single-valued backward map. However, using a local analysis, some potentially interesting equilibria may be ignored. And as Medio (1992, pp. 222–23) notes, the backward map solution is not entirely satisfactory either because the backward map gives trajectories that go backward into the infinite past, whereas equilibria are trajectories that lead off into the infinite future.

For certain properties of the dynamical system (e.g., establishing periodic orbits), the backward map is entirely satisfactory. However, it is less clear that one can use the backward map for other properties of the multi-valued forward map that may be of interest like *Li-Yorke chaos*. To be more concrete, suppose the set of equilibria in the model include trajectories $\{x_1, x_2, \ldots\}$ where $x_i = f(x_{i+1})$ and $f : I \to I$ where I is a closed interval of the real line. If one can establish that f has a 3-cycle, then the backward map f is chaotic in the sense of Li-Yorke. In this case, the Li-Yorke chaos is a property of forward orbits of backward map f which run backward in time. What, if anything, does a 3-cycle of f say about the *backward* orbits of f which run forward in time and correspond to equilibria in the model.

The term chaotic when applied to dynamical systems has been defined in several non-equivalent ways.[2] The focus of this paper is on one of the first and more commonly used definitions of chaos, namely that of Li and Yorke (1975). In this celebrated paper, Li and Yorke (1975) show that a cycle of period 3 implies chaos as they define it. This condition is sufficient, but not necessary and can be weakened to any cycle of period n not equal to 2^k for some $k \geq 0$. In this paper, I offer a definition of Li-Yorke chaos for dynamical systems

[1]See Grandmont (1985) for the OG model, Michener and Ravikumar (1998) for the CIA model and Gu and Wright (2011) for a model of credit with limited commitment.

[2]See, for example Robinson (1995, pp. 83–84).

with backward dynamics and provide sufficient conditions for such a dynamical system to be chaotic. I utilize the theory of *inverse limits* to establish the two main results:

- Let $f : I \to I$ be continuous with $I : [a, b] \subset \mathbb{R}$ with $a < b$. If f has a periodic point of order not equal to 2^k for $k = 0, 1, 2, \ldots$, then f^{-1} is chaotic in the sense of Li-Yorke. Corollary: if f has 3-cycle, then f^{-1} is Li-Yorke chaotic.

- Let $f : X \to X$ is continuous, where X is compact metric. If f has positive topological entropy, then f^{-1} is Li-Yorke chaotic.

The inverse limit of a dynamical system is a subset of an infinite dimensional space (e.g. the Hilbert cube) where each point in the inverse limit corresponds to a *backward* solution (backward orbit) of the dynamical system. Using the backward map f from say the CIA model as our dynamical system, a point in the inverse limit, being a backward orbit of f, corresponds to a *forward* orbit in the model. The backward map f can be used to induce a *homeomorphism* F on the inverse limit space. The dynamical properties of this induced homeomorphism are closely related to those of the backward map. The dynamical properties of the inverse of this homeomorphism F^{-1} (a single-valued function) are closely related to those of f^{-1} (the forward multi-valued dynamical system).

The use of inverse limits to analyzing models with backward dynamics is a relatively new approach. Medio and Raines (2007, 2006) use inverse limits to analyze the long-run behavior of an OG model. Even though the forward dynamics are multi-valued, they show that "typical" long-run behavior of equilibria in the model corresponds to an "attractor" of the shift map on the inverse limit space. In particular, they persuasively argue that these equilibria associated with an attractor should be the ones of interest in models with backward dynamics since these are the ones that one can expect to occur. Kennedy et al. (2007) investigate the topological structure on the inverse limit space associated with the CIA model of Lucas and Stokey (1987). The complexity of the dynamical system and the complexity of the inverse limit space are connected.[3] In economics, models with backward dynamics do occur, and it is important to have a framework consisting of tools and results for analyzing such systems.

The paper is organized as follows. To motivate the problem, in section 2 I show that the CIA model, OG model and a credit model with limited commitment may have an implicitly-defined difference equation characterizing equilibria that is sometimes multi-valued forward in time, but single-valued going backward in time, i.e., these models may exhibit backward

[3]See for example, Ingram and Mahavier (2004).

dynamics. In section 3, I discuss inverse limits and the induced homeomorphism on this space by the dynamical system. I also define Li-Yorke chaos for a multi-valued dynamical system and provide sufficient conditions for the multi-valued forward dynamics to be Li-Yorke chaotic. I conclude in section 4.

2 Models with Backward Dynamics

Some models in economics have the property that the dynamical system characterizing the equilibrium conditions is multi-valued going forward in time, but single-valued going backward in time, i.e., the model has backward dynamics. To be more concrete, an equilibrium in the model must satisfy an implicitly-defined difference equation $G(x_{t+1}, x_t) = 0$. Given x_t there is more than one value for x_{t+1} that satisfies $G(x_{t+1}, x_t) = 0$. However, given x_{t+1}, there is a unique value for x_t satisfying $G(x_{t+1}, x_t) = 0$, i.e., one can solve for the backward map $x_t = F(x_{t+1})$. In this section I briefly discuss three models that may exhibit backward dynamics.

2.1 Cash-in-Advance Model of Money

The model is the standard endowment CIA model of Lucas and Stokey (1987). I closely follow the exposition of Michener and Ravikumar (1998), hereafter [MR]. Since our intent is only to show that the model has backward dynamics and that for certain parameter values, the backward map is chaotic, I will focus on a particular family of utility functions and parameterizations used in [MR].[4] It is an endowment economy with both cash and credit goods. There is a representative agent and a government. The government consumes nothing and sets monetary policy using a money growth rule.

The household has preferences over sequences of the cash good (c_{1t}) and credit good (c_{2t}) represented by a utility function of the form

$$\sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t}), \tag{1}$$

with the discount factor $0 < \beta < 1$. The utility function is assumed to take the following form:

$$U(c_1, c_2) := \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\gamma}}{1-\gamma},$$

with $\sigma > 0$ and $\gamma > 0$. To purchase the cash good c_{1t} at time t the household must have cash m_t. This cash is carried forward from $t-1$. The credit good c_{2t} does not require cash,

[4]See Michener and Ravikumar (1998) for more details and a more general framework.

but can be bought on credit. The household has an endowment y each period that can be transformed into the cash and credit goods according to $c_{1t} + c_{2t} = y$. Since this technology allows the cash good to be substituted for the credit good one-for-one, both goods must sell for the same price p_t in equilibrium and the endowment must be worth this price per unit as well.

The household seeks to maximize (1) by choice of $\{c_{1t}, c_{2t}, m_{t+1}\}_{t=0}^{\infty}$ subject to the constraints $c_{1t}, c_{2t}, m_{t+1} \geq 0$,

$$\begin{aligned} p_t c_{1t} &\leq m_t, && (2) \\ m_{t+1} &\leq p_t y + (m_t - p_t c_{1t}) + \theta M_t - p_t c_{2t}, && (3) \end{aligned}$$

taking as given m_0 and $\{p_t, M_t\}_{t=0}^{\infty}$. The money supply $\{M_t\}$ is controlled by the government and follows a constant growth path $M_{t+1} = (1+\theta)M_t$ where θ is the growth rate and $M_0 > 0$ given. Each period the household receives a transfer of cash from the government in the amount θM_t. A *perfect foresight equilibrium* is defined in the usual.

Let $x_t := m_t/p_t$ denote the level of real money balances and c be the unique solution to $U_1(x, y - x) = U_2(x, y - x)$. If the cash-in-advance constraint (2) binds, then $c_{1t} = x_t$. If not, then $c_{1t} = c$. It then follows that $c_{1t} = \min[x_t, c]$ for all t. [MR] use this relationship to get a difference equation in x alone that characterizes equilibria in the model:

$$x_t U_2(\min[x_t, c], y - \min[x_t, c]) = \frac{\beta}{1+\theta} x_{t+1} U_1(\min[x_{t+1}, c], y - \min[x_{t+1}, c])$$

or

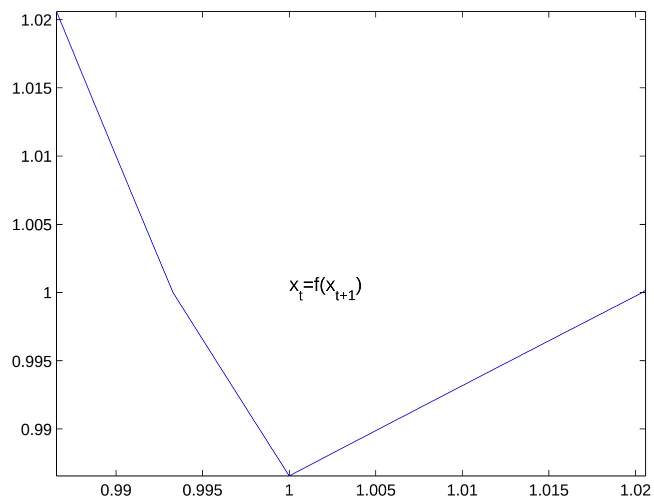
$$B(x_t) = A(x_{t+1}), \tag{4}$$

where

$$\begin{aligned} B(x) &:= xU_2(\min[x, c], y - \min[x, c]), \\ A(x) &:= \frac{\beta}{1+\theta} xU_1(\min[x, c], y - \min[x, c]). \end{aligned}$$

The function B is invertible so one can always solve for the backward map $f := B^{-1} \circ A$ giving $x_t = f(x_{t+1})$. Whether or not the dynamics going forward are multi-valued depends on whether or not $A(\cdot)$ is invertible. In one parameterization, [MR, p. 1129] set $\beta = 0.98$, $\sigma = 0.5$, $\gamma = 4$, $y = 2$ and consider θ equal to 0, 0.5 and 1.0. In this case the function A is not invertible and there exists an invariant set $[x_l, x_h]$ such that the the backward map has a three cycle. The backward map for this parameterization (with $\theta = 0$) is in Figure 1. One sees that for this parameterization, the CIA model has backward dynamics.

