Derivatives Markets

THIRD EDITION

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Suppose you have the opportunity to spend $1 one year from today to receive $2 two years from today. What is the value of this opportunity? To answer this question, you need to know the appropriate interest rates for discounting the two cash flows. This comparison is an example of the most basic concept in finance: using interest rates to compute present values. Once we find a present value for one or more assets, we can compare the values of cash flows from those assets even if the cash inflows and cash outflows occur at different times. In order to perform these calculations, we need information about the set of interest rates prevailing between different points in time.

We begin the chapter by reviewing basic bond concepts—coupon bonds, yields to maturity, and implied forward rates. Any reader of this book should understand these basic concepts. We then look at interest rate forwards and forward rate agreements, which permit hedging interest rate risk. Finally, we look at bond futures and the repo market.

7.1 BOND BASICS

Table 7.1 presents information about current interest rates for bonds maturing in from 1 to 3 years. *Identical information is presented in five different ways in the table.* Although the information appears differently across columns, it is possible to take the information in any one column of Table 7.1 and reproduce the other four columns.[1]

In practice, a wide range of maturities exists at any point in time, but the U.S. government issues Treasury securities only at specific maturities—typically 3 months, 6 months, and 1, 2, 5, 10, and 30 years.[2] Government securities that are issued with less than 1 year to maturity and that make only a single payment, at maturity, are called Treasury *bills*. *Notes* and *bonds* pay coupons and are issued at a price close to their maturity value (i.e., they are issued at par). Notes have 10 or fewer years to maturity and bonds have more than 10

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1. Depending upon how you do the computation, you may arrive at numbers slightly different from those in Table 7.1. The reason is that all of the entries except those in column 1 are rounded in the last digit, and there are multiple ways to compute the number in any given column. Rounding error will therefore generate small differences among computations performed in different ways.

2. Treasury securities are issued using an auction. In the past the government also issued bonds with maturities of 3 and 7 years. Between 2002 and 2005 the government issued no 30-year bonds.

TABLE 7.1	Five ways to present equivalent information about default-free interest rates. All rates but those in the last column are effective annual rates.				
Years to Maturity	(1) Zero-Coupon Bond Yield	(2) Zero-Coupon Bond Price	(3) One-Year Implied Forward Rate	(4) Par Coupon	(5) Continuously Compounded Zero Yield
1	6.00%	0.943396	6.00000%	6.00000%	5.82689%
2	6.50	0.881659	7.00236	6.48423	6.29748
3	7.00	0.816298	8.00705	6.95485	6.76586

years to maturity. The distinctions between bills, notes, and bonds are not important for our purposes; we will refer to all three as bonds. Treasury inflation protected securities are bonds for which payments are adjusted for inflation. Finally, the most recently issued government bonds are called **on-the-run**; other bonds are called **off-the-run**. These terms are used frequently in talking about government bonds since on-the-run bonds generally have lower yields and greater trading volume than off-the-run bonds. Appendix 7.A discusses some of the conventions used in bond price and yield quotations.

In addition to government bonds there are also STRIPS. A **STRIPS**—Separate Trading of Registered Interest and Principal of Securities—is a claim to a single interest payment or the principal portion of a government bond. These claims trade separately from the bond. STRIPS are zero-coupon bonds since they make only a single payment at maturity. "STRIPS" should not be confused with the forward strip, which is the set of forward prices available at a point in time.

We need a way to represent bond prices and interest rates. Interest rate notation is, unfortunately and inevitably, cumbersome, because for any rate we must keep track of three dates: the date on which the rate is quoted, and the period of time (this has beginning and ending dates) over which the rate prevails. We will let $r_t(t_1, t_2)$ represent the interest rate from time t_1 to time t_2, prevailing on date t. If the interest rate is current—i.e., if $t = t_1$—and if there is no risk of confusion, we will drop the subscript.

Zero-Coupon Bonds

We begin by showing that the zero-coupon bond yield and zero-coupon bond price, columns (1) and (2) in Table 7.1, provide the same information. A **zero-coupon bond** is a bond that makes only a single payment at its maturity date. Our notation for zero-coupon bond prices will mimic that for interest rates. The price of a bond quoted at time t_0, with the bond to be purchased at t_1 and maturing at t_2, is $P_{t_0}(t_1, t_2)$. As with interest rates, we will drop the subscript when $t_0 = t_1$.