Figure 1: Backward map $f : [x_l, x_h] \to [x_l, x_h]$ from the cash-in-advance model.



2.2 Overlapping Generations Model

In this subsection I describe the basic overlapping generations model to illustrate the possibility of backward dynamics in this model as well.[5] Consider a discrete time economy with constant population equally divided into two equally numerous cohorts, labeled "young" and "old ." At the beginning of each period t a new cohort is born and lives for two periods (being "young" at t, and "old" at $t+1$). At $t = 0$, it is assumed that an "old" cohort already exists. Because individuals in each class are assumed to be identical, one can describe the situation in terms of "the young (old) agent." There is no production, but fixed amounts of the homogeneous, perishable consumption good are distributed at the beginning of each period to young and old.

Let $c_t \geq 0$ be the young agent's consumption at time t, and let $g_t \geq 0$ be the old agent's consumption at time t. Let $w_0 \geq 0$ and $w_1 \geq 0$ be the young and old agent's endowment respectively of the perishable good (there is no production). Let $\rho_t > 0$ be the interest rate at time t, i.e. the exchange rate between present and future consumption. Define the utility functions by

$$U(c_t, g_{t+1}) := u_1(c_t) + u_2(g_{t+1}), \tag{5}$$

with $u_i'(\cdot) > 0$ and $u_i''(\cdot) \leq 0$ for $i = 1, 2$ (the same for all agents).

[5]The OG model literature is vast. Some relevant papers include Benhabib and Day (1982), Gale (1973) and Grandmont (1985). See Azariadis (1993) for more references and a text-book exposition of the model.

The young agent's problem is to maximize (5) by choice of $\{c_t, g_{t+1}\}$ subject to the intertemporal budget constraint $c_t + g_{t+1}/\rho_t \leq w_0 + w_1/\rho_t$ and non-negativity constraints $c_t, g_{t+1} \geq 0$ taking $\{w_0, w_1, \rho_t\}$ as given. The market-clearing condition (for all t):

$$c_t + g_t = w_0 + w_1. \tag{6}$$

A competitive equilibrium is defined in the usual way as a collection of sequences of $\{c_t, g_t\}_{t=0}^{\infty}$ and $\{\rho_t\}_{t=0}^{\infty}$ satisfying optimality (each agent's utility maximization problem is solved) and the market-clearing condition (6). From the first-order conditions of the young agent's problem and the market-clearing condition (6), it is clear that an *equilibrium* in the model is an infinite sequence of numbers $\{g_t\}$ satisfying $0 \leq g_t \leq w_0 + w_1$ and the equation:

$$H(g_{t+1}, g_t) := \mathcal{U}(g_{t+1}) + \mathcal{V}(g_t) = 0, \tag{7}$$

where $\mathcal{U}(g) := u_2'(g)(g - w_1)$ and $\mathcal{V}(g) := u_1'(w_0 + w_1 - g)(w_1 - g)$.

Whether or not from (7) one can derive a difference equation moving forward in time depends on whether the function $\mathcal{U}$ is invertible. Consider the following specific example:

$$u_1(c) = c; \quad u_2(g) = ag - (b/2)g^2, \tag{8}$$

with a and b positive constants. Furthermore, set $w_0 > 0$, $w_1 = 0$, and $a = b$. In this case, $\mathcal{U}$ is not invertible and one gets

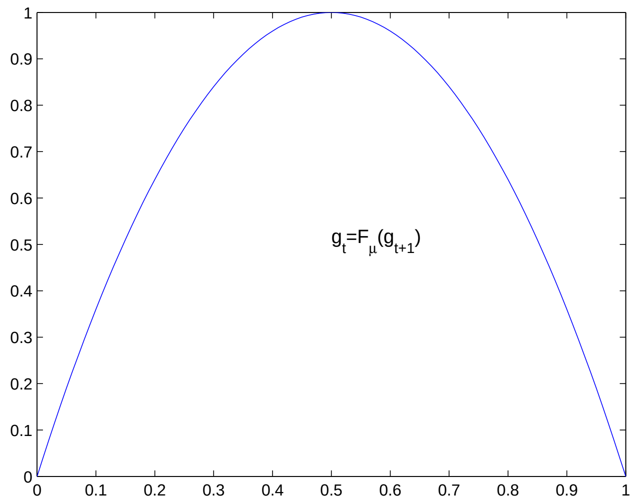
$$g_t = F_\mu(g_{t+1}) := \mu g_{t+1}(1 - g_{t+1}). \tag{9}$$

The function F_μ is the much-studied "logistic map" in dynamical systems (see Figure 2). Note that this map is typically multi-valued going forward. However, the dynamics are single-valued going backward in time (given g_{t+1} there is a unique g_t that satisfies (9)). One sees that for this parameterization, the OG model has backward dynamics. For $\mu = 4$, F_μ is chaotic on the entire interval $[0, 1]$.

2.3 Credit Model with Limited Commitment

In Gu and Wright (2011), time is discrete and each period is divided into two sub-periods. There are two types of agents (each type has measure 1) and two types of goods. Type 1 consumes good 1 but produces good 2. Type 2 consumes good 2 but produces good 1. Both goods are produced in the first sub-period. Good 1 is consumed in the first sub-period and good 2 is consumed in the second sub-period. A type 1 consumer who produces y units good

Figure 2: Backward map $F_\mu : [0,1] \to [0,1]$ with $\mu = 4$ ($a = b = 4$).



2 delivers y units of good 2 in the *second* sub-period. During a given period, utility over (x, y) (goods 1 and 2 respectively) are given by

$$U^1(x, y) \text{ and } U^2(x, y)$$

with $U^j(0, 0) = 0$ (normalization), strictly increasing in the consumption good, strictly decreasing in the production good and twice differentiable. In the second sub-period, a type one agent can choose not to deliver y and instead consume the output with a payoff λy added to the utility $U^1(x, y)$. If a type 1 chooses not to deliver, then with probability π the type 1 agent goes to autarky (receiving 0 utility forever).

Let V_t^j be the expected (lifetime) utility of agent of type j at the beginning of period t entering into a contract (x_t, y_t). If contracts are honored, then

$$\begin{aligned} V_t^1 &= U^1(x_t, y_t) + \beta V_{t+1}^1, \\ V_t^2 &= U^2(x_t, y_t) + \beta V_{t+1}^2. \end{aligned}$$

Participation constraints are given by

$$U^1(x_t, y_t) \geq 0 \text{ and } U^2(x_t, y_t) \geq 0. \tag{10}$$

The repayment constraint for type 1 is

$$\lambda y_t + (1 - \pi)\beta V_{t+1}^1 \leq \beta V_{t+1}^1.$$

Define

$$\phi_t := \frac{\beta\pi}{\lambda} V^1_{t+1}.$$

Then the repayment constraint for type 1 can be written as

$$y_t \leq \phi_t. \tag{11}$$

If contracts are honored, one gets the following recursive expression for ϕ:

$$\phi_{t-1} = \frac{\beta\pi}{\lambda} U^1(x_t, y_1) + \beta\phi_t. \tag{12}$$

Gu and Wright (2011) model two different types of market structures: generalized Nash bargaining and Walrasian price taking. The Nash bargaining problem is the following:

$$\max_{x,y} U^1(x,y)^\theta U^2(x,y)^{1-\theta},$$

subject to (10) and (11). Let (x^N, y^N) solve the Nash bargaining problem. Then the necessary first-order conditions to the Nash bargaining problem without the repayment constraint are:

$$\theta U^1_x(x_t, y_t) U^2(x_t, y_t) + (1-\theta) U^1(x_t, y_t) U^2_x(x_t, y_t) = 0, \tag{13}$$

$$\theta U^1_y(x_t, y_t) U^2(x_t, y_t) + (1-\theta) U^1(x_t, y_t) U^2_y(x_t, y_t) = 0. \tag{14}$$

If $\phi_t \geq y^N$, then $x_t = x^N$ and $y_t = y^N$ can be implemented. If $\phi_t < y^N$, then $y_t = \phi_t$ and using $y_t = \phi_t$ in equation (13), one can solve for x_t which defines a function $x_t = h(\phi_t)$.

To recap, what matters for (x_t, y_t) in equilibrium is how ϕ_t compares to y^N (the unconstrained allocation):

$$y_t = \begin{cases} \phi_t & \text{if } \phi_t < y^N \\ y^N & \text{if } \phi_t \geq y^N \end{cases} \quad \text{and } x_t = h(y_t). \tag{15}$$

Note that $y_t := y(\phi_t)$ is continuous, but not differentiable at $\phi_t = y^N$.

An equilibrium is defined as a bounded sequence of non-negative credit limits $\{\phi_t\}_{t=1}^\infty$ and contracts $\{x_t, y_t\}_{t=1}^\infty$ satisfying (15) given $\{\phi_t\}_{t=1}^\infty$ and the sequence $\{\phi_t\}_{t=1}^\infty$ satisfies (12). One can simplify these conditions to get the following characterization: equilibria correspond to non-negative and bounded sequences $\{\phi_t\}_{t=1}^\infty$ that satisfy:

$$\phi_{t-1} = f(\phi_t) := \begin{cases} \frac{\beta\pi}{\lambda} U^1(h(\phi_t), \phi_t) + \beta\phi_t & \text{if } \phi_t < y^N \\ \frac{\beta\pi}{\lambda} U^1(x^N, y^N) + \beta\phi_t & \text{otherwise} \end{cases}. \tag{16}$$

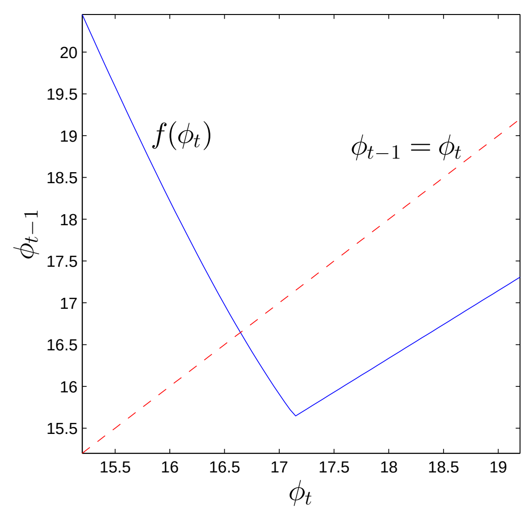
Note that f gives the backward dynamics of ϕ and the model has *backward dynamics* if f is not invertible.

Gu and Wright (2011) provide numerical examples using the following specific utility functions:

$$U^1(x,y) := \frac{(x+b)^{1-\alpha} - b^{1-\alpha}}{1-\alpha} - y \text{ and } U^2(x,y) := y - \frac{Ax^{1+\gamma}}{1+\gamma}.$$

Example 5 from their paper has $\alpha = 2.25$, $b = 0.082$, $A = 1.3$, $\beta = 0.81$, $\pi/\lambda = 40/3$, $\theta = 0.01$, $\gamma = 0$. Under Nash bargaining, one gets a steady state $\phi^* = 16.65$, $y^N = 17.14$ along with a three cycle $\phi^1 = 15.73$, $\phi^2 = 17.094$ and $\phi^3 = 18.93$. The backward map f for this parameterization is in Figure 3. Note that f is not invertible so the model has *backward dynamics*. Moreover, a period 3 for f implies that the dynamics are Li-Yorke chaotic going *backward* in time.

Figure 3: Backward map f from the credit model of Gu and Wright (2011).



2.4 Concluding Remarks

Having illustrated that there are economic models with backward dynamics, two questions naturally arise (1) what does it mean for a multi-valued dynamical system with backward dynamics to be *chaotic*? and (2) how does one determine if such a system is chaotic? I address these two questions in the next section.

3 Li-Yorke Chaos for Dynamical Systems with Backward Dynamics

In this section, I extend the definition of *Li-Yorke chaos* to a model with backward dynamics. I use the theory of inverse limits to provide sufficient conditions on the backward map to imply the multi-valued dynamical system going forward in time is Li-Yorke chaotic. For the case where the state space is a compact interval, I show that period 3 for the backward map implies Li-Yorke chaos for the multi-valued dynamical system going forward.

3.1 Inverse Limits

Let X be a nonempty compact metric space with metric d and suppose $f : X \to X$ is a continuous function. In the context of the economic model, one should think of X as the state space and f as the backward map. Let X^∞ be the infinite product of X with itself endowed with the usual product topology. Recall that the product topology is generated by the following basic open sets. Let $\{u_1, u_2, \ldots, u_n\}$ be a finite collection of open sets in X. Define $< u_1, u_2, \ldots u_n >:= \{\mathbf{x} = (x_1, x_2, \ldots) \in X^\infty : x_i \in u_i \text{ for } 1 \leq i \leq n\}$. The collection $B := \{< u_1, u_2, \ldots u_n > : \{u_1, u_2, \ldots u_n\}$ is a finite collection of open sets in $X\}$ is the collection of *basic open sets*. For a metric on X^∞ I will use the following function $\tilde{d} : X^\infty \to \mathbb{R}_+$ given by

$$\tilde{d}(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}.$$

The product topology is compatible with the topology generated by the metric $\tilde{d}$. Note the following standard results from topology: (1) If X is a metric space, then X^∞ is a metric space; and (2) If X is compact, then X^∞ is compact.

Let $\mathbb{N}$ denote the natural numbers. The space X is called the *factor space* and the function f is called the *bonding map*. The pair (X, f) is called an *inverse system*. The set of points

$$\varprojlim(X, f) := \{\mathbf{x} = (x_1, x_2, \ldots) \in X^\infty \mid x_i = f(x_{i+1}) \text{ for } i \in \mathbb{N}\},$$

is the *inverse limit* of the inverse system (X, f). If $m \in \mathbb{N}$, the map $\pi_m : \varprojlim(X, f) \to X$ defined by $\pi_m(\mathbf{x}) = x_m$ is called the *projection map* (or the m^{th} *projection map*).

Note that $\varprojlim(X, f)$ is a subset of X^∞ and each point in the inverse limit corresponds to a *backward* solution to the dynamical system $f : X \to X$. Note that since f is the backward map from an economic model, the points in the inverse limit space correspond to forward solutions of the implicit difference equation characterizing an equilibrium in the model (i.e, backward orbits of the backward map). In other words, the set of equilibria in the model *is* an inverse limit space.

Let $Y := \varprojlim(X, f)$. A natural map is induced on the inverse limit space by the bonding map f: for $\mathbf{x} = (x_1, x_2, \ldots) \in Y$, define

$$F(\mathbf{x}) \equiv F((x_1, x_2, \ldots)) := (f(x_1), f(x_2), \ldots) \equiv (f(x_1), x_1, x_2, \ldots).$$

The induced map F is a homeomorphism from Y onto Y. The inverse $\sigma := F^{-1}$ of F, is then defined by $\sigma(\mathbf{x}) \equiv \sigma((x_1, x_2, \ldots)) := (x_2, x_3, \ldots)$. The map σ is called the *shift homeomorphism*. Thus, the pair (Y, F) forms a dynamical system that runs both forward and backward. This is one of the main advantages of the inverse limit approach to backward dynamics. By treating an equilibrium in the model as a single point in a larger space, one can gain insight into the dynamics of f^{-1} (which is multi-valued) by analyzing the shift map σ on Y.

3.2 Chaos for Dynamical Systems with Backward Dynamics

When confronted with a model with backward dynamics, what does it mean to say the model is *Li-Yorke chaotic*? Li-Yorke chaos requires the existence of a *scrambled set*:

Definition 1. *Suppose X is a compact metric space and $f : X \to X$ is continuous and $\delta \geq 0$. $S \subset X$ is a* scrambled set *of f such that for any $x, y \in S$, $x \neq y$ and any periodic point p of f:*

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta, \tag{17}$$

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \tag{18}$$

$$\limsup_{n\to\infty} d(f^n(x), f^n(p)) > \delta. \tag{19}$$

If $\delta > 0$, S is called a δ-scrambled set of f.

Definition 2. *The function f is called chaotic in the sense of Li-Yorke (Li-Yorke chaotic) if f has an uncountable scrambled set.*

Next, I define a dynamical system as a collection of trajectories and then extend the notion of a scrambled set an Li-Yorke chaos to to this collection of trajectories.

Definition 3. *Let X be a compact metric space and $X^\infty : \{x_1, x_2, \ldots \,|x_i \in X\}$, the $D \subset X^\infty$ is a dynamical system. We say that f generates D if for each $x_1, x_2, \ldots \in D$ we have*

$x_{i+1} = f(x_i)$ for $i \in \mathbb{N}$. We say that f^{-1} generates D if $x_i = f(x_{i+1})$ for $i \in \mathbb{N}$. We say that $x \in D$ is periodic, if there exists an $n \in \mathbb{N}$ such that $x = \sigma^n(x)$. The period of a periodic point x is the smallest $k \in \mathbb{N}$ such that $\sigma^k(x) = x$.

Consider the following definition of scrambled set for D:

Definition 4. *Let X be a compact metric space and $X^\infty := \{x_1, x_2, \ldots \ |x_i \in X\}$, the $D \subset X^\infty$ and $\delta \geq 0$. $S \subset D$ is a* scrambled set *of D such that for any $x, y \in D$, $x \neq y$ and any periodic point p of D:*

$$\limsup_{n\to\infty} d(\pi_n(x), \pi_n(y)) > \delta, \tag{20}$$
$$\liminf_{n\to\infty} d(\pi_n(x), \pi_n(y)) = 0, \tag{21}$$
$$\limsup_{n\to\infty} d(\pi_n(x), \pi_n(p)) > \delta. \tag{22}$$

If $\delta > 0$, S is called a δ-scrambled set of D.