The 1-year zero-coupon bond price of $P(0, 1) = 0.943396$ means that you would pay $0.943396 today to receive $1 in 1 year. You could also pay $P(0, 2) = 0.881659$ today to receive $1 in 2 years and $P(0, 3) = 0.816298$ to receive $1 in 3 years.

The **yield to maturity** (or *internal rate of return*) on a zero-coupon bond is simply the percentage increase in dollars earned from the bond. For the 1-year bond, we end up

with $1/0.943396 - 1 = 0.06$ more dollars per \$1 invested. If we are quoting interest rates as effective annual rates, this is a 6% yield.

For the zero-coupon 2-year bond, we end up with $1/0.881659 - 1 = 0.134225$ more dollars per \$1 invested. We could call this a 2-year effective interest rate of 13.4225%, but it is conventional to quote rates on an annual basis. If we want this yield to be comparable to the 6% yield on the 1-year bond, we could assume annual compounding and get $(1 + r(0, 2))^2 = 1.134225$, which implies that $r(0, 2) = 0.065$. In general,

$$P(0, n) = \frac{1}{[1 + r(0, n)]^n} \tag{7.1}$$

Note from equation (7.1) that *a zero-coupon bond price is a discount factor*: A zero-coupon bond price is what you would pay today to receive \$1 in the future. If you have a future cash flow at time t, C_t, you can multiply it by the price of a zero-coupon bond, $P(0, t)$, to obtain the present value of the cash flow. Because of equation (7.1), multiplying by $P(0, t)$ is the same as discounting at the rate $r(0, t)$, i.e.,

$$C_t \times P(0, t) = \frac{C_t}{[1 + r(0, t)]^t}$$

The inverse of the zero-coupon bond price, $1/P(0, t)$, provides a future value factor.

In contrast to zero-coupon bond prices, interest rates are subject to quoting conventions that can make their interpretation difficult (if you doubt this, see Appendix 7.A). Because of their simple interpretation, we can consider zero-coupon bond prices as the building block for all of fixed income.

A graph of annualized zero-coupon yields to maturity against time to maturity is called the zero-coupon **yield curve**. A yield curve shows us how yields to maturity vary with time to maturity. In practice, it is common to present the yield curve based on coupon bonds, not zero-coupon bonds.

Implied Forward Rates

We now see how column (3) in Table 7.1 can be computed from either column (1) or (2). The 1-year and 2-year zero-coupon yields are the rates you can earn from year 0 to year 1 and from year 0 to year 2. There is also an *implicit* rate that can be earned from year 1 to year 2 that must be consistent with the other two rates. This rate is called the **implied forward rate**.

Suppose we could today guarantee a rate we could earn from year 1 to year 2. We know that \$1 invested for 1 year earns $[1 + r_0(0, 1)]$ and \$1 invested for 2 years earns $[1 + r_0(0, 2)]^2$. Thus, the time 0 forward rate from year 1 to year 2, $r_0(1, 2)$, should satisfy

$$[1 + r_0(0, 1)][1 + r_0(1, 2)] = [1 + r_0(0, 2)]^2$$

or

$$1 + r_0(1, 2) = \frac{[1 + r_0(0, 2)]^2}{1 + r_0(0, 1)} \tag{7.2}$$

Figure 7.1 shows graphically how the implied forward rate is related to 1- and 2-year yields. If $r_0(1, 2)$ did not satisfy equation (7.2), then there would be an arbitrage opportunity.