Using this I can now define Li-Yorke chaos for D:

Definition 5. *Let X be a compact metric space and $X^\infty := \{x_1, x_2, \ldots \ |x_i \in X\}$, the $D \subset X^\infty$. We say D is Li-Yorke chaotic if there exists an uncountable scrambled set S. If D is generated by f^{-1} we say f^{-1} is Li-Yorke chaotic.*

N.B. If D is generated by f, then D is Li-Yorke chaotic iff f is Li-Yorke chaotic. The important point here is that Li-Yorke chaos is a property defined for a collection of trajectories generated by a dynamical system. With f, each point $x \in X$ is mapped to a single trajectory $\{x, f(x), f^2(x), \ldots\}$. However with f^{-1}, there may be an infinite number of trajectories associated with x as an initial condition. Define the direct limit space of (X, f) as

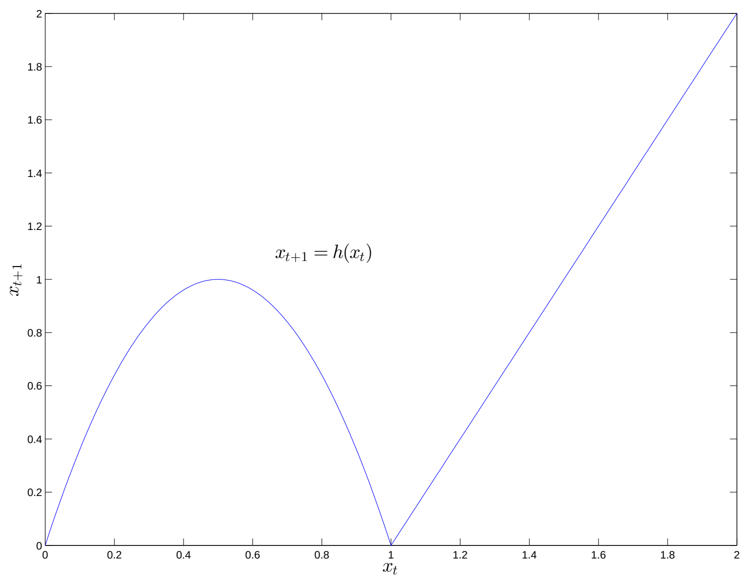
$$\hat{Y} := \lim_{\rightarrow}\{X, f\} := \{\mathbf{x} \in X^\infty \mid x_{i+1} = f(x_i),\ i \in \mathbb{N}\}.$$

One then thinks of $\hat{Y}$ as generated by f and Y being generated by f^{-1}. These are *different* subsets of X^∞ (and when chaos is present, they are *very* different spaces topologically). A scrambled set for f will be a subset of $\hat{Y}$ and a scrambled set for f^{-1} will be a subset of Y. To illustrate that the relationship between these sets is non-trivial, consider the following example.

Example 1. *Let $h : [0, 2] \to [0, 2]$ be defined as*

$$h(x) := \begin{cases} 4x(1-x) & 0 \leq x \leq 1 \\ 2(x-1) & 1 < x \leq 2. \end{cases}$$

Figure 4: Graph of function $h : [0, 2] \to [0, 2]$.



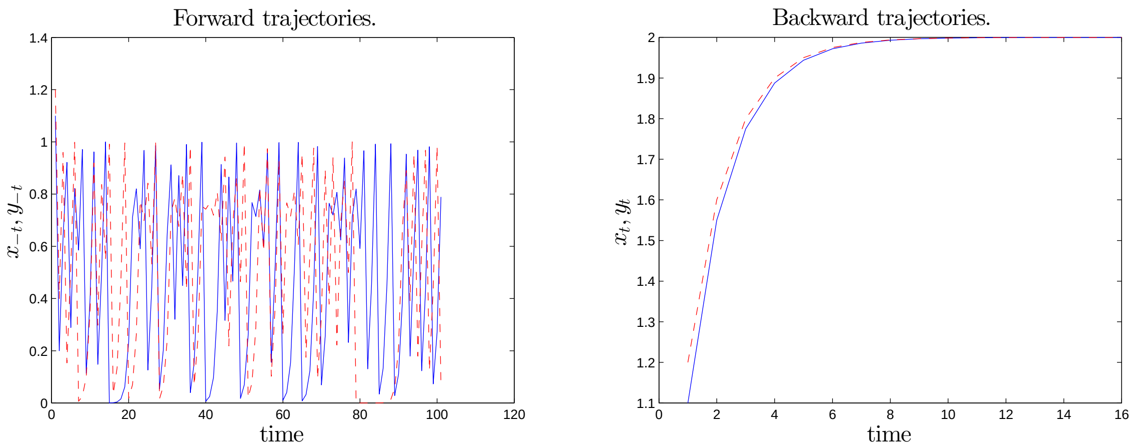
Note that $h|_{[0,1]}$ is the logistic map, $[0, 1]$ is invariant under h and that $h|_{[0,1]}$ is Li-Yorke chaotic. See Figure 4 for the graph of h.

Let $x_1, y_1 \in S$ for f so we have

$$\begin{aligned} \limsup_{n\to\infty} d(h^n(x), h^n(y)) &> \delta, \\ \liminf_{n\to\infty} d(h^n(x), h^n(y)) &= 0. \end{aligned}$$

This can be done so that $1 < x_1 < 3/2$ and $1 < y_1 < 3/2$. Each trajectory in $\underset{\longleftarrow}{\lim}(X, h)$ that start at x and y typically involves selecting from the set generated by f^{-1}. It is not the case that any $\mathbf{x}, \mathbf{y} \in Y$ with $\pi_1(\mathbf{x} = x$ and $\pi_t(\mathbf{y}) = y$ will be part of a scrambled set $\tilde{S}$ for h^{-1} (if one even exists). In this particular example, there is only one preimage for x_1 and y_1 under h since $h|_{(3/2,2]}$ is 1-to-1 and maps onto $(1, 2]$. This implies there is only one point $\mathbf{x} \in Y$ and $\mathbf{y} \in Y$ with $\pi_1(\mathbf{x} = x_1$ and $\pi_t(\mathbf{y}) = y_1$. Moreover $x_j \to 2$ and $y_j \to 2$ implying that these trajectories are convergent and asymptotic, i.e., not part of a scrambled set for h^{-1}. Yet, going backward in time, the trajectories are part of a scrambled set and look highly erratic. See Figure 5 for forward and backward trajectories of h with initial conditions $x_1 = 1.1$ and $y_1 = 1.2$. We see that looking backwards in times, the dynamics look chaotic, yet going forward in time (the relevant direction according to the economic model) dynamics are monotonic and convergent.

Figure 5: Forward and backward trajectories of h with the same initial conditions.



We now have defined what it would mean for the CIA, OG or credit model with backward dynamics to be chaotic. The equilibria in these models are given by iterating a relation (see Figure 6).

The main result in this section is that f^{-1} is Li-Yorke chaotic if and only if σ is Li-Yorke chaotic. I also show that f is Li-Yorke chaotic if and only if F is Li-Yorke chaotic. The following lemmas will be used.

Lemma 1. *Let d be the metric on X and $\tilde{d}$ be the metric on X^∞ induced by d. Then for $\mathbf{x}, \mathbf{y} \in X^\infty$, we have given $r > 0$, if $\tilde{d}(\sigma^k(\mathbf{x}), \sigma^k(\mathbf{y})) > r$ for some $k \geq 0$, then there exists $m \geq k$ such that $d(x_{m+1}, y_{m+1}) \geq r/2$.*