FIGURE 7.1

An investor investing for 2 years has a choice of buying a 2-year zero-coupon bond paying $[1 + r_0(0, 2)]^2$ or buying a 1-year bond paying $1 + r_0(0, 1)$ for 1 year, and reinvesting the proceeds at the implied forward rate, $r_0(1, 2)$, between years 1 and 2. The implied forward rate makes the investor indifferent between these alternatives.
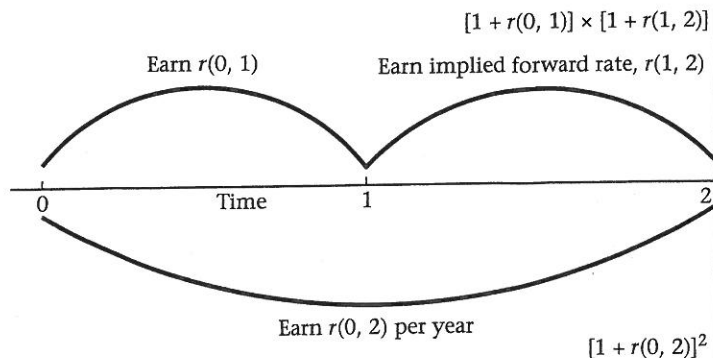
Problem 7.15 asks you to work through the arbitrage. In general, we have

$$\left[1 + r_0(t_1, t_2)\right]^{t_2 - t_1} = \frac{[1 + r_0(0, t_2)]^{t_2}}{[1 + r_0(0, t_1)]^{t_1}} = \frac{P(0, t_1)}{P(0, t_2)} \qquad (7.3)$$

Corresponding to 1-year and 2-year interest rates, $r_0(0, 1)$ and $r_0(0, 2)$, we have prices of 1-year and 2-year zero-coupon bonds, $P_0(0, 1)$ and $P_0(0, 2)$. Just as the interest rates imply a forward 1-year interest rate, the bond prices imply a 1-year forward zero-coupon bond price. The implied forward zero-coupon bond price must be consistent with the implied forward interest rate. Rewriting equation (7.3), we have

$$P_0(t_1, t_2) = \frac{1}{[1 + r_0(t_1, t_2)]^{t_2 - t_1}} = \frac{[1 + r_0(0, t_1)]^{t_1}}{[1 + r_0(0, t_2)]^{t_2}} = \frac{P(0, t_2)}{P(0, t_1)} \qquad (7.4)$$

The implied forward zero-coupon bond price from t_1 to t_2 is simply the ratio of the zero-coupon bond prices maturing at t_2 and t_1.

Example 7.1   Using information in Table 7.1, we want to compute the implied forward interest rate from year 2 to year 3 and the implied forward price for a 1-year zero-coupon bond purchased in year 2.
    The implied forward interest rate, $r_0(2, 3)$, can be computed as

$$1 + r_0(2, 3) = \frac{[1 + r_0(0, 3)]^3}{[1 + r_0(0, 2)]^2} = \frac{(1 + 0.07)^3}{(1 + 0.065)^2} = 1.0800705$$

or equivalently as

$$1 + r_0(2, 3) = \frac{P_0(0, 2)}{P_0(0, 3)} = \frac{0.881659}{0.816298} = 1.0800705$$

The implied forward 1-year zero-coupon bond price is

$$\frac{P_0(0, 3)}{P_0(0, 2)} = \frac{1}{1 + r_0(2, 3)} = 0.925865 \quad \blacksquare$$

Coupon Bonds

Given the prices of zero-coupon bonds—column (1) in Table 7.1—we can price coupon bonds. We can also compute the **par coupon**—column (4) in Table 7.1—the coupon rate at which a bond will be priced at par. To describe a coupon bond, we need to know the date at which the bond is being priced, the start and end date of the bond payments, the number and amount of the payments, and the amount of principal. Some practical complexities associated with coupon bonds, not essential for our purposes, are discussed in Appendix 7.A.

We will let $B_t(t_1, t_2, c, n)$ denote the time t price of a bond that is issued at t_1, matures at t_2, pays a coupon of c per dollar of maturity payment, and makes n evenly spaced payments over the life of the bond, beginning at time $t_1 + (t_2 - t_1)/n$. We will assume the maturity payment is \$1. If the maturity payment is different than \$1, we can just multiply all payments by that amount.

Since the price of a bond is the present value of its payments, at issuance time t the price of a bond maturing at T must satisfy

$$B_t(t, T, c, n) = \sum_{i=1}^{n} c P_t(t, t_i) + P_t(t, T) \tag{7.5}$$

where $t_i = t + i(T - t)/n$, with i being the index in the summation. Using equation (7.5), we can solve for the coupon as

$$c = \frac{B_t(t, T, c, n) - P_t(t, T)}{\sum_{i=1}^{n} P_t(t, t_i)}$$

A par bond has $B_t = 1$, so the coupon on a par bond is given by

$$\boxed{c = \frac{1 - P_t(t, T)}{\sum_{i=1}^{n} P_t(t, t_i)}} \tag{7.6}$$

Example 7.2 Using the information in Table 7.1, the coupon on a 3-year coupon bond that sells at par is

$$c = \frac{1 - 0.816298}{0.943396 + 0.881659 + 0.816298}$$
$$= 6.95485\% \quad \blacksquare$$

Equation (7.5) computes the bond price by discounting each bond payment at the rate appropriate for a cash flow with that particular maturity. For example, in equation (7.5), the coupon occuring at time t_i is discounted using the zero-coupon bond price $P_t(t, t_i)$; an alternative way to write the bond price is using the yield to maturity to discount all payments.

Suppose the bond makes m payments per year. Denoting the per-period yield to maturity as y_m, we have

$$B_t(t, T, c, n) = \sum_{i=1}^{n} \frac{c}{(1+y_m)^i} + \frac{1}{(1+y_m)^n} \qquad (7.7)$$

It is common to compute the quoted annualized yield to maturity, y, as $y = m \times y_m$. Government bonds, for example, make two coupon payments per year, so the annualized yield to maturity is twice the semiannual yield to maturity.

The difference between equation (7.5) and equation (7.7) is that in equation (7.5), each coupon payment is discounted at the appropriate rate for a cash flow occurring at that time. In equation (7.7), one rate is used to discount all cash flows. By definition, the two expressions give the same price. However, equation (7.7) can be misleading, since the yield to maturity, y_m, is not the return an investor earns by buying and holding a bond. Moreover, equation (7.7) provides no insight into how the cash flows from a bond can be replicated with zero-coupon bonds.

Zeros from Coupons

We have started with zero-coupon bond prices and deduced the prices of coupon bonds. In practice, the situation is often the reverse: We observe prices of coupon bonds and must infer prices of zero-coupon bonds. This procedure in which zero coupon bond prices are deduced from a set of coupon bond prices is called **bootstrapping**.

Suppose we observe the par coupons in Table 7.1. We can then infer the first zero-coupon bond price from the first coupon bond as follows:

$$1 = (1 + 0.06)P(0, 1)$$

This implies that $P(0, 1) = 1/1.06 = 0.943396$. Using the second par coupon bond with a coupon rate of 6.48423% gives us

$$1 = 0.0648423 P(0, 1) + 1.0648423 P(0, 2)$$

Since we know $P(0, 1) = 0.943396$, we can solve for $P(0, 2)$:

$$P(0, 2) = \frac{1 - 0.0648423 \times 0.943396}{1.0648423}$$
$$= 0.881659$$

Finally, knowing $P(0, 1)$ and $P(0, 2)$, we can solve for $P(0, 3)$ using the 3-year par coupon bond with a coupon of 6.95485%:

$$1 = (0.0695485 \times P(0, 1)) + (0.0695485 \times P(0, 2)) + (1.0695485 \times P(0, 3))$$

which gives us

$$P(0, 3) = \frac{1 - (0.0695485 \times 0.943396) - (0.0695485 \times 0.881659)}{1.0695485}$$
$$= 0.816298$$

There is nothing about the procedure that requires the bonds to trade at par. In fact, we do not even need the bonds to all have different maturities. For example, if we had a 1-year

bond and two different 3-year bonds, we could still solve for the three zero-coupon bond prices by solving simultaneous equations.

Interpreting the Coupon Rate

A coupon rate—for example the 6.95485% coupon on the 3-year bond—determines the cash flows the bondholder receives. However, except in special cases, it does not correspond to the rate of return that an investor actually earns by holding the bond.