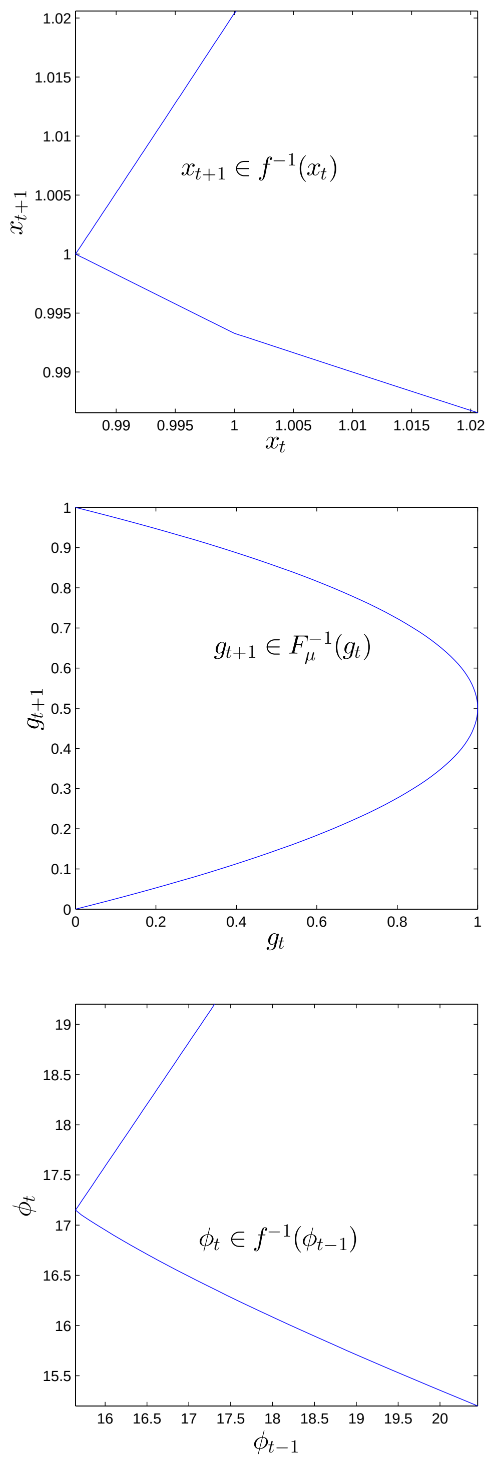
Proof. Suppose $\mathbf{x}, \mathbf{y} \in X^\infty$, $r > 0$ and there exists $k \geq 0$ such that $\tilde{d}(\sigma^k(\mathbf{x}), \sigma^k(\mathbf{y})) > r$. Suppose that $d(x_{m+1}, y_{m+1}) < r/2$ for all $m \geq k$. Then we have

$$r < \tilde{d}(\sigma^k(\mathbf{x}), \sigma^k(\mathbf{y})) = \sum_{i=1}^{\infty} \frac{d(x_{k+i}, y_{k+i})}{2^{i-1}} < (r/2) \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = r.$$

which is a contradiction. □

Lemma 2. *Let d be the metric on X and $\tilde{d}$ be the metric on X^∞ induced by d. Suppose $f : X \to X$ is continuous and onto, and and $Y := \varprojlim(X, f)$. Let $K := \max_{x,y \in X} d(x, y)$. Then for $\mathbf{x} := (x_1, x_2, \ldots), \mathbf{y} := (y_1, y_2, \ldots) \in Y$, and given $r > 0$, if $\tilde{d}(F^k(\mathbf{x}), F^k(\mathbf{y})) > r$ for some $k > 0$ with $\frac{K}{2^{k-1}} < r/2$, then there exists $1 \leq m \leq k$ such that $d(f^m(x_1), f^m(y_1)) > r/4$.*

Figure 6: Forward relations from the CIA, OG and credit models.



Proof. Let $k > 0$ with $\frac{K}{2^{k-1}} < r/2$ and $\tilde{d}(F^k(\mathbf{x}), F^k(\mathbf{y}) > r$. Then

$$\begin{aligned} r &< \tilde{d}(F^k(\mathbf{x}), F^k(\mathbf{y}) := \sum_{j=1}^{k} \frac{d(f^{k+1-j}(x_1), f^{k+1-j}(y_1))}{2^{j-1}} + \sum_{j=k+1}^{\infty} \frac{d(x_{j-k}, y_{j-k})}{2^{j-1}} \\ &\leq \sum_{j=1}^{k} \frac{d(f^{k+1-j}(x_1), f^{k+1-j}(y_1))}{2^{j-1}} + \frac{K}{2^{k-1}} \\ &< \sum_{j=1}^{k} \frac{d(f^{k+1-j}(x_1), f^{k+1-j}(y_1))}{2^{j-1}} + \frac{r}{2}. \end{aligned}$$

This implies

$$\frac{r}{2} < \sum_{j=1}^{k} \frac{d(f^{k+1-j}(x_1), f^{k+1-j}(y_1))}{2^{j-1}}.$$

Suppose $d(f^j(x_1), f^j(y_1)) \leq r/4$ for $j = 1, 2, \ldots k$. Then we have

$$\frac{r}{2} < \sum_{j=1}^{k} \frac{d(f^{k+1-j}(x_1), f^{k+1-j}(y_1))}{2^{j-1}} \leq \frac{r}{4} \frac{1-(1/2)^k}{1-(1/2)} \leq \frac{r}{2},$$

a contradiction. Then there exists m with $1 \leq m \leq k$ such that $d(f^m(x_1), f^m(y_1)) > r/4$. ☐

Theorem 1. *Suppose that $f : X \to X$ is continuous and X is a compact metric space with metric d. Let $Y := \varprojlim(X, f)$ with metric*

$$\tilde{d}(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}$$

and $\sigma : Y \to Y$ be the shift homeomorphism. If f^{-1} is Li-Yorke chaotic then σ is Li-Yorke chaotic.

Proof. Let $S \subset Y$ be a scrambled set for f^{-1}. Let $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{q} \in Y$ be periodic for σ. Since

$$d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) \leq \tilde{d}(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{y})), \text{ and } d(\pi_n(\mathbf{x}), \pi_n(\mathbf{q})) \leq \tilde{d}(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{q})),$$

It follows that

$$\begin{aligned} 0 &< \limsup d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) \leq \limsup \tilde{d}(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{y})), \\ 0 &< \limsup d(\pi_n(\mathbf{x}), \pi_n(\mathbf{q})) \leq \limsup \tilde{d}(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{q})). \end{aligned}$$

We now want to show that

$$\liminf_{n\to\infty} \tilde{d}(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{y})) = 0.$$

To do this, I will show that for any $\epsilon > 0$, there exists a subsequence $n_1, n_2, \ldots$ with $n_j \to \infty$ such that

$$\tilde{d}(\sigma^{n_j}(\mathbf{x}), \sigma^{n_j}(\mathbf{y})) < \epsilon.$$

Let $K := \max_{x,y \in X} d(x, y)$. Then

$$\tilde{d}(x, y) := \sum_{i=1}^{T} \frac{d(x_i, y_i)}{2^{i-1}} + \sum_{i=T+1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}} \leq \sum_{i=1}^{T} \frac{d(x_i, y_i)}{2^{i-1}} + K \sum_{i=T+1}^{\infty} \frac{1}{2^{i-1}} = \sum_{i=1}^{T-1} \frac{d(x_i, y_i)}{2^{i-1}} + \frac{K}{2^{T-1}}.$$

Pick $T \in \mathbb{N}$ sufficiently large so that $(K/2^{T-1}) < \epsilon/2$. Pick $\delta > 0$ such that

$$\sum_{i=1}^{T} \frac{\delta}{2^{i-1}} = \frac{1 - (1/2)^T}{1 - 1/2} \delta < \epsilon/2.$$

Since f is continuous and X is compact, f^n is uniformly continuous. For the given $\delta > 0$, for $n = 1, 2, \ldots, T$ there exists $\tau_n > 0$ such that $d(f^n(x), f^n(y)) < \delta$ if $d(x, y) < \tau_n$. Let $\tau := \min\{\delta, \tau_1, \tau_2, \ldots, \tau_T\}$. Since $\liminf d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) = 0$, there exists infinitely many m such that $d(\pi_m(\mathbf{x}), \pi_m(\mathbf{y})) < \tau$. Call these $m_1, m_2, \ldots$. These can be picked so that $m_1 > T + 1$ and $m_{i+1} - m_i > T - 1$. Let $n_i = m_i - (T + 1)$. Then

$$\begin{aligned}
\sigma^{n_i}(\mathbf{x}) &= (f^T(x_{m_i}), f^{T-1}(x_{m_i}), \ldots, f^2(x_{m_i}), f(x_{m_i}), x_{m_i}, x_{m_i+1}, \ldots), \\
\sigma^{n_i}(\mathbf{y}) &= (f^T(y_{m_i}), f^{T-1}(y_{m_i}), \ldots, f^2(y_{m_i}), f(y_{m_i}), y_{m_i}, y_{m_i+1}, \ldots).
\end{aligned}$$

Since $d(x_{m_i}, y_{m_i})) < \tau$, we have have $d(f^k(x_{m_i}), f^k(y_{m_i})) < \delta$ for $k = 1, 2, \ldots, T$. It then follows that

$$\tilde{d}(\sigma^{n_i}(\mathbf{x}), \sigma^{n_i}(\mathbf{y})) < \epsilon/2 + \epsilon/2 = \epsilon.$$

It follows that

$$\liminf_{n \to \infty} \tilde{d}(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{y})) < \epsilon.$$

Since this is true for any $\epsilon > 0$, we have

$$\liminf_{n \to \infty} \tilde{d}(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{y})) = 0.$$

□

Theorem 2. *Suppose that $f : X \to X$ is continuous and X is a compact metric space with metric d. Let $Y := \varprojlim(X, f)$ with metric*

$$\tilde{d}(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}},$$

and $\sigma : Y \to Y$ be the shift homeomorphism. If σ is Li-Yorke chaotic then f^{-1} is Li-Yorke chaotic. Moreover, if S is δ-scrambled for σ then S is $(\delta/2)$-scrambled for D where D is the set of trajectories generated by f^{-1}.

Proof. First note that the set $D \subset X^\infty$ generated by f^{-1} is Y. Since σ is Li-Yorke chaotic, there exists an uncountable scrambled set $S \subset Y \equiv D$ for σ. We need to show that S is δ-scrambled for D. Let $x, y \in S$, $x \neq y$ and q be periodic for σ.

$$\limsup_{n\to\infty} \tilde{d}(\sigma^n(x), \sigma^n(y)) \quad > \quad \delta, \tag{23}$$
$$\liminf_{n\to\infty} \tilde{d}(\sigma^n(x), \sigma^n(y)) \quad = \quad 0, \tag{24}$$
$$\limsup_{n\to\infty} \tilde{d}(\sigma^n(x), \sigma^n(p)) \quad > \quad \delta. \tag{25}$$

Note that by definition of $\tilde{d}$ and the fact that d is a metric, we have

$$\tilde{d}(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{y})) = \sum_{i=1}^{\infty} \frac{d(x_{n+i}, y_{n+i})}{2^{i-1}} \geq d(x_{n+1}, y_{n+1}).$$

This implies that

$$\liminf_{n\to\infty} d(\pi_{n+1}(x), \pi_{n+1}(y)) \leq \liminf_{n\to\infty} \tilde{d}(\sigma^n(x), \sigma^n(y)) = 0.$$

This implies

$$\liminf_{n\to\infty} d(\pi_n(x), \pi_n(y)) = 0.$$

Let

$$c := \limsup_{n\to\infty} \tilde{d}(\sigma^n(x), \sigma^n(y)) > 0.$$

This implies the existence of a subsequence with the property

$$c = \lim_{j\to\infty} \tilde{d}(\sigma^{n_j}(x), \sigma^{n_j}(y)) > 0.$$

For $0 < \epsilon < c$, there exists an N such that for all $j \geq N$ we have

$$0 < c - \epsilon < \tilde{d}(\sigma^{n_j}(x), \sigma^{n_j}(y)).$$

I want to construct a subsequence of $d(\pi_n(x), \pi_n(y))$ strictly bounded away from 0. Since X is compact, this subsequence will have a convergent sub-subsequence converging to something strictly positive. This will imply the $\limsup d(\pi_n(x), \pi_n(y)) > 0$. For $j = N$, let $t_1 = n_j$. Then we have $0 < c - \epsilon < \tilde{d}(\sigma^{t_1}(x), \sigma^{t_1}(y))$. By Lemma 1, there exists an $m_1 \geq t_1$ such that $d(\pi_{m_1}(x), \pi_{m_1}(y)) \geq (c-\epsilon)/2$. Next, pick an $n_j > m_1$, call it t_2. Then by the same reasoning, there exists an $m_2 \geq t_2$ with $d(\pi_{m_2}(x), \pi_{m_2}(y)) \geq (c-\epsilon)/2$. By repeating this process, we get a sequence m_i with $m_i < m_{i+1}$ and $m_i \to \infty$ with the property that $d(\pi_{m_i}(x), \pi_{m_i}(y)) \geq (c-\epsilon)/2$ for all i. Since X is compact, there exists a $K > 0$ (finite) with $d(\pi_{m_i}(x), \pi_{m_i}(y)) \in [(c-\epsilon)/2, K]$ for all $i \in \mathbb{N}$. Since $[(c-\epsilon)/2, K]$ is compact, this sequence $\{d(\pi_{m_i}(x), \pi_{m_i}(y))\}_{i=1}^{\infty}$ contains a convergent subsequence converging to $p^* \in [(c-\epsilon)/2, K]$. This implies that there exists a subsequence $\{d(\pi_{k_i}(x), \pi_{k_i}(y))\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} d(\pi_{k_i}(x), \pi_{k_i}(y)) = p^* \geq (c-\epsilon)/2$. This implies

$$\limsup d(\pi_n(x), \pi_n(y)) \geq p^* \geq (c-\epsilon)/2 > 0.$$

Since this is true for all $0 < \epsilon < c$, we have

$$\limsup d(\pi_n(x), \pi_n(y)) \geq p^* \geq c/2 > 0.$$

A similar argument shows

$$\limsup d(\pi_n(x), \pi_n(q)) \geq c/2 > 0.$$

Note that $c > \delta$ implies $c/2 > \delta/2$. This implies that if S is δ-scrambled for σ with $\delta > 0$, then S is $(\delta/2)$-scrambled for D. □

Theorem 3. *Suppose that $f : X \to X$ is continuous and X is a compact metric space with metric d. Let $Y := \varprojlim(X, f)$ with metric*

$$\tilde{d}(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}.$$

and $F : Y \to Y$ be the induced homeomorphism. The f is Li-Yorke chaotic if and only if F is Li-Yorke chaotic.

Proof. ($\Rightarrow$) Suppose that f is Li-Yorke chaotic. Let S be a scrambled set for f. For each $x \in S$, $\pi_1^{-1}(x) \neq \varnothing$. Construct $\tilde{S}$ as follows: for each $x \in S$, select one point in $\pi_1^{-1}(x)$. There are typically many such $\tilde{S}$ subsets of Y. They all have the following property: for all $y, y' \in \tilde{S}$ with $y \neq y'$ we have $\pi_1(y) \in S$, $\pi_1(y') \in S$ and $\pi_1(y) \neq \pi_1(y')$. Note that $\tilde{S}$ is uncountable since S is uncountable.

Let $\mathbf{x} = (x_1, x_2, \ldots), \mathbf{y} = (y_1, y_2, \ldots) \in \tilde{S}$ with $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{q} = (q_1, q_2, q_3, \ldots, q_M, q_1, q_2, \ldots)$ be periodic for F. Then $x_1 \neq y_1$ and $x_1, y_1 \in S$. The we have

$$\limsup_{n\to\infty} d(f^n(x_1), f^n(y_1)) > \delta$$

and

$$\limsup_{n\to\infty} d(f^n(x_1), f^n(q_1)) > \delta$$

Since

$$\tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y})) \geq d(f^n(x_1), f^n(y_1)) \text{ and } \tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{q})) \geq d(f^n(x_1), f^n(q_1))$$

it follows that

$$\limsup_{n\to\infty} \tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y})) > \delta \text{ and } \limsup_{n\to\infty} \tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{q})) > \delta.$$

We now want to show that

$$\liminf_{n\to\infty} \tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y})) = 0.$$

To do this, I will show that for any $\epsilon > 0$, there exists a subsequence $n_1, n_2, \ldots$ with $n_j \to \infty$ such that

$$\tilde{d}(F^{n_j}(\mathbf{x}), F^{n_j}(\mathbf{y})) < \epsilon.$$

Let $K := \max_{x,y\in X} d(x, y)$. Then

$$\tilde{d}(x, y) := \sum_{i=1}^{T} \frac{d(x_i, y_i)}{2^{i-1}} + \sum_{i=T+1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}} \leq \sum_{i=1}^{T} \frac{d(x_i, y_i)}{2^{i-1}} + K \sum_{i=T+1}^{\infty} \frac{1}{2^{i-1}} = \sum_{i=1}^{T-1} \frac{d(x_i, y_i)}{2^{i-1}} + \frac{K}{2^{T-1}}.$$

Pick T sufficiently large so that $(K/2^{T-1}) < \epsilon/2$. Pick $\delta > 0$ such that

$$\sum_{i=1}^{T} \frac{\delta}{2^{i-1}} = \frac{1-(1/2)^T}{1-1/2}\delta < \epsilon/2.$$

Since f is continuous and X is compact, f^n is uniformly continuous. For the given $\delta > 0$, for $n = 1, 2, \ldots, T$ there exists $\tau_n > 0$ such that $d(f^n(x), f^n(y)) < \delta$ if $d(x, y) < \tau_n$. Let $\tau := \min\{\delta, \tau_1, \tau_2, \ldots, \tau_T\}$. Since $\liminf d(f^n(x_1), f^n(y_1)) = 0$, there exists infinitely many m such that $d(f^m(x_1), f^m(y_1)) < \tau$. Call these $m_1, m_2, \ldots$. These can be picked so that $m_{i+1} - m_i > T$. Let $n_i = m_i + T$. Then

$$\begin{aligned} F^{n_i}(\mathbf{x}) &= (f^T \circ f^{m_i}(x_1), f^{T-1} \circ f^{m_i}(x_1), \ldots, f^2 \circ f^{m_i}(x_1), f \circ f^{m_i}(x_1), f^{m_i}(x_1), \ldots), \\ F^{n_i}(\mathbf{y}) &= (f^T \circ f^{m_i}(y_1), f^{T-1} \circ f^{m_i}(y_1), \ldots, f^2 \circ f^{m_i}(y_1), f \circ f^{m_i}(y_1), f^{m_i}(y_1), \ldots). \end{aligned}$$

Since $d(f^{m_i}(x_1), f^{m_i}(y_1)) < \tau$, we have have $d(f^k \circ f^{m_i}(x_1), f^k \circ f^{m_i}(y_1)) < \delta$ for $k = 1, 2, \ldots, T$. It then follows that

$$\tilde{d}(F^{n_i}(\mathbf{x}), F^{n_i}(\mathbf{y})) < \epsilon/2 + \epsilon/2 = \epsilon.$$

It follows that

$$\liminf_{n\to\infty} \tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y})) < \epsilon.$$

Since this is true for any $\epsilon > 0$, we have

$$\liminf_{n\to\infty} \tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y})) = 0.$$

($\Leftarrow$) Suppose that F is Li-Yorke chaotic. Let $\tilde{S}$ be a scrambled set for F. Let $S := \pi_1(\tilde{S})$. Then S must be uncountable. If it is not, then there exists a countable set $V \subset Y$ such that for each $\mathbf{x} \in \tilde{S}$, there exists a $\mathbf{y} \in V$ such that $\tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y})) \to 0$. This implies that there must exist $\mathbf{x}, \mathbf{x}' \in \tilde{S}$ such that $\tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{x}')) \to 0$ – a contradiction that $\tilde{S}$ is a scrambled set.