Suppose for a moment that interest rates are certain; i.e., the implied forward rates in Table 7.1 are the rates that will actually occur in years 1 and 2. Imagine that we buy the 3-year bond and hold it to maturity, reinvesting all coupons as they are paid. What rate of return do we earn? Before going through the calculations, let's stop and discuss the intuition. We are going to invest an amount at time 0 and to reinvest all coupons by buying more bonds, and we will not withdraw any cash until time 3. *In effect, we are constructing a 3-year zero-coupon bond.* Thus, we should earn the same return as on a 3-year zero: 7%. This buy-and-hold return is different than the yield to maturity of 6.95485%. The coupon payment is set to make a par bond fairly priced, but it is not actually the return we earn on the bond except in the special case when the interest rate is constant over time.

Consider first what would happen if interest rates were certain, we bought the 3-year bond with a $100 principal and a coupon of 6.95485%, and we held it for 1 year. The price at the end of the year would be

$$B_1 = \frac{6.95485}{1.0700237} + \frac{106.95485}{(1.0700237)(1 + 0.0800705)}$$
$$= 99.04515$$

The 1-period return is thus

$$1\text{-period return} = \frac{6.95485 + 99.04515}{100} - 1$$
$$= 0.06$$

We earn 6%, since that is the 1-year interest rate. Problem 7.13 asks you to compute your 2-year return on this investment.

By year 3, we have received three coupons, two of which have been reinvested at the implied forward rate. The total value of reinvested bond holdings at year 3 is

$$6.95485 \times [(1.0700237)(1.0800705) + (1.0800705) + 1] + 100 = 122.5043$$

The 3-year yield on the bond is thus

$$\left(\frac{122.5043}{100}\right)^{1/3} - 1 = 0.07$$

As we expected, this is equal to the 7% yield on the 3-year zero and different from the coupon rate.

This discussion assumed that interest rates are certain. Suppose that we buy and hold the bond, reinvesting the coupons, and that interest rates are not certain. Can we still expect to earn a return of 7%? The answer is yes if we use interest rate forward contracts to guarantee the rate at which we can reinvest coupon proceeds. Otherwise, the answer in general is no.

The belief that the implied forward interest rate equals the expected future spot interest rate is a version of the **expectations hypothesis**. We saw in Chapters 5 and 6 that forward prices are biased predictors of future spot prices when the underlying asset has a risk premium; the same is true for forward interest rates. When you own a coupon bond, the rate at which you will be able to reinvest coupons is uncertain. If interest rates carry a risk premium, then the expected return to holding the bond will not equal the 7% return calculated by assuming interest rates are certain. The expectations hypothesis will generally not hold, and you should not expect implied forward interest rates to be unbiased predictors of future interest rates.

In practice, you can guarantee the 7% return by using forward rate agreements to lock in the interest rate for each of the reinvested coupons. We discuss forward rate agreements in Section 7.2.

Continuously Compounded Yields

Any interest rate can be quoted as either an effective annual rate or a continuously compounded rate. (Or in a variety of other ways, such as a semiannually compounded rate, which is common with bonds. See Appendix 7.A.) Column (5) in Table 7.1 presents the continuously compounded equivalents of the rates in the "zero yield" column.

In general, if we have a zero-coupon bond paying $1 at maturity, we can write its price in terms of an annualized continuously compounded yield, $r^{cc}(0, t)$, as[3]

$$P(0, t) = e^{-r^{cc}(0,t)t}$$

Thus, if we observe the price, we can solve for the yield as

$$r^{cc}(0, t) = \frac{1}{t} \ln[1/P(0, t)]$$

We can compute the continuously compounded 3-year zero yield, for example, as

$$\frac{1}{3} \ln(1/0.816298) = 0.0676586$$

Alternatively, we can obtain the same answer using the 3-year zero yield of 7%:

$$\ln(1 + 0.07) = 0.0676586$$

Any of the zero yields or implied forward yields in Table 7.1 can be computed as effective annual or continuously compounded. The choice hinges on convention and ease of calculation.

7.2   FORWARD RATE AGREEMENTS, EURODOLLAR FUTURES, AND HEDGING

We now consider the problem of a borrower who wishes to hedge against increases in the cost of borrowing. We consider a firm expecting to borrow $100m for 91 days, beginning 120

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3. In future chapters we will denote continuously compounded interest rates simply as r, without the cc superscript.