Let $x_1, y_1 \in S$, $\mathbf{x} = \pi_1^1(x_1) \cap \tilde{S}$ and $\mathbf{y} = \pi_1^1(y_1) \cap \tilde{S}$. Let q_1 be periodic with orbit $(q_1, q_2, \ldots, q_M, q_1, q_2, \ldots)$. Note that by definition of $\tilde{d}$ and the fact that d is a metric, we have

$$\tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y})) \geq d(f^n(x_1), f^n(y_1)).$$

This implies that

$$\liminf_{n\to\infty} d(f^n(x_1), f^n(y_1) \leq \liminf_{n\to\infty} \tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y}) = 0.$$

This implies

$$\liminf_{n\to\infty} d(f^n(x_1), f^n(y_1)) = 0.$$

Let

$$c := \limsup_{n\to\infty} \tilde{d}(F^n(\mathbf{x}), F^n(\mathbf{y})) > 0.$$

This implies the existence of a subsequence with the property

$$c = \lim_{j\to\infty} \tilde{d}(F^{n_j}(x), F^{n_j}(y)) > 0,$$

for $j = 1, 2, \ldots$. Let $n_0 = 1$. For $0 < \epsilon < c$, there exists an N such that for all $j \geq N$ we have

$$0 < r := c - \epsilon < \tilde{d}(F^{n_j}(\mathbf{x}), F^{n_j}(\mathbf{y})).$$

Let $k > 1$ such that $K/2^{k-1} < r/2$. Without loss of generality assume that $n_1 > k$ and $n_{j+1} - n_j > k$. Let $k_1 = n_1$ and $k_{i+1} := n_{i+1} - n_i$ for $i \in \mathbb{N}$. Note for each k_i, we have $K/2^{k_i - 1} < r/2$. Since

$$\tilde{d}(F^{n_j}(\mathbf{x}), F^{n_j}(\mathbf{y})) > r,$$

By Lemma 2, there exists m_j with $n_{j-1} \le m_j \le n_j$ with

$$d(f^{m_j}(x_1), f^{m_j}(y_1) > r/4.$$

This implies that

$$\limsup d(f^n(x_1), f^n(y_1) \ge r/4 > 0.$$

A similar argument shows

$$\limsup d(f^n(x_1), f^n(q_1)) \ge r/4 > 0.$$

Hence S is a scrambled set for f and f is Li-Yorke chaotic. □

The main result of Li and Yorke (1975) is that for a continuous map on a compact interval of the reals, a period-three implies chaos in the sense of Li-Yorke. To make this connection, I will discuss how periodic orbits are related to topological entropy (another measure of complicated dynamics) and then how topological entropy is related to Li-Yorke chaos.

Topological entropy involves the concept of an (n, ϵ)-separated set.

Definition 6. Let $f : X \to X$ be a continuous function on a compact metric space X. For $n \in \mathbb{N}$ and $\epsilon > 0$, a set $E \subset X$ is (n, ϵ)-*separated* under f provided for distinct $x, y \in E$, $\exists 0 \le k < n$ such that $d(f^k(x), f^k(y)) > \epsilon$.

Let $f : X \to X$ be a continuous map on a compact metric space X with metric d. Let $A, E \subset X$. We that E is (d, ϵ, A)-*spanning* if E is finite and for every $y \in A$, there exists an $x \in E$ such that $d(x, y) < \epsilon$. Given f, for $n \in \mathbb{Z}^+$ we define a new metric d_n^f on X given by $d_n^f(x, y) := \max_{0 \le i \le n-1} d(f^i(x), f^i(y))$. For $n \in \mathbb{Z}^+$ and $\epsilon > 0$, let $S(d_n^f, \epsilon, A)$ denote the minimum cardinality of all (d_n^f, ϵ, A)-spanning sets. Heuristically, the (d_n^f, ϵ, A)-spanning set E denote the number of initial conditions in A that an observer of the dynamical system f can distinguish given orbits of length n and an ability to measure the system with accuracy no greater than ϵ.

Definition 7. For $A \subset X$, we define

$$h(f, A, \epsilon) := \limsup_{n \to \infty} (1/n)[\log S(d_n^f, \epsilon, A)].$$

The *topological entropy of f on A* is defined by

$$h(f, A) := \lim_{\epsilon \to 0} h(f, A, \epsilon).$$

The *topological entropy* of f is defined by $h(f) := h(f, X)$.

Note that since f is a single-valued map, we can associate each finite orbit of length n from our dynamical system $(x_1, x_2, \dots, x_n)$ with its initial condition x_1. Intuitively our set E is (d_n^f, ϵ, X)-spanning if for every orbit $\mathbf{y}$ under f of length n there is some orbit $\mathbf{x}$ under f of length n starting in E such that each point in time x_i and y_i are no more than ϵ apart. If one cannot distinguish orbits that are never more than ϵ apart then $S(d_n^f, \epsilon, X)$ tells us the number of distinct orbits there are of length n. An alternative (and equivalent) formulation, motivated by the inverse limits approach to dynamical systems, is the following. Let

$$\varinjlim\{X, f, n\} = \{(x_1, x_2, \dots, x_n) \in X^n | x_i = f(x_{i-1}), i = 2, \dots, n\}.$$

Note that this is just the truncation after the first n coordinates of the direct limit space

$$\varinjlim\{X, f\} := \{(x_1, x_2, \dots) \in X^\infty | x_{i+1} = f(x_i), i \in \mathbb{N}\}.$$

We can define a metric on D_n on X^n given by $D_n(\mathbf{x}, \mathbf{y}) := \max_{1 \le i \le n} d(x_i, y_i)$. Then we call a subset $C \subset \varinjlim\{X, f, n\}$ a $(D_n, \epsilon, \varinjlim\{X, f, n\})$-spanning set if C is finite and for each $\mathbf{y} \in \varinjlim\{X, f, n\}$, there exists an $\mathbf{x} \in C$ such that $D_n(\mathbf{x}, \mathbf{y}) < \epsilon$. For $\mathbf{x} \in \varinjlim\{X, f, n\}$ and $\delta > 0$, let $B(\mathbf{x}, \delta) := \{\mathbf{y} \in \varinjlim\{X, f, n\} | D_n(\mathbf{x}, \mathbf{y}) < \delta\}$. This is just an open ball in $\varinjlim\{X, f, n\}$ with center $\mathbf{x}$ and radius δ. Then we see that if $C = \{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^M\}$ is a $(D_n, \epsilon, \varinjlim\{X, f, n\})$-spanning set then $\varinjlim\{X, f, n\} = \cup_{i=1}^M B(\mathbf{x}_i, \epsilon)$, i.e, ϵ neighborhoods of the points in C form an open cover of $\varinjlim\{X, f, n\}$. If $C \subset \varinjlim\{X, f, n\}$ is a $(D_n, \epsilon, \varinjlim\{X, f, n\})$-spanning set then the projection of the first coordinate $E := \pi_1(C) \subset X$ is (d_n^f, ϵ, X)-spanning. Conversely, if $E \subset X$ is a (d_n^f, ϵ, X)-spanning set then there exists a (unique) finite subset $C \subset \varinjlim\{X, f, n\}$ such that C is $(D_n, \epsilon, \varinjlim\{X, f, n\})$-spanning and $E = \pi_1(C)$. So the notion of spanning can be applied to either a subset $E \subset X$ ("initial conditions") or a subset of orbits $C \subset \varinjlim\{X, f, n\}$. The difference here is entirely superficial since f is single-valued there is a one-to-one mapping between orbits and initial conditions. Given the equivalence of the cardinality of these sets, we can use the direct limit space approach to give an equivalent definition of topological entropy for f:

$$h(f) := \lim_{\epsilon \to 0} \left[\limsup_{n \to \infty} \frac{1}{n} \log S(D_n, \epsilon, \varinjlim\{X, f, n\}) \right].$$

When f^{-1} is multi-valued, we can no longer associate a unique orbit to each initial condition. However, we can still talk about the space of orbits under the action of f^{-1} with length n and what it would mean for a finite subset of this space to form a spanning set. Let

$$\varprojlim(X, f, n) := \{\mathbf{x} \in X^n | x_{i-1} = f(x_i), i =, 2, \dots, n\}.$$

This set is just the truncation after the first n coordinates of the inverse limit space $\varprojlim(X, f)$ onto X^n. Let D_n be a metric on $\varprojlim(X, f, n)$ as defined above (this makes sense since both $\varprojlim(X, f, n)$ and $\varinjlim\{X, f, n\}$ are subsets of X^n). We say that a finite set $\bar{C} \subset \varprojlim(X, f, n)$ is $(D_n, \epsilon, \varprojlim(X, f, n))$-spanning if for every $\bar{y} \in \varprojlim(X, f, n)$, there exists some $\bar{x} \in \bar{C}$ such that $D_n(\mathbf{y}, \mathbf{x}) < \epsilon$. Having defined an $(D_n, \epsilon, \varprojlim(X, f, n))$-spanning sets for f^{-1}, $h(f^{-1})$ can be defined as above:

$$h(f^{-1}) := \lim_{\epsilon \to 0} \left[\limsup_{n \to \infty} \frac{1}{n} \log S(D_n, \epsilon, \varprojlim(X, f, n)) \right].$$

The next theorem generalizes the well-known result for entropy when f is a homeomorphism.

Theorem 4 (Kennedy et al. (2008))**.** *Let $f : X \to X$ be a continuous onto map on a compact metric space X (not necessarily a homeomorphism). Then $h(f) = h(f^{-1})$.*

The next theorem links the topological entropy of f and that of the induced homeomorphism F.

Theorem 5 (Bowen (1970), Proposition 5.2, p. 35)**.** *Let $f : X \to X$ be a continuous onto map on a compact metric space X, $Y := \varprojlim(X, f)$ and $F : Y \to Y$ the induced homeomorphism. Then $h(f) = h(F)$.*

Note that since $\sigma = F^{-1}$, we have $h(F) = h(\sigma)$ and $h(f) = h(\sigma)$. The following theorem show that positive topological entropy implies Li-Yorke chaos.

Theorem 6 (Blanchard et al. (2002))**.** *Let $f : X \to X$ be continuous, X a compact metric space. If $h(f) > 0$, then f is Li-Yorke chaotic.*

The converse of this theorem is not true. Smítal (1986)) constructs Li-Yorke chaotic functions with zero topological entropy. We now have all of the necessary pieces for the following:

Theorem 7. *Let $f : X \to X$ be continuous and X a compact metric space. Then if $h(f) > 0$, then f^{-1} is Li-Yorke chaotic.*

Proof. By Theorem 5, $h(f) > 0$ implies $h(\sigma) > 0$. By Theorem 6 we have σ is Li-Yorke chaotic. By Theorem 2, if σ is Li-Yorke chaotic, then f^{-1} is Li-Yorke chaotic. □

Theorem 8 (Misiurewicz (1979))**.** *Suppose $X = [a, b] \subset \mathbb{R}$ with $a < b$ and the usual Euclidean metric and $f : X \to X$ is continuous. Then f has a cycle of order not equal to 2^k for $k = 0, 1, 2, \ldots,$ if and only if $h(f) > 0$.*

This gives the following.

Theorem 9. *Let $f : I \to I$ be continuous, I a compact interval in $\mathbb{R}$. If f has a periodic point of order not equal to 2^k for $k = 0, 1, 2, \ldots$, then f^{-1} is Li-Yorke chaotic.*

Proof. Since f has a periodic point of order not equal to 2^k for $k = 0, 1, 2, \ldots$, then it follows from Theorem 8 that $h(f) > 0$. This implies $h(\sigma) > 0$ and hence σ is Li-Yorke chaotic by Theorem 6. It follows from Theorem 2 that f^{-1} is Li-Yorke chaotic. □

Corollary 1. *If f has a period 3, then f^{-1} is Li-Yorke chaotic.*

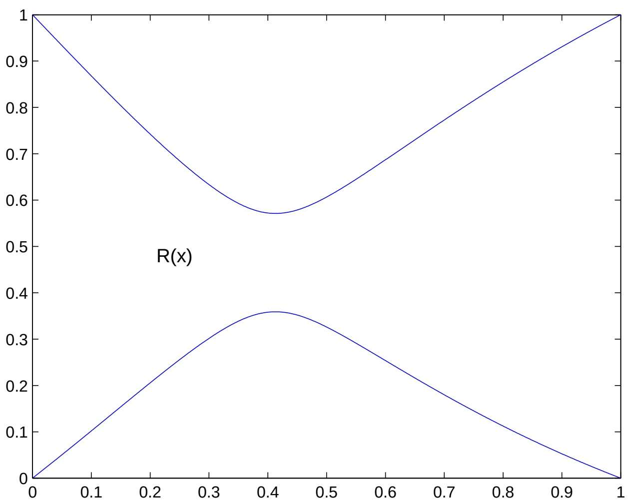
4 Conclusion

In some economic models, the dynamical system characterizing equilibria in the model has multi-valued forward dynamics but single-valued backward dynamics. One says that such a dynamical system has backward dynamics. In this paper, I have offered a definition of chaos for such a dynamical system that is in the spirit of Li and Yorke (1975). Furthermore, by utilizing the inverse limit space, I have been able to provide sufficient condition for the set of equilibria to be Li-Yorke chaotic. An open question is the following:

- Suppose f is Li-Yorke chaotic with zero topological entropy. Does f being Li-Yorke chaotic imply that f^{-1} is Li-Yorke chaotic?

In future research, I would like to investigate more general MVDSs represented by iterated relations (set-valued functions). Such systems are a natural generalization and extension of those represented by iterated maps. Akin (1993) and McGehee (1992) make this argument and contain some initial results for such dynamical systems. Even on an interval, continuous (closed) relations can behave dynamically quite differently from continuous maps. As noted earlier, the theorems of Sharkovskiĭ (1995) and Li and Yorke (1975) do not extend to continuous relations on an interval. My interest in such systems is motivated by the work of Christiano and Harrison (1999). Their model is the standard real business cycle model with a production externality. Equilibria in this model correspond to forward orbits of an MVDS (see Figure 7) that is multi-valued going forward *and* backward in time.

Figure 7: MVDS from Christiano and Harrison (1999), $R : (0, 1) \to (0, 1)$.



References

Akin, E., 1993. The general topology of dynamical systems. Vol. 1 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI.

Azariadis, C., 1993. Intertemporal Macroeconomics. Blackwell, Cambridge, MA.

Benhabib, J., Day, R., 1982. A characterization of erratic dynamics in the overlapping generations model. Journal of Economic Dynamics and Control 4, 37–55.

Blanchard, F., Glasner, E., Kolyada, S., Maass, A., 2002. On Li-Yorke pairs. J. Reine Angew. Math. 547, 51–68.

Bowen, R., 1970. Topological entropy and axiom A. In: Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968). Amer. Math. Soc., Providence, R.I., pp. 23–41.

Christiano, L. J., Harrison, S. G., August 1999. Chaos, sunspots and automatic stabilizers. Journal of Monetary Economics 44 (1), 3–31.

Gale, D., 1973. Pure exchange equilibrium of dynamic economic models. Journal of Economic Theory 6, 12–36.

Grandmont, J.-M., 1985. On endogenous competitive business cycles. Econometrica 53, 995–1045.

Gu, C., Wright, R., October 2011. Endogenous credit cycles. Working Paper 17510, National Bureau of Economic Research.

Ingram, W. T., Mahavier, W. S., 2004. Interesting dynamics and inverse limits in a family of one-dimensional maps. Amer. Math. Monthly 111 (3), 198–215.

Kennedy, J. A., Stockman, D. R., Yorke, J. A., 2007. Inverse limits and an implicitly defined difference equation from economics. Topology and Its Applications 154, 2533–2552.

Kennedy, J. A., Stockman, D. R., Yorke, J. A., 2008. The inverse limits approach to models with chaos. Journal of Mathematical Economics 44, 423–444.

Li, T.-Y., Yorke, J. A., 1975. Period three implies chaos. American Mathematical Monthly 82, 985–992.

Lucas, R. E., Stokey, N. L., 1987. Money and interest in a cash-in-advance economy. Econometrica 55, 491–513.

McGehee, R., 1992. Attractors for closed relations on compact hausdorff spaces. Indiana University Mathematics Journal 41 (4), 1165–1209.

Medio, A., 1992. Chaotic dynamics: Theory and applications to economics. Cambridge University Press, Cambridge.

Medio, A., Raines, B. E., 2006. Inverse limit spaces arising from problems in economics. Topology and Its Applications 153, 3439–3449.

Medio, A., Raines, B. E., 2007. Backward dynamics in economics. the inverse limit approach. Journal of Economic Dynamics & Control, 31, 1633–1671.

Michener, R., Ravikumar, B., 1998. Chaotic dynamics in a cash-in-advance economy. Journal of Economic Dynamics and Control 22, 1117–1137.

Misiurewicz, M., 1979. Horseshoes for mappings of the interval. Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (2), 167–169.

Robinson, C., 1995. Dynamical Systems – Stability, Sybolic Dynamics and Chaos. CRS Press, Boca Raton, FL.

Sharkovskiĭ, A. N., 1995. Coexistence of cycles of a continuous map of the line into itself. In: Proceedings of the Conference "Thirty Years after Sharkovskiĭ's Theorem: New Perspectives" (Murcia, 1994). Vol. 5. pp. 1263–1273, translated from the Russian [Ukrain. Mat. Zh. **16** (1964), no. 1, 61–71] by J. Tolosa.

Smítal, J., 1986. Chaotic functions with zero topological entropy. Trans. Amer. Math. Soc. 297 (1), 269–282